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ALGEBRAIC FERMI CURVES
[after Gieseker, Trubowitz and Knörrer]

by Chris PETERS

0. INTRODUCTION

We give an overview of the work [GKT2] of Gieseker, Trubowitz and Knörrer on the theory of algebraic Fermi curves. A technical summary of the results can be found in [GKT1]. Here we concentrate more on the background from solid state physics (to be recalled in §1) and a more leisurely account of their main results and techniques.

In the discrete approximation one can use techniques from algebraic geometry, while even in the original independent electron approximation one works with highly non algebraic analytic varieties where both the geometry and the analysis are very difficult.

More results in the discrete case can be found in [B1], [B2], [K] and [PS]. Some related results in the continuum case can be found in [BKT], [G] and [KT]. See also §8.

1. BACKGROUND FROM PHYSICS

The following model from solid state physics is called *the independent electron model.* Details can be found e.g. in [AM]. Fix a lattice $\Gamma \subset \mathbb{R}^d$ $(d \leq 3)$ of ions (so we assume that the ions don't move) and a gas of electrons, which move independently under the influence of a potential $q(x)$ which is periodic in Γ (this potential describes the total effect of ions plus electrons). Each individual electron is given by its wave function which is a superposition of solutions $\Psi : \mathbb{R}^d \to \mathbb{C}$ of the Schrödinger equation

$$(-\Delta + q(x))\Psi = \lambda\Psi, \lambda \in \mathbb{R} \tag{1.1}$$

with boundary conditions

$$\Psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\Psi(x) \ \ \forall\gamma \in \Gamma,\ k \in \mathbb{R}^d \text{ fixed.} \tag{1.2}$$

Assuming that $q \in L^2(\mathbb{R}^d/\Gamma)$, (1.1) and (1.2) determine a self adjoint boundary value problem yielding a discrete spectrum

$$E_1(k) \leq E_2(k) \leq \dots,$$

where every $E_j(k)$ (the *energy for crystal momentum* k) depends continuously on k and is called the j-th band function. It is periodic in the dual lattice

$$\Gamma^\sharp = \{b \in (\mathbb{R}^d)^* \mid \langle b, \Gamma\rangle \subset 2\pi\mathbb{Z}\}.$$

To explain what physicists mean by Fermi-surface, let F be a fundamental domain for the lattice Γ and let us approximate solid matter by a box $L \cdot F \subset \mathbb{R}^d$, $L \in \mathbb{Z}_+$ containing $L^d n = N$ electrons (n=*electronic density* is kept fixed) and take the limit when L goes to infinity. In other words restrict solutions of the preceding problem to L^2-functions on the box $L \cdot F$ having the same values on corresponding boundary points. Then the crystal momenta belong to $(1/L) \cdot \Gamma^\sharp$. The Pauli exclusion principle dictates that each electron can be placed in exactly one energy level (two if we take spin into account) so that the lowest energy level for the system consisting of N electrons in the box of solid matter is the sum of the first N eigenvalues for these momenta. The largest energy of an individual electron for this *ground state* of the box has a finite limit when L goes to infinity and is called the Fermi-energy. The surface in crystal momentum k-space with this energy is the Fermi-surface. It separates occupied states from non-occupied states at absolute zero. The shape of the Fermi-surface can be measured experimentally. Its properties in turn predict qualitative behaviour of matter, e.g. whether it acts as a conductor, semi-conductor or insulator.

More generally one can define the Fermi-hypersurface in $\mathbb{R}^d$ for energy λ as the hypersurface in the space of crystal momenta with energy precisely λ. In view of the periodicity with respect to $\Gamma^\sharp$ one can replace this Fermi-hypersurface by its image in $\mathbb{R}^d/\Gamma^\sharp$. For simplicity, let us assume that $\Gamma = \mathbb{Z}^d$. Then $\mathbb{R}^d/\Gamma^\sharp$ is a direct product of d circles which we can view as unit circles in the complex plane. This leads to the description of the Fermi-variety as

$$\{(\xi_1, \dots, \xi_d) \in (S^1)^d \mid \exists \Psi \neq 0 \text{ solving (1.1) with}$$
$$\Psi(x_1, \dots, x_i + 1, \dots, x_d) = \xi_i \Psi(x) \text{ for } i = 1, \dots, d\}$$

This Fermi-variety can be considered as the fiber over λ of a hypersurface inside $(S^1)^d \times \mathbb{R}$ under projection onto the last factor.

Going one step further, we can consider solutions (ξ, λ) varying in $(\mathbb{C}^*)^d \times \mathbb{C}$, yielding a complex analytic variety B fibered in $d-1$-dimensional varieties on which the Fermi-hypersurfaces are real cycles of real dimension $d-1$.

