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Astérisque, tome 177-178 (1989), Séminaire Bourbaki,
exp. nº 704, p. 95-126
<http://www.numdam.org/item?id=SB_1988-1989__31__95_0>

Séminaire BOURBAKI
41e année, 1988-89, n° 704

Novembre 1988

CONFORMAL FIELD THEORY

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One of the main trends in contemporary theoretical physics is the gradual convergence of quantum mechanics and geometry. The appearence of quantum gauge theories of elementary particles, then, more recently, of string theory (a hypothetical quantum gravity theory) marked the milestones of this process. The geometrization of quantum physics promises not only to reveal more fundamental physical laws based on geometrical principles. It also produces novel insights which ultimately may lead to new geometry. As such, it deserves a close scrutiny by the mathematicians.

Conformal field theory (CFT) is a recent example of a physical theory undergoing the geometrization process. It has already revealed new relations between different branches of mathematics, to mention only representation theory of infinite-dimensional groups, theory of finite sporadic groups, algebraic geometry of moduli spaces of complex curves and of vector bundles over them and knot theory. Some of these relations will be the topic of the present exposition.

0. 1. Among the original motivations behind CFT was an effort to understand the asymptotic behaviour of statistical mechanical systems like the celebrated Ising model in d dimensions. The basic object of that model is a probability measure on the space of "spin configurations" (σ_x), $\sigma_x = \pm 1$, $x \in \Lambda_R = \{x \in \mathbf{Z}^d |\ |x| < R\}$. The properties of the measure are encoded in the correlation functions

$$< \prod_{i \in I} \sigma_{x_i} >_{\beta,R} = \sum_{\substack{\sigma_x = \pm 1 \\ x \in \Lambda_R}} \prod_{i \in I} \sigma_{x_i} \exp[\beta \sum_{\substack{x,y \in \Lambda_R \\ |x-y|=1}} \sigma_x \sigma_y] / \sum_{\substack{\sigma_x = \pm 1 \\ x \in \Lambda_R}} \exp[\beta \sum_{\substack{x,y \in \Lambda_R \\ |x-y|=1}} \sigma_x \sigma_y]$$

which exist also in the infinite volume limit $R \to \infty$. It is expected that for $d > 1$ and an appropriate choice of β and η, the scaling limit of the correlation functions

$$\lim_{n \to \infty} n^{\eta |I|/2} < \prod_{i \in I} \sigma_{n x_i} >_{\beta,\infty} \equiv \mathbf{G}(x_i) \tag{0.1}$$

exists, is non-trivial and gives smooth functions of (x_i) for $x_i \neq x_{i'}$ ($|I|$ denotes the number of elements of I).

It was realized quite early that, besides the scaling covariance, functions $\mathbf{G}$, called sometimes *Green functions*, should possess a richer conformal symmetry imposing strong restrictions on them [67,58]. However, a real breakthrough which has started the present wave of renewed interest in CFT came with the paper of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [6] who showed that in two dimensions the conformal symmetry determines the Green functions for the Ising and many other statistical-mechanical models completely.

0. 2. Another boost for (two-dimensional) CFT originated in the realization (see e.g. [28]) of the role which its techniques play in *string theory*, an attempt at a quantum theory unifying gravity with other forces of nature [47]. In some sense, CFT can be viewed as a theory of classical and perturbative string-theory solutions. This broadens considerably the field of its possible physical applications.

0. 3. As in any quantum theory, the mathematics of CFT may be described by giving a Hilbert space H (of states), a projective representation of a relevant (symmetry) group and certain algebras of operators in H. The distinctive feature of CFT is that it is symmetric under a group of conformal transformations which in two dimensions is the infinite-dimensional (hence the strength of the symmetry!) group $Diff_+S^1 \times Diff_+S^1$. $Diff_+S^1$ is the group of the orientation preserving C^∞-smooth diffeomorphisms of the circle S^1. $S^1 \times S^1$ may be viewed as a compactification of the two-dimensional Minkowski space M_2 via the embedding

$$M_2 \ni (x^0, x^1) \longmapsto \left(\frac{1+ix^+}{1-ix^+}, \frac{1+ix^-}{1-ix^-}\right) \in S^1 \times S^1$$

where $x^\pm = \frac{1}{2}(x^0 \pm x^1)$. $Diff_+S^1 \times Diff_+S^1$ preserves the (pseudo-)conformal structure of $S^1 \times S^1$ inherited from that of Minkowski space.

0. 4. The plan of this exposition is as follows. We shall start by discussing the representations of $Diff_+S^1$ which play an important role in CFT. This shall give us an occasion to describe one of the most important techniques of CFT, the *"coset construction"* [43], which played an important role in the development of the representation theory of $Diff_+S^1$.