For $d = 1$ the variety B is a hyperelliptic curve which generically has infinitely many branch points. It has been studied by McKean and Trubowitz in [MT]. For higher d there are some partial results available, see §8. These are motivated by certain results in a discrete approximation, to which we will turn in a moment. Here the analogue of the Fermi variety is an algebraic variety and so one can make use of the rich arsenal of results and methods from Algebraic Geometry.

2. A DISCRETE MODEL

In order to focus on the geometric aspects of the problem Gieseker, Knörrer and Trubowitz turn to a discrete approximation, which we now describe. Inside $\mathbb{Z}^d$ one takes the lattice

$$\Gamma = \oplus_{j=1}^d \mathbb{Z} \cdot a_j \mathbf{e}_j,$$

where $\mathbf{e}_j$ is the j-th standard basis vector. Introduce a fundamental domain for Γ:

$$F := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid 1 \leq x_j \leq a_j, j = 1, \ldots, d\}.$$

The vector space of complex valued Γ-periodic functions on $\mathbb{Z}^d$ with the usual inner product

$$\langle \varphi, \psi \rangle = 1/(a_1 \cdots a_d) \sum_{x \in F} \varphi(x)\overline{\psi}(x)$$

is denoted by

$$L^2(\mathbb{Z}^d/\Gamma).$$

Potentials $q(x)$ are in this set $L^2(\mathbb{Z}^d/\Gamma)$, in particular they are allowed to be complex valued. Define the shift operators S_j acting on functions $\Psi : \mathbb{Z}^d \to \mathbb{C}$ by

$$S_j\Psi(x_1, \ldots, x_j, \ldots, x_d) = \Psi(x_1, \ldots, x_j + 1, \ldots, x_d)$$

and the discrete Laplacian by

$$\Delta = \sum_{j=1}^{d} S_j + S_j^{-1}.$$

The corresponding discrete problem translates into

$$(2.1) \qquad \begin{cases} (-\Delta + q - \lambda)\Psi = 0 & (\lambda \in \mathbb{C}) \\ \qquad\qquad S_j^{a_j}\Psi = \xi_j\Psi & (\xi_j \in \mathbb{C}^*), j = 1, \ldots, d \end{cases}$$

and one introduces

$$B(q) = \{(\xi_1, \ldots, \xi_d, \lambda) \in (\mathbb{C}^*)^d \times \mathbb{C} \mid \exists \Psi \neq 0,\ \Psi \text{ solves } (2.1)\}.$$

In [G-K-T] this variety is called *Bloch variety*. By the projection $\pi : B \to \mathbb{C}$ onto the second factor it is fibered into varieties of dimension $d - 1$, the *(complex) Fermi-varieties.* The fiber $\pi^{-1}(\lambda)$ is denoted by F_λ. The analogue of the Fermi-surface from physics is the intersection of F_λ with $(S^1)^d \times \{\lambda\}$ for real values of λ.

Since $S_j^{a_j}\Psi = \xi_j\Psi$, the function Ψ is determined by its A values on the fundamental domain F, where

$$A = a_1 \cdots a_d.$$

Now (2.1) translates into the eigenvalue problem for the $A \times A$-matrix $\mathcal{P}$ one gets by writing out $-\Delta + q$ as acting on the A-dimensional space $\{\Psi(x), x \in F\}$. Since this matrix has entries which are linear functions in the variables $\xi_1, \xi_1^{-1}, \dots, \xi_d, \xi_d^{-1}$, we conclude that $B(q)$ is an *algebraic hypersurface* of degree A in $(\mathbb{C}^*)^d \times \mathbb{C}$ given by the characteristic equation $P(\xi_1, \dots, \xi_d, \lambda) = 0$ of the matrix $\mathcal{P}$:

$$B(q) = \{(\xi_1, \dots, \xi_d, \lambda) \in (\mathbb{C}^*)^d \times \mathbb{C} \mid P(\xi_1, \dots, \xi_d, \lambda) = 0\}. \tag{2.2}$$

3. THE DENSITY OF STATES

In solid state physics another experimentally observable quantity plays an important rôle : the (integrated) density of states. If H_n is the operator $-\Delta + q$ acting on the space of complex valued functions on $\mathbb{Z}^n$ with periodicity $n\Gamma$, we can define the integrated density of states function as $\rho(\lambda) = \lim_{n\to\infty} \rho_n(\lambda)$, with

$$\rho_n(\lambda) = 1/(n^d A) \cdot \text{ number of eigenvalues of } H_n \text{ less than or equal to } \lambda.$$

An easy computation shows that the (non-integrated) density of states function $d\rho/d\lambda$ is equal to

$$\frac{1}{(2\pi)^d A} \sum_{j=1}^{A} \int_{\mathbb{R}^d/2\pi\mathbb{Z}^d} \delta(\lambda - E_j(k))dk,$$

where δ is the delta-function and $E_j(k)$ is the (analogue of the) j-th band function. Since the region over which we integrate is precisely the 'real' Fermi-surface $\phi_\lambda = F_\lambda \cap (S^1)^d \times \{\lambda\}$ we get

$$\frac{d\rho}{d\lambda} = \frac{1}{(2\pi)^d A} \sum_{j=1}^{A} \int_{E_j(k)=\lambda} \frac{dS}{|\nabla_k E_j|} = \frac{1}{(2\pi)^d A} \int_{\phi_\lambda} \omega_\lambda, \tag{3.1}$$

where

$$\omega_\lambda = (-1)^i \frac{\frac{\partial}{\partial\lambda} P d\xi_1 \wedge \ldots \wedge \widehat{d\xi_i} \wedge \ldots d\xi_d}{\xi_1 \cdots \xi_d \frac{\partial}{\partial \xi_i} P}$$

is the restriction to F_λ of the relative $d-1$-form ω on the Bloch variety defined by

$$\pi^*(d\lambda) \wedge \omega = \frac{d\xi_1}{\xi_1} \wedge \ldots \wedge \frac{d\xi_d}{\xi_d}.$$

So the density of states appears as an integral of a holomorphic $d-1$-form over a real $d-1$-cycle on F_λ. Note that one can define this only for q real valued and λ real. If now λ is a regular value for the projection π, the fibration π is differentiably a product near λ so the cycle ϕ_λ extends naturally to fibres near the fibre we started with and $\frac{d\rho}{d\lambda}$ defines a germ of an analytic function near λ.

The preceding observations formed the starting point for [G-K-T] and led to some striking results which we can now formulate.

4. MAIN RESULTS IN DIMENSION $d = 2$

We assume from now on that

$$a := a_1 \text{ and } b := a_2 \text{ are distinct odd primes.}$$

(4.1) **Theorem** *There is a Zariski open dense set $L_1 \subset L^2(\mathbb{Z}^2/\Gamma)$ such that $B(q) = B(q')$, $q \in L_1$, $q' \in L^2$ implies that there exists some $(x_0, y_0) \in \mathbb{Z}^2/\Gamma$ with $q'(x,y) = q(\pm x + x_0, \pm y + y_0)$.*

In other words for generic potentials the Bloch-variety determines the potential up to the obvious symmetries.

To formulate the next results, we need to introduce precise genericity conditions.

(4.2) **Definition**
(1) A potential $q \in L^2(\mathbb{Z}^2/\Gamma)$ is generic for the Bloch-projection $B(q) \to \mathbb{C}$ if π has exactly

$$\nu = 2a^2b^2 + 6ab(a+b) + 12ab - 12(a^2+b^2) - 2(a+b) - 12$$

critical values.
(2) A real valued potential is generic with respect to the density function if the analytic continuation of the germ of the density of states function (3.1) near a real value where it is analytic (that is, near a real non-critical value of the Bloch-projection) has precisely ν ramification points.

The next result states that generic potententials have good properties and that there are many of them.

(4.3) **Proposition** *(1) The potentials generic with respect to the Bloch projection form a Zariski open dense subset $L_2 \subset L^2(\mathbb{Z}^2/\Gamma)$.*
(2) A real potential which is generic for the Bloch projection is generic with respect to the density of states function and conversely.
(3) Moreover for potentials in L_2 we have
- The Bloch variety is smooth.
- The Fermi curves over $4ab$ of the critical values of π have exactly one ordinary double point and exactly two ordinary double points over the remaining critical values.

Finally we have

(4.4) **Theorem** *Let $q, q' \in L^2(\mathbb{Z}^2)$ be real valued potentials and assume that the germs of the density of states functions for q and q' near a real* *point coincide. If $q, q' \in L_2$ either $B(q) = B(q')$ or $B(q') = jB(q)$, where j is the involution $j(\xi_1, \xi_2, \lambda) = (\xi_1^{-1}, \xi_2, \lambda)$.*

Combining the previous results we see that the density of states function for a generic real potential essentially determines the potential.