Next, we shall explain how the richer operator framework of CFT arises as a natural extension of its symmetry structure. The consistency of this framework leads to important restrictions on the symmetry content of CFT. We shall discuss the resulting *"A-D-E classification"* [13] of the *unitary series* of CFT models.

The last part of the exposition is devoted to the concept of *rational CFT*. We shall try to give a precise definition of the rational CFT and to comment briefly on the recent exciting work in this fast developing field.

0. 5. CFT is a vast subject and we shall discuss here only a limited circle of its problems. We have left completely aside its "stringy" trend (some of its mathematical aspects were discussed in [10]), its application to the theory of sporadic groups (the CFT construction of the Monster

group was reviewed in [79]), its supersymmetric and non-unitary extensions and, last but not least, its multiple relations to statistical mechanics.

Even with such a limited program, our survey risked to become much too heavy if we attempted presenting complete proofs of the main mathematical results discussed here. Instead, we have decided to refer in most cases to the original work.

Our exposition being ahistorical, we were not able to give a proper credit to many important contributions to the field. We would like to appologize for that to their authors.

1. UNITARY REPRESENTATIONS OF $Diff_+S^1$

1. 1. Virasoro algebra and its representations

1. 1. 1. $Diff_+S^1$ is a (Fréchet) Lie group [59] with the Lie algebra $VectS^1$ of smooth vector fields on S^1. Projective action of conformal symmetries in Hilbert space H requires considering central extensions of $Diff_+S^1$ and $VectS^1$. The (universal) central extension $\widehat{VectS^1}$ of $VectS^1$ by $\mathbf{R}$ is $VectS^1 \oplus \mathbf{R}Z$ with the bracket

$$(1.1)\qquad [f\frac{d}{d\theta}, g\frac{d}{d\theta}] = (fg' - f'g)\frac{d}{d\theta} + \frac{1}{24\pi}\int_0^{2\pi} d\theta(f' + f''')g\ Z, \quad [Z, f\frac{d}{d\theta}] = 0$$

as established by I.M. Gelfand and D.B. Fuchs [38].

In particular, for the elements $L_n = ie^{in\theta}\frac{d}{d\theta}$ of the complexification $\widehat{Vect}^{\mathbf{C}}S^1$ of $\widehat{VectS^1}$,

$$(1.2)\qquad [L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0}\, iZ\ .$$

The subalgebra $Vir = (\bigoplus_{n\in\mathbf{Z}} \mathbf{C}L_n) \oplus \mathbf{C}Z$ is known as the *Virasoro algebra* and it is a central extension of the algebra of (complex) polynomial vector fields on S^1. It will be useful to introduce subalgebras $T = \mathbf{C}Z \oplus \mathbf{C}L_0$, $N_+ = \bigoplus_{n\geq 1} \mathbf{C}L_n$ and $B_+ = T \oplus N_+$ of *Vir*.

1. 1. 2. The first mathematical tool of CFT is the representation theory of conformal symmetries. Its algebraic aspects can be exposed by studying representations of the Virasoro algebra. We shall be specially interested in its *unitary, highest weight (HW)* representations.

DEFINITION 1.1. *A representation (π, V) of Vir in a complex vector space V is called unitary if there exists a (positive) scalar product $\mathcal{H}(.,.)$ on V s.t.*

$$(1.3)\qquad \mathcal{H}(\pi(L_n)v, w) = \mathcal{H}(v, \pi(L_{-n})w)$$

and

$$\mathcal{H}(\pi(Z)v, w) + \mathcal{H}(v, \pi(Z)w) = 0\ .$$

DEFINITION 1.2. *(π, V) is called a HW representation if*

i/. *there exists* $v_0 \in V$ *s.t.*
$\pi(N_+)v_0 = 0,$
$\pi(iZ)v_0 = cv_0,\ \pi(L_0)v_0 = hv_0$ *with* $c, h \in \mathbf{C}$,
ii/. $\pi(\mathcal{U}(Vir))v_0 = V$ *where* $\mathcal{U}(Vir)$ *is the universal enveloping algebra of Vir.*

REMARK 1.1. In a unitary representation, the real part $Vir \cap \widehat{Vect}S^1$ of Vir is represented by skew-symmetric operators on V. This is what is needed if we want to consider ultimately the unitary projective representations of $Diff_+S^1$.

REMARK 1.2. In a HW representation, $\pi(iZ)$ acts on the whole V by multiplication by c called its *central charge*. v_0, unique up to a factor, is called a *HW vector* of the representation and (c,h) its *HW*.