Remark

One would also want to see from properties of the germ f at λ of the density of states function alone whether the fibre over λ is smooth. In fact this can be read off from the lattice Γ_λ generated by analytic continuation of f inside the ring of germs of holomorphic functions at λ. If some continuation of f yields a germ which is multivalued at λ, the fibre over this point certainly is singular, so we may assume that this is not the case. The result complementing Proposition 4.3 says that the Fermi curve over λ is smooth when Γ_λ has rank $2ab$. Moreover, if this is the case, one only has to find ν branch points for f since one can show that there cannot be more of them in this case. Concerning the proofs we make a few preliminary remarks. The proof of theorem 4.1 is rather straightforward and uses the geometry of a suitable compactification of the Bloch variety. We give a sketch in section 6 after we discuss an intrinsic compactification in the next section. The proof of Proposition 4.3 is surprizingly subtle and uses several delicate degeneration arguments, which also play a rôle in the proof of Theorem 4.4. We don't say anything about the proof of Proposition 4.3, but we give a sketch of the long and intricate arguments employed in the proof of Theorem 4.4.

5. A COMPACTIFICATION OF THE BLOCH VARIETY

Motivated by an idea of Mumford (see [M]), Bättig in [B1] has constructed the following intrinsic compactification of the Bloch variety .

Consider the algebraic torus $T = (\mathbb{C}^*)^3 \subset (\mathbb{C}^*)^2 \times \mathbb{C}$. We let T_Σ be the toroidal compactification of T corresponding to the fan Σ in $\mathbb{R}^3$ consisting of the cones (with vertex the origin) over the faces of the prism of Fig. 1.

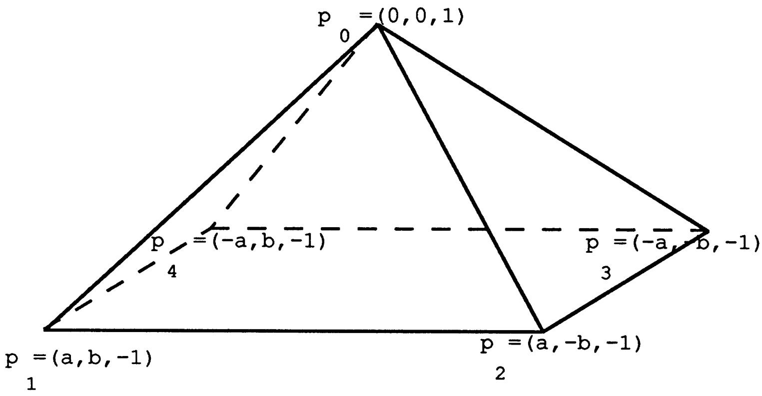

Figure 1. Fan Σ.

The corresponding 'cradle' (Fig. 2) is a singular complete algebraic variety with one-dimensional singular locus. The latter is stratified into nine T-orbits, four of dimension 1 and five of dimension 0. The one-dimensional orbits correspond to the codimension one cones over the four horizontal edges of the prism. These four curves have transversal A_k type, two with $k = 2a - 1$ and two with $k = 2b - 1$. The zero dimensional orbits in the closures of the one dimensional orbits correspond to the zero codimensional faces. Observe that $(\mathbb{C}^*)^2 \times \mathbb{C}$ is embedded in T_Σ as the open chart defined by the torus embedding T_σ, where σ is the cone $\mathbb{R}_{\geq 0}(0,0,1)$. We therefore can take the closure of $B(q)$ in the cradle T_Σ. The resulting variety is always singular in the four points P_{ij} where it meets the singular locus of the cradle (see Fig. 2). Blow up these singular points in the cradle and form the proper transform $\overline{B(q)}'$ of $B(q)$.

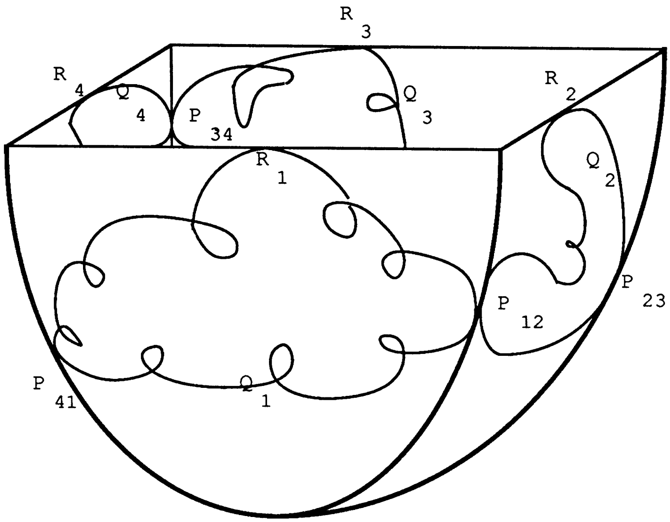

Figure 2. Cradle T_Σ.