1\. 1. 3. An example of a HW representation of Vir is a *Verma module*. Consider the one-dimensional representation of B_+ in $\mathbf{C}$ with iZ and L_0 acting by multiplication by c and h respectively and N_+ acting trivially. The Verma module is defined as $V_{c,h} = \mathcal{U}(Vir)\bigotimes_{\mathcal{U}(B_+)}\mathbf{C}$ with the left action π of Vir. $v_0 = 1 \bigotimes 1$ is its HW vector of HW (c,h). $V_{c,h}$ is spanned by the basis vectors

$$\pi(L_{-n_I})^{p_I} \dots \pi(L_{-n_1})^{p_1} v_0 \ , \qquad 1 \leq n_1 < \dots < n_I, \ \ p_1, \dots, p_I \geq 1 \ . \tag{1.4}$$

For (c,h) real, there exists a unique hermitian form $\mathcal{H}(.,.)$ on $V_{c,h}$ satisfying (1.3) and normalized so that $\mathcal{H}(v_0, v_0) = 1$. This form may be degenerate. Denote by $Null$ the (invariant) subspace of vectors in $V_{c,h}$ orthogonal to all other ones. Passing to the quotient gives a unique (up to the equivalence) irreducible HW representation $(\pi_{c,h}, W_{c,h} = V_{c,h}/Null)$ of (real) HW (c,h) [23].

1\. 1. 4. The central question arises when is the (non-degenerate) form $\mathcal{H}(.,.)$ on $W_{c,h}$ positive i.e. when is $(\pi_{c,h}, W_{c,h})$ a unitary representation. Since for $n > 0$

$$\begin{aligned} &\mathcal{H}(\pi(L_{-n})v_0, \pi(L_{-n})v_0) = \mathcal{H}(\pi([L_n, L_{-n}])v_0, v_0) \\ &= \mathcal{H}((2nL_0 + \frac{1}{12}(n^3 - n)c)v_0, v_0) = 2nh + \frac{1}{12}(n^3 - n)c \ , \end{aligned}$$

$c, h \geq 0$ is a necessary condition. The tool for a more complete answer is the formula for the determinant of the matrix of $\mathcal{H}(.,.)$-products of vectors (1.4) given by V.G. Kac [52] and proven by B.L. Feigin and D.B. Fuchs [22].

Consider the subspace $V_{c,h}(\ell)$ of $V_{c,h}$ corresponding to the eigenvalue $h+\ell$ of $\pi(L_0)$ ($\ell = 0, 1, \dots$). It is spanned by vectors (1.4) with $\sum_{i=1}^{I} p_i n_i = \ell$. $\mathcal{H}(V_{c,h}(\ell), V_{c,h}(\ell')) = 0$ if $\ell \neq \ell'$. For fixed ℓ, put

$$D_\ell(c,h) \ = \ \det \mathcal{H}(\pi(L_{-n_I})^{p_I} \dots \pi(L_{-n_1})^{p_1} v_0 \ , \ \pi(L_{-n'_{I'}})^{p'_{I'}} \dots \pi(L_{-n'_1})^{p'_1} v_0)$$

where $\Sigma p_i n_i = \Sigma p'_{i'} n'_{i'} = \ell$. Kac's formula is

(1.5)
$$D_\ell(c,h) = \text{const.} \prod_{\substack{r,s\geq 1\\ rs\leq \ell}} (h - h_{rs}(m))^{P(\ell - rs)}$$

where m is a root of the equation

(1.6)
$$c = c_m \equiv 1 - \frac{6}{m(m+1)},$$

(1.7)
$$h_{rs}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

and $P(k)$ is the number of partitions of k.

From (1.5), it is easy to see that for $c > 1$, $h > 0$ the form $\mathcal{H}(.,.)$ is non-degenerate on $V_{c,h}$. By considering the limit $h \to +\infty$, one checks that it is positive. Thus it stays non-negative for $c \geq 1$, $h \geq 0$. As a result, all the irreducible HW representations are unitary for $c \geq 1$, $h \geq 0$ (they are equal to the Verma module if and only if $c > 1$ or $c = 1$ and $h \neq \frac{m^2}{4}$, $m = 0, 1, \ldots$).

1\. 1. 5. For $0 \leq c < 1$, the situation is more interesting providing the first example of the selective power of conformal invariance.

THEOREM 1.1. *For $0 \leq c < 1$, the irreducible HW representations $(\pi_{c,h}, W_{c,h})$ are unitary if and only if*

$c = c_m$ for $m = 2, 3, \ldots$

and

$h = h_{rs}(m)$ *for* $1 \leq r \leq m-1$, $1 \leq s \leq r$.

The "only if" part of Theorem 1.1 is due to D. Friedan, Z. Qiu and S. Shenker [29,30]. It has been proven by a careful analysis of the geometry of lines $(c(m), h_{r,s}(m))$ in the (c,h) plane. This, in conjunction with Kac's determinant formula, allowed subsequent elimination of portions of the (c,h) plane were negative norm vectors appear, by an induction on ℓ. At the end, only the points listed above were left.

The "if" part of Theorem 1.1 has been proven by P. Goddard, A. Kent and D. Olive [43,44] by an explicit realization of the irreducible HW representations of the above series in a space of positive metric. We shall present here this so called "coset construction" as it plays an important role in CFT, reaching beyond its representation theory aspect.

1. 2. Affine Lie algebras and their representations

1\. 2. 1. For a compact, connected, simply connected, semi-simple Lie group G, consider its *loop group* LG (composed of smooth maps from S^1 to G). It is an infinite-dimensional Lie group with Lie algebra $L\mathrm{g}$ where g is the Lie algebra of G [68].

Consider the central extension $\widehat{L\mathbf{g}} = L\mathbf{g} \oplus \mathbf{R}K$ of $L\mathbf{g}$ with the bracket

$$(1.8) \qquad [X,Y] \;=\; [X,Y]_{\mathbf{g}} + \frac{1}{2\pi}\int_0^{2\pi} d\theta < X', Y >_{\mathbf{g}} K, \qquad [K,X] = 0$$

for $X, Y \epsilon L\mathbf{g}$, where $[.,.]_{\mathbf{g}}$ stands for the (point-wise) bracket in $\mathbf{g}$ and $< .,. >_{\mathbf{g}}$ for a (negative) Killing form on $\mathbf{g}$. In particular, for $x, y \in \mathbf{g}^{\mathbf{C}}$ and $J_n(x) = xe^{in\theta} \in \widehat{L\mathbf{g}}^{\mathbf{C}}$, we obtain

$$(1.9) \qquad [J_n(x), J_m(y)] = J_{n+m}([x,y]_{\mathbf{g}}) + n\delta_{n+m,0} < x, y >_{\mathbf{g}} iK \ .$$

The subalgebra $\hat{\mathbf{g}} = (\bigoplus_{n\in\mathbf{Z}} J_n(\mathbf{g}^{\mathbf{C}})) \oplus \mathbf{C}K \subset \widehat{L\mathbf{g}}^{\mathbf{C}}$ is called an *affine Lie algebra*. It is a central extension of the algebra of polynomial loops in $\mathbf{g}^{\mathbf{C}}$.

DEFINITION 1.3. *A representation* (ρ, V) *of* $\hat{\mathbf{g}}$ *is called unitary if with respect to a scalar product* $\mathcal{H}(.,.)$ *on* V

$$(1.10) \qquad \mathcal{H}(\rho(J_n(x))v, w) + \mathcal{H}(v, \rho(J_{-n}(x))w) = 0 \quad for \;\; x \in \mathbf{g}$$

and

$$\mathcal{H}(\rho(K)v, w) + \mathcal{H}(v, \rho(K)w) = 0 \ .$$

This corresponds to the unitarity of projective representations of LG.

1. 2. 2. Let us fix a Cartan subalgebra $\mathbf{t} \subset \mathbf{g}$ and a set of positive roots $\Delta_+ \subset \mathbf{t}'$ of $\mathbf{g}$ so that

$$\mathbf{g}^{\mathbf{C}} = \mathbf{t}^{\mathbf{C}} \bigoplus (\bigoplus_{\alpha\in\Delta_+} \mathbf{C}e_\alpha) \bigoplus (\bigoplus_{\alpha\in\Delta_+} \mathbf{C}e_{-\alpha})$$

where $e_{\pm\alpha}$ are common eigenvectors of the adjoint action of $\mathbf{t}$ in $\mathbf{g}^{\mathbf{C}}$ corresponding to roots $\pm\alpha$. Let $\mathbf{g} = \bigoplus_{i=1}^{I} \mathbf{g}_i$ be the decomposition of $\mathbf{g}$ into the simple components. We shall denote as $-2k_i^{-1}$ the lengths squared of the long roots of $\mathbf{g}_i$ with respect to the dual Killing form $< .,. >_{\mathbf{g}'}$. $k \equiv (k_i)$, called the *level*, fixes the Killing form. Besides $< .,. >_{\mathbf{g}}$ and $< .,. >_{\mathbf{g}'}$, we shall also use the modified Killing forms $\ll .,. \gg_{\mathbf{g}}$, $\ll .,. \gg_{\mathbf{g}'}$ corresponding to level $(k_i + g_i^\vee)$ where $g_i^\vee$ are the dual Coxeter numbers (i.e. adjoint representation Casimirs) of simple components $\mathbf{g}_i$. Let us also introduce subalgebras $\hat{\mathbf{t}} = J_0(\mathbf{t}^{\mathbf{C}}) \oplus \mathbf{C}K$, $\hat{\mathbf{n}}_+ = (\bigoplus_{\alpha\in\Delta_+} J_0(\mathbf{C}e_\alpha)) \bigoplus (\bigoplus_{n\geq 1} J_n(\mathbf{g}^{\mathbf{C}}))$ and $\hat{\mathbf{b}}_+ = \hat{\mathbf{t}} \oplus \hat{\mathbf{n}}_+$ of $\hat{\mathbf{g}}$.