Before stating the next proposition, recall the notation

$$\mu_a := \text{ group of } a\text{-th roots of unity}$$
$$\mu_a^\times := \text{ the set of primitive } a\text{-th roots of unity.}$$

(5.1) **Proposition** *(i) The four exceptional surfaces on the blown up cradle intersect the proper transform* $\overline{B(q)}'$ *of* $B(q)$ *in four curves* H_{ij}*. These curves are hyperelliptic. They are isomorphic in pairs. The two (isomorphism classes of) curves are the Bloch varieties for the one dimensional potential obtained by averaging over each of the two coordinates.*
(ii) There are precisely four other curves Q_j *added at infinity* $\overline{B(q)}' \setminus B(q)$*. These curves are singular rational curves with* $(a-1)(b-1)$ *ordinary double points naturally indexed by the elements of* $\mu_a^\times \times \mu_b^\times$*. The curve* Q_j *intersects the* T*-orbit corresponding to the one codimensional cone over the line through* p_0 *and* p_j *(see Fig. 1) in a point* R_j *which is a smooth point*

on $\overline{B(q)}'$.

(iii) The Bloch projection $\pi : B(q) \to \mathbb{C}$ *extends to a rational map* $\overline{\pi}' : \overline{B(q)}' - - \to \mathbb{P}^1$ *which is everywhere defined except at the points* R_j, $j = 1,2,3,4$. *After blowing up* $\overline{B(q)}'$ *at these points we obtain the* *<u>compactified Bloch variety</u>* $\overline{B(q)}$ *and the map* $\overline{\pi}'$ *extends as a morphism.*

$$\overline{\pi} : \overline{B(q)} \to \mathbb{P}^1.$$

The four exceptional curves are sections for this fibration.

(5.2) **Remark** In [GKT] the compactification is described in a more complicated, but equivalent way.

6. SKETCH OF THE PROOF OF THEOREM 4.1

Recall the definition of the Fourier coefficients of a potential $q \in L^2(\mathbb{Z}^2/\Gamma)$. For each $\rho = (\rho_1, \rho_2) \in \mu_a \times \mu_b$ we set

$$\hat{q}(\rho) = \frac{1}{ab} \sum_{(m,n)\in F} q(m,n)\overline{\rho}_1^m \overline{\rho}_2^n.$$

The Bloch-variety $B(q) = \{(\xi_1, \xi_2, \lambda) \mid P(\xi_1, \xi_2, \lambda) = 0\}$ determines:
(i) The polynomial $P(1,1,\lambda)$, whose roots are the periodic spectrum of the Schrödinger operator,
(ii) The values at q of the function

$$f_\rho(q) = \hat{q}(\rho)\hat{q}(\overline{\rho}), \quad \rho \in \mu_a^\times \times \mu_b^\times.$$

The last assertion can be shown by looking at the equation of the Bloch-variety near one of the points Q_j (see Proposition 5.1). It turns out that there are local coordinates (x, y, z) near Q_j independent of the potential q such that the Bloch-variety has an equation

$$f_\rho \, x^2 = y^2 + z^2 + \text{ higher order terms }.$$

The polynomial $P(1,1,\lambda)$ belongs to the affine ab-dimensional space

$$\mathcal{P} = \text{ polynomials of degree } ab \text{ with } -1 \text{ as leading coefficient.}$$

Define a family of algebraic morphisms

$$\varphi_\epsilon : L^2 \to \mathcal{P},$$

sending q to the determinant of $-\epsilon\Delta + q - \lambda$ acting on the periodic functions L^2. Of course $\varphi_1 = P(1,1,\lambda)$ and $\varphi_0 = \prod_{(m,n)\in F}((q(m,n)-\lambda)$, so a fibre of φ_0 consists of precisely those potentials that are related by the obvious action of the symmetric group $\mathfrak{S}_{ab}$ on L^2:

$$\sigma(q)(m,n) = q(\sigma(m,n)), \ \ \sigma \in \mathfrak{S}_{ab}.$$

The subgroup $\mathfrak{D} \subset \mathfrak{S}_{ab}$ generated by $(n,m) \mapsto (n+1,m)$, $(n,m) \mapsto (n,m+1)$, $(n,m) \mapsto (-n,-m)$ has the property that any element σ in it preserves the Bloch-variety: $B(\sigma(q)) = B(q)$. We need to see that for generic q conversely $B(q') = B(q)$ implies that $q' = \sigma(q)$, $\sigma \in \mathfrak{D}$. To this end fix $\rho \in \mu_a^\times \times \mu_b^\times$ and set $f = f_\rho$. This function takes the same values in the points of a fibre of φ_1 which are related by an element of $\mathfrak{D}$. To decide whether $\sigma \in \mathfrak{S}_{ab}$ belongs to $\mathfrak{D}$ we use

(6.1) **Lemma** *If $f(\sigma(q)) = f(q)$ for all potentials q, σ belongs to $\mathfrak{D}$.*

The proof of this lemma is relatively easy if one uses some carefully chosen potentials.