DEFINITION 1.4. *A representation* (ρ, V) *of* $\hat{\mathbf{g}}$ *is called a HW one if*

i/. *there exists* $v_0 \in V$ *s.t.*
$\rho(\hat{\mathbf{n}}_+)v_0 = 0,$
$\rho(iK)v_0 = v_0,$
$\rho(J_0(t))v_0 = i < t, \lambda > v_0$

for some $\lambda \in \mathbf{t}'^{\mathbf{C}}$ *and all* $t \in \mathbf{t}$,

ii/. $\rho(\mathcal{U}(\hat{\mathbf{g}}))v_0 = V$.

v_0 *is called the HW vector and* (k,λ) *the HW of the representation.*

1\. 2. 3. We may define Verma modules also for the affine algebras. Letting iK act on $\mathbf{C}$ as 1, $J_0(t)$ for $t \in \mathbf{t}^{\mathbf{C}}$ as multiplication by $i < t, \lambda >$ and $\hat{\mathbf{n}}_+$ as zero, we obtain a one-dimensional representation of $\hat{\mathbf{b}}_+$. The Verma module is $V_{k,\lambda} = \mathcal{U}(\hat{\mathbf{g}}) \otimes_{\mathcal{U}(\hat{\mathbf{b}}_+)} \mathbf{C}$ with the left action ρ of $\hat{\mathbf{g}}$ and HW vector $v_0 = 1 \otimes 1$ of HW (k,λ). For k real and $\lambda \in \mathbf{t}'$, a hermitian form $\mathcal{H}(.,.)$ on $V_{k,\lambda}$ can be defined uniquely by imposing relation (1.10) and requiring that $\mathcal{H}(v_0, v_0) = 1$. Dividing by its null space gives, as previously, a unique irreducible HW representation $(\rho_{k,\lambda}, W_{k,\lambda})$ of HW (k,λ).

1\. 2. 4. The question of positivity of $\mathcal{H}(.,.)$ on $W_{k,\lambda}$ proved easier to decide.

PROPOSITION 1.1. (Essentially [35]). *Irreducible representations* $(\rho_{k,\lambda}, W_{k,\lambda})$ *are unitary if and only if*

i/. *all* k_i *are positive integers,*

ii/. λ *is a weight and the image of* $-\lambda - \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ *in* $\mathbf{t}$ *induced by the Killing form* $\ll .,. \gg_{\mathbf{g}}$ *is in the positive Weyl alcove.*

REMARK 1.3. The positive Weyl alcove is the component of $\mathbf{t} \setminus \{ \ t \mid < t, \alpha > \in \mathbf{Z}$ for some $\alpha \in \Delta_+\}$ which has zero in its closure and on which $< ., \alpha >$ are positive for $\alpha \in \Delta_+$. For example, for algebra A_1, at level $k = 1, 2, \ldots$, the weights λ admitted by ii/. are $0, \ldots, k$ in the natural identification of the weight lattice of A_1 with $\mathbf{Z}$.

1\. 3. <u>Coset construction</u>

1.3.1. $Diff_+S^1$ acts canonically as a group of automorphisms of LG. There is an associated homomorphism D of Vir, trivial on the center, into the algebra of derivations of the affine algebra $\hat{\mathbf{g}}$. It is given by

$$D(L_n)J_m(x) = -mJ_{n+m}(x) \ . \tag{1.11}$$

In the space of a HW representation (ρ, V) of $\hat{\mathbf{g}}$ of HW (k,λ), D may be implemented by a representation of Vir in V given by the so called *Sugawara construction* [78,25].

Note first that each $\rho(J_n)$ can be considered as an element of $\mathbf{g}'^{\mathbf{C}} \otimes End\ V$. $\ll \rho(J_n), \rho(J_m) \gg_{\mathbf{g}'}$ will be viewed as an element of $\rho(\mathcal{U}(\hat{\mathbf{g}})) \subset End\ V$. Put for $v \in V$

$$\begin{aligned} \pi_{\mathbf{g}}(L_n)v \ &= \ \frac{1}{2}\sum_{m\geq\frac{n}{2}} \ll \rho(J_{n-m}), \rho(J_m) \gg_{\mathbf{g}'} v \\ &+ \ \frac{1}{2}\sum_{m<\frac{n}{2}} \ll \rho(J_m), \rho(J_{n-m}) \gg_{\mathbf{g}'} v \ . \end{aligned} \tag{1.12}$$

Notice that the sum in (1.12) involves only a finite number of non-vanishing terms. Explicit computation verifies that (1.12) defines a representation of *Vir* in V with the central charge

$$c_k^{\mathbf{g}} = \sum_{i=1}^{I} \frac{k_i \dim \mathbf{g}_i}{k_i + g_i^{\vee}} \ . \tag{1.13}$$

Moreover

$$\begin{aligned} &[\pi_{\mathbf{g}}(L_n), \rho(J_m(x))] = -m\rho(J_{n+m}(x)) \ , \\ &\pi_{\mathbf{g}}(L_n)v_0 = 0 \quad \text{for} \quad n > 0 \ , \end{aligned} \tag{1.14}$$

and

$$\pi_{\mathbf{g}}(L_0)v_0 = -\frac{1}{2} \ll \lambda, \lambda + \sum_{\alpha\in\Delta_+} \alpha \gg v_0 \ \equiv \ h_k^{\mathbf{g}}\, v_0 \ . \tag{1.15}$$

$(\pi_{\mathbf{g}}, V)$ is not however a HW representation.

1. 3. 2. Let $\mathbf{h}$ be a semi-simple Lie subalgebra of $\mathbf{g}$. Let $\mathbf{h} = \bigoplus_{j=1}^{J} \mathbf{h}_j$ be its decomposition into the simple factors. Each representation (ρ, V) of $\hat{\mathbf{g}}$ induces by restriction a representation $(\tilde{\rho}, V)$ of the affine algebra $\hat{\mathbf{h}}$ corresponding to the restricted Killing form. If (ρ, V) is HW representation then in V one can define two Sugawara actions $\pi_{\mathbf{g}}$ and $\pi_{\mathbf{h}}$ of *Vir*. Their central charges are $c_k^{\mathbf{g}}$ and $c_{\tilde{k}}^{\mathbf{h}}$, respectively, where $\tilde{k} = (\tilde{k}_j)$ is the level of the g-Killing form restricted to $\mathbf{h}$.

Let us put, following P. Goddard, A. Kent and D. Olive [44]

$$\pi_{\mathbf{g}/\mathbf{h}} = \pi_{\mathbf{g}} - \pi_{\mathbf{h}} \ . \tag{1.16}$$

PROPOSITION 1.2. *$\pi_{\mathbf{g}/\mathbf{h}}$ defines a representation of Vir corresponding to central charge $c_k^{\mathbf{g}/\mathbf{h}} = c_k^{\mathbf{g}} - c_{\tilde{k}}^{\mathbf{h}}$ and commuting with representation $\tilde{\rho}$ of $\hat{\mathbf{h}}$.*

Proof. Commutation with $\tilde{\rho}$ follows immediately from (1.14). Hence $\pi_{\mathbf{g}/\mathbf{h}}$ commutes also with $\pi_{\mathbf{h}}$ and consequently

$$\begin{aligned} [\pi_{\mathbf{g}/\mathbf{h}}(L_n), \pi_{\mathbf{g}/\mathbf{h}}(L_m)] \ &= \ [\pi_{\mathbf{g}}(L_n), \pi_{\mathbf{g}}(L_m)] \ - \ [\pi_{\mathbf{h}}(L_n), \pi_{\mathbf{h}}(L_m)] \\ &= \ (n-m)\ \pi_{\mathbf{g}/\mathbf{h}}(L_{n+m}) \ + \ \frac{1}{12}(n^3 - n)\delta_{n+m,0}(c_k^{\mathbf{g}} - c_{\tilde{k}}^{\mathbf{h}}) \ . \end{aligned}$$

1. 3. 3. Although the Sugawara construction gives always representations of *Vir* with central charges ≥ 1, Proposition 1.2 allows to obtain central charges < 1. To this end let us take $\mathbf{g} = A_1 \oplus A_1$ with the Killing form of level $k = (m-2, 1)$ and a unitary HW representation $(\rho_{k,\lambda}, W_{k,\lambda})$ with $\lambda = (\mu, \varepsilon)$. By Proposition 1.1, μ may be equal $0, 1, ..., m-2$ and $\varepsilon = 0$ or 1. Let $\mathbf{h}$ be the diagonal subalgebra of $A_1 \oplus A_1$. The induced level $\tilde{k} = m - 1$. We obtain

$$c_k^{\mathbf{g}} = 3(m-2)/m + 1 \ , \quad c_{\tilde{k}}^{\mathbf{h}} = 3(m-1)/(m+1)$$

and

$$c_k^{\mathbf{g}/\mathbf{h}} = 1 - \frac{6}{m(m+1)} \,.$$

The latter runs through the same set of values as in Theorem 1.1 except for $m = 2$ corresponding to the trivial representation of *Vir.*

1. 3. 4. The coset representations $\pi_{A_1 \bigoplus A_1/\mathrm{diag}A_1}$ are not irreducible as they commute with the $\tilde{\rho}$ representations of $\mathrm{diag}A_1$. Let

$$W^0_{k,\lambda} = \{v \in W_{k,\lambda} |\ \tilde{\rho}(\hat{\mathrm{n}}_+)v = 0\} \,.$$

$W^0_{k,\lambda}$ is an invariant subspace for $\pi_{\mathbf{g}/\mathbf{h}}$.