Next, we remark that

$$\varphi_\epsilon(q)(\lambda) = \epsilon^{ab}\varphi_1(\frac{q}{\epsilon})(\frac{\lambda}{\epsilon}).$$

So if $\{q_1,\ldots,q_N\}$ is a fiber of φ_ϵ, the points $q_1/\epsilon,\ldots,q_N/\epsilon$ form a fiber of φ_1 and conversely. So it suffices to show that the function f separates points in a fibre of φ_ϵ, $(\epsilon \neq 0)$ not related by an element of $\mathfrak{D}$. It is easily seen that for $|\epsilon|$ small, the map φ_ϵ is a finite dominant morphism of degree $N = |\mathfrak{S}_{ab}| = (ab)!$. Note that φ_ϵ factors as $\hat{\varphi}_\epsilon \cdot p$, where $p : L^2 \to L^2/\mathfrak{D}$ is the is the quotient map. Likewise f factors as $\hat{f} \cdot p$. There is a Zariski-open subset U of $\mathbb{C}$ and a Zariski-open subset $\mathcal{P}_1$ of $\mathcal{P}$ such that $P \in \mathcal{P}_1$ if and only if there exists $\epsilon \in U$ with $P = \varphi_\epsilon(q)$ for some $q \in L^2$ with the following two properties:

i. The fibre of $\varphi_\epsilon^{-1}(P)$ consists of N points,

ii. The function $\hat{f}$ separates the points of the fibre of $\hat{\varphi}_\epsilon^{-1}(P)$.

Since by Lemma 6.1 the function $\hat{f}$ separates the fibres of $\hat{\varphi}_0$ it follows that U and $\mathcal{P}_1$ are non-empty. ∎

7. SKETCH OF THE PROOF OF THEOREM 4.4

Step 1 : the monodromy representation

On the Bloch variety we have an involution $\imath$ given by

$$\imath(\xi_1, \xi_2, \lambda) = (\xi_1^{-1}, \xi_2^{-1}, \lambda).$$

We first divide out $\overline{B(q)}$ by this involution, obtaining a variety $\overline{Y(q)}$ with fibres $Y(q)_\lambda$. If q is generic with respect to the Bloch projection one can show:

1) $Y(q)_\lambda$ is smooth whenever $\lambda \in \mathbb{P}^1 \setminus D$, ($D$ a finite set of points, called the *Van Hove singularities*),

2) The fibre over a finite Van Hove singularity contains precisely one ordinary double point which either is smooth on $\overline{Y(q)}$ or an ordinary double point. The second type of singularity is called a *spectral Van Hove singularity*, because it lies over a point of the periodic-anti-periodic spectrum.

3) Y_∞ has four components, the non-isomorphic components of F_∞ : two hyperelliptic curves corresponding to the one-dimensional averaged potentials and two rational curves. Each of the four points where a rational curve and a hyperelliptic curve meet is an ordinary double point for Y_∞ and smooth on $\overline{Y(q)}$. They define four vanishing cycles which are easily seen to be homologous and so yield a class $\gamma_\infty \in H_1(Y_\lambda, \mathbb{Z})$ for λ close to ∞.

These facts suggest to study the monodromy representation

$$r : \pi_1(\mathbb{P}^1 \setminus D, \lambda) \to \mathrm{Aut}\ H_1(Y_\lambda, \mathbb{Z}) \tag{7.1}$$

by means of a Lefschetz-type argument. This turns out to be surprizingly difficult. One lets q degenerate to a generic separable potential (i.e. of the form $q_1(x_1) + q_2(x_2)$) where the monodromy can be computed explicitly and then one has to use connectedness of the set of good potentials. The final result is as follows

(7.2) **Proposition** *i) The monodromy representation (7.1) is absolutely irreducible.*
ii) For λ a negative real number close to ∞ the smallest r-invariant sublattice of $H_1(Y_\lambda, \mathbb{Q})$ generated by $\frac{1}{2}(\gamma_\infty - r(\alpha)\gamma_\infty)$, $\alpha \in \pi_1(\mathbb{P}^1 \setminus D, \lambda)$ is equal to $H_1(Y_\lambda, \mathbb{Z})$.

Step 2: Recovering γ_∞.