THEOREM 1.2. [44]

$$(\pi_{\mathbf{g}/\mathbf{h}}, W^0_{k,\lambda}) \;\cong \bigoplus_{\substack{\nu=0,1,\ldots,m-1 \\ \mu+\epsilon-\nu=0 \bmod 2}} (\pi_{c_m, h_{(\mu+1)(\nu+1)}}, W_{c_m, h_{(\mu+1)(\nu+1)}}) \,.$$

Since $\pi_{\mathbf{g}/\mathbf{h}}$ satisfies (1.3) with respect to the scalar product of $W_{k,\lambda}$, this establishes the "if" part to Theorem 1.1.

1. 3. 5. Theorem 1.2 was proven in [44] by showing a decomposition

$$W_{k,\lambda} \cong \bigoplus_{\nu} (W_{\tilde{k},\nu} \bigotimes W_{c_m, h_{(\mu+1)(\nu+1)}}) \tag{1.17}$$

with the property that representation $\tilde{\rho}$ of $\mathrm{diag}A_1$ acts on the first factors by $\rho_{\tilde{k},\nu}$ and representation $\pi_{\mathbf{g}/\mathbf{h}}$ of *Vir* on the second ones by $\pi_{c_m, h_{(\mu+1)(\nu+1)}}$. Proof of (1.17) is easily reduced to an identity between the representation *characters.*

DEFINITION 1.5. i/. *The character of a HW representation* (π, V) *of Vir is the function*

$$\chi(\tau) = \sum_{\ell=0}^{\infty} e^{2\pi i \tau (h+\ell)} \dim V(\ell) \tag{1.18}$$

where $V(\ell)$ *denotes the eigenspace of* $\pi(L_0)$ *of eigenvalue* $h + \ell$ *and* $\tau \in \mathbf{C}$, $Im\tau > 0$.
ii/. *The (affine) character of a HW representation* (ρ, V) of $\hat{\mathbf{g}}$ *is the function*

$$\chi^{\mathbf{g}}(\tau, t) = \sum_{\ell=0}^{\infty} e^{2\pi i \tau (h^{\mathbf{g}}+\ell)}\ Tr\ (e^{\rho(J_0(t))} \Big|\ V(\ell)) \tag{1.19}$$

where $t \in \mathrm{t}$, *and* $V(\ell)$ *corresponds to the eigenvalue* $h^{\mathbf{g}} + \ell$ *of* $\pi_{\mathbf{g}}(L_0)$.

REMARK 1.4. $\dim V(\ell) < \infty$ and the sums in (1.18) and (1.19) converge absolutely. For the unitary representations, we may write

$$\chi(\tau) = Tr \exp[2\pi i \tau\ \pi(L_0)^-]$$

and

$$\chi^{\mathbf{g}}(\tau,t) = Tr\exp[2\pi i\tau\ \pi_{\mathbf{g}}(L_0)^- + \rho(J_0(t))^-]$$

where $\pi_{(\mathbf{g})}(L_0)^-$, $\rho(J_0(t))^-$ denote the closures of the operators in the Hilbert space completion V^- of V.

We shall denote by $\chi_{c,h}$ the characters of representations $(\pi_{c,h}, W_{c,h})$. They are known. In particular for the discrete series of Theorem 1.1, they were computed by A. Rocha-Caridi [71]. The affine characters $\chi^{\mathbf{g}}_{k,\lambda}$ of representations $(\pi_{k,\lambda}, W_{k,\lambda})$ were obtained by V.G. Kac in [53].

1. 3. 6. In order to prove (1.17), Goddard, Kent and Olive have checked explicitly the identity

$$\begin{aligned}\chi^{A_1\oplus A_1}_{k,\lambda}(\tau, t\oplus t)\ \ (= \chi^{A_1}_{m-2,\mu}(\tau,t)\ \chi^{A_1}_{1,\epsilon}(\tau,t))\\ = \sum_{\substack{\nu=0,1\ldots,m-1\\ \mu+\epsilon-\nu=0 \bmod 2}} \chi^{A_1}_{m-1,\nu}(\tau,t)\ \chi_{c_m,h_{(\mu+1)(\nu+1)}}(\tau)\ .\end{aligned} \tag{1.20}$$

Looking for the vector contributing the lowest power of $q = e^{2\pi i z}$ to (1.20) allows to identify a joint HW vector for $\tilde{\rho}$ and $\pi_{\mathbf{g}/\mathbf{h}}$ in $W_{k,\lambda}$ and thus the first component in the decomposition (1.17). Subtracting its contribution from (1.20) and repeating the procedure gives an inductive construction of (1.17).