Since ω is $\imath$-invariant it descends to a relative holomorphic one form for the family $Y(q) \to \mathbb{C}$ denoted by the same symbol. Let α be a non bounding 1-cycle on Y_λ. Displace α to a cycle on Y_s for s in a small open neighbourhood U of λ, where the fibration is provided with some differentiable trivialisation. This cycle is still denoted as α. On U we have a germ $f_\alpha = \int_\alpha \omega_s$ of an analytic function. Let $\mathcal{O}$ denote the ring of germs of holomorphic functions near λ. Analytic continuation of f_α over paths in $\mathbb{P}^1 \setminus D$ defines an injection

$$I : H_1(Y_\lambda, \mathbb{Z}) \longrightarrow \mathcal{O}.$$

A special loop l_∞ is defined as follows. First, starting from λ, go to left along the negative real axis, make a big circle which clockwise encircles all Van Hove singularities, then go back to λ along the real axis. If $g \in \mathcal{O}$ we let g_∞ be the germ obtained from g by analytically continuation along the loop l_∞.

Let q be a generic real potential and λ not a Van Hove singularity. Then

$$f := \int_{\phi_\lambda + i(\phi_\lambda)} \omega_\lambda$$

is twice the density of states. Let H_λ be the lattice inside $\mathcal{O}$ generated by all analytic continuations of the density of states function $\frac{1}{2}f$. Choose $g \in H_\lambda$ such that

1 g is invariant under complex conjugation,

2 g_∞ differs from g.

Let $h := g - g_\infty$. It can be shown that

$$I(\gamma_\infty) = \frac{2\pi ab}{\lim\limits_{t \to -\infty} h(t)} h,$$

where we take the limit along the negative real axis. So from monodromy alone we have recovered γ_∞. From Proposition 7.2 we see that we can recover $H_1(Y_\lambda, \mathbb{Z})$.

Step 3: Invoking Torelli's theorem

If now q' is another real potential with the same density of states function near λ we can define for s near λ an r-equivariant isomorphism

$$\alpha : H_1(Y(q)_s, \mathbb{Z}) \to H_1(Y(q')_s, \mathbb{Z}) \tag{7.3}$$

upon setting

$$\int_\gamma \omega_s(q) = \int_{\alpha(\gamma)} \omega_s(q'),\ \ \forall \gamma \in H_1(Y(q)_s, \mathbb{Z}).$$

If one collects the periods for a basis of the regular one forms for Y_s with respect to a homology basis in a $2g$ by g matrix, one gets the period matrix Ω_s for Y_s. Different bases correspond to equivalent period matrices. Recall

(7.4) **Torelli's theorem** *Two Riemann surfaces are isomorphic if and only if their period matrices are equivalent.*

This suggest that we should find the periods of all one forms rather than only those for the density of states form ω_s. To do this we have to use irreducibility of the monodromy representation (7.1) again together with a deep result: the *Theorem of the fixed part* due to Deligne. To explain the consequence of this theorem needed in the proof we have to recall the notion of *Hodge structure of weight* w on a finitely generated $\mathbb{Z}$-module H. It is nothing but a direct sum decomposition of the $\mathbb{C}$-vector space $H \otimes_{\mathbb{Z}} \mathbb{C}$ into subspaces $H^{k,w-k}$ *of Hodge type* $(k, w-k)$ with the property that $H^{w-k,k} = \overline{H^{k,w-k}}$ (conjugation with respect to the natural complex structure on $H \otimes_{\mathbb{Z}} \mathbb{C}$).
The $\mathbb{Z}$-module $H^1(F(q)_s.\mathbb{Z})$ carries a weight one Hodge structure: $H^{1,0}F(q)_s$ consists of the g rows of Ω_s, each considered as an element of $H^1(F(q)_s, \mathbb{C})$ $= \mathrm{Hom}_{\mathbb{C}}(H_1(F(q), \mathbb{Z}), \mathbb{C})$ written out in the basis dual to the given homology basis.
The usual linear algebra constructions applied to Hodge structures yield new Hodge structures. E.g. if H and H' carry Hodge structures of weight w the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(H, H')$ carries a weight zero Hodge structure: a $\mathbb{C}$-linear $\psi : H \to H'$ has type $(-i, i)$ if $\psi(H^{k,w-k}) \subset (H')^{k-i,w-k+i}$.
The consequence of Deligne's Theorem of the Fixed Part [D, Cor. 4.1.2] reads as follows:

(7.5) **Theorem** *The weight zero Hodge structure on the $\mathbb{Z}$-module* $\mathrm{Hom}(H^1(Y(q')_\lambda,\mathbb{Z}),H^1(Y(q)_\lambda,\mathbb{Z})$ *induces one on the sublattice of homomorphisms which are equivariant with respect to the monodromy representation (7.1).*

Since monodromy acts irreducibly on cohomology with complex coefficients, Schur's lemma implies that these equivariant homomorphisms form a rank one lattice and Deligne's theorem implies that they all have type (0,0), i.e they preserve the Hodge types. But then the homomorpism α (see (7.3)) also preserves the Hodge types, i.e the Riemann surfaces $Y(q)_\lambda$ and $Y(q')_\lambda$ have the same period matrices and Torelli's theorem implies that they must be isomorphic. From this point on it is not difficult to get a global isomorphism between the Bloch varieties $B(q)$ and $B(q')$ respecting the Bloch-projections.

8. RELATED RECENT RESULTS

8.1 Kappeler in [K] generalizes Theorem 4.1 to any dimension. His proof is similar to the one from [GKT2] and which is presented in §6, but the use of the compactification is eliminated since another function is used to separate the fibres of $\hat{\varphi}_\epsilon$ (see §6 for notation).

8.2 The density of states function can be used to distinguish the spectral Van Hove singularities (see the beginning of §7) from the other Van Hove singularities. So for a real potential which is generic in the sense of definition 4.2. the density of states function determines the periodic-antiperiodic spectrum.

In the continuum case, if the lattice Γ is generic in the sense that there are at most two lattice points on any sphere centered at the origin, the periodic-antiperiodic spectrum is known to determine the Bloch variety (Theorem 6.2 in [ERT]). So in the continuum case one would like to show that the periodic-antiperiodic spectrum can be recovered from the density of states function alone.

8.3 Observe that in the discrete case, the Bloch variety B is the locus of points in $(\mathbb{C}^*)^d \times \mathbb{C}$ where the $d+1$ commuting operators

$$S_\lambda := \Delta + q - \lambda,\ T_{\xi_j} := S_j^{a_j} - \xi_j,\ j = 1, \ldots, d,$$

have a common kernel in the space

$$V^d = \{\Psi : \mathbb{Z}^d \to \mathbb{C}\}.$$

In other words B is the support of the subsheaf $\mathcal{L}$ of the trivial bundle $B \times V^d$ given by

$$\mathcal{L} := \{((\xi_1, \ldots, \xi_d), \lambda, \Psi) \in (\mathbb{C}^*)^d \times \mathbb{C} \times V^d \mid S_\lambda \Psi = 0 = T_{\xi_j} \Psi,\ j = 1, \ldots, d\}.$$

Bättig shows in [B1] how for $d = 2$ one can extend $\mathcal{L}$ to a sheaf over his compactification $\overline{B}$ (see §5). Moreover he rewrites the spectral problem on certain coordinate patches in such a way that one immediately sees that the curves M_{ij} (loc. cit.) are the supports of a sheaf defining the one-dimensional problem for the potential obtained by averaging q over one of the two coordinates.
For $d = 3$ Bättig describes in [B2] a toroidal compactification of the complex Fermi surface F_λ. Here one gets four curves at infinity independent of q and twelve hyperelliptic curves independent of λ, but depending on q. The twelve curves come in three quadruples of mutually isomorphic curves and each of these curves is isomorphic to the Bloch variety for the potential obtained from q by averaging over two of the three coordinates.
It turns out that in the continuum case for $d = 3$ a certain directional compactification exists with similar properties. See [BKT] for details (see also [KT] for the two dimensional continuum case).

8.4 A more or less standard argument shows that the data $(B, \mathcal{L}|B)$ ($\mathcal{L}$ is the sheaf considered in §8.3; it is a line bundle on B) gives back the potential q. See [Mu, §2] and [VKN] for a related situation.

If one could characterize $\mathcal{L}|B$ intrinsically from the geometry of B it would follow that B gives back the potential. In this respect it is useful to note that one can show that the compactified Bloch variety for $d = 2$ has first Betti number zero so that a line bundle is completely characterized by its Chern class in $H^2(\overline{B})$. The problem however is to find a good description of this cohomology group which would single out Chern classes coming from potentials in this way.

8.5 Gerard in [G] studies the singularities of the resolvent $r(\lambda)$ of $-\Delta + q$ on $\mathbb{R}^d$ and shows that one can analytically locally extend $r(\lambda)$ around a spectral value λ and the singularities one gets are Van Hove singularities. Here one has to redefine the Van Hove singularities as being critical values for the Bloch projection π restricted to a stratum of a Thom stratification for π. Also the global problem is considered. A similar statement holds, but one possibly must add some extra singularities. It would be interesting to see whether the compactification from [BKT] can be used to see whether these singularities actually are present, at least for generic potentials.

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