1. 4. Integration of the affine and Virasoro algebras representations

1. 4. 1. As proven by R. Goodman and N.R. Wallach [45], the unitary HW representations $(\rho_{k,\lambda}, W_{k,\lambda})$ of $\hat{\mathbf{g}}$ integrate to unitary projective representations of the loop group LG in the Hilbert space completion $W^-_{k,\lambda}$ of $W_{k,\lambda}$. First, $(\rho_{k,\lambda}, W_{k,\lambda})$ may be extended to a representation of $\widehat{L\mathbf{g}}^{\mathbf{C}}$ on an invariant domain $\tilde{W}_{k,\lambda}$, $W_{k,\lambda} \subset \tilde{W}_{k,\lambda} \subset W^-_{k,\lambda}$. $\widehat{L\mathbf{g}}$ appears to be represented by essentially skew-adjoint operators whose exponentiation leads to the representation of LG (the exponential map on $L\mathbf{g}$ is a local homeomorphism with a dense image).

1. 4. 2. Similarly, each unitary HW representation $(\pi_{c,h}, W_{c,h})$ of Vir extends to a representation of $V\widehat{ect}^{\mathbf{C}}S^1$ on an invariant domain $\tilde{W}_{c,h}$, $W_{c,h} \subset \tilde{W}_{c,h} \subset W^-_{c,h}$ such that $V\widehat{ect}S^1$ is represented by essentially skew-adjoint operators ($\tilde{W}_{c,h}$ is the space of C^∞-vectors for $\pi(L_0)^-$). Lifting of the representation to $Diff_+S^1$ is however a more subtle problem as the image of the exponential map of $V\widehat{ect}S^1$ is nowhere dense. Nevertheless one has

THEOREM 1.3. [46] *There exists a projective unitary representation* $(U_{c,h}, W^-_{c,h})$ *of* $Diff_+S^1$ *with* $\tilde{W}_{c,h}$ *as a common invariant subspace such that for* $X \in VectS^1$

$$e^{\pi_{c,h}(X)^-} = U_{c,h}(e^X)$$

and for $X \in Vect^{\mathbf{C}}S^1$ *and* $D \in Diff_+S^1$

$$U_{c,h}(D)\ \pi_{c,h}(X) = (\pi_{c,h}(D_*X) + c\ \alpha(X,D))\ U_{c,h}(D) \tag{1.21}$$

where α is a function on $Vect^{\mathbf{C}}S^1 \times Diff_+S^1$.

1. 4. 3. It is convenient to extend $(U_{c,h}, W^-_{c,h})$ to diffeomorphisms changing the orientation of S^1 by representing $e^{i\theta} \to e^{-i\theta}$ by an anti-unitary involution P of $W^-_{c,h}$ such that

$$Pv_0 = v_0 \qquad \text{and} \qquad P\pi_{c,h}(X)P = \pi_{c,h}(\bar{X}^\vee)$$

where $X^\vee$ denotes X transformed by $e^{i\theta} \mapsto e^{-i\theta}$.

1. 4. 4. One can easily exponentiate an even richer class of operators than in Theorem 1.3. For example, for $\tau \in \mathbf{C}$, $Im\tau > 0$, $\exp[2\pi i\tau\ \pi_{c,h}(L_0)^-]$ is a contraction semigroup in $W^-_{c,h}$, mapping $\tilde{W}_{c,h}$ into itself. Consider now for $q = e^{2\pi i\tau}$ the annulus $\Sigma_q = \{|q| \le |z| \le 1\}$ with the boundary loops parametrized by S^1 via

$$p_1(e^{i\theta}) = qD_1(e^{i\theta}) \quad , \quad p_2(e^{i\theta}) = D_2(e^{i\theta})$$

where $D_1, D_2 \in Diff_+S^1$ and are assumed real analytic. Put

$$A_{c,h}(\Sigma_q, p_i) = \text{const. } U_{c,h}(D_2^{-1})\ \exp[2\pi i\tau\ \pi_{c,h}(L_0)^-]\ U_{c,h}(D_1)\ . \tag{1.22}$$

Then, for a holomorphic vector field X on Σ_q,

$$A_{c,h}(\Sigma_q, p_i)\ \pi_{c,h}(p_1^*X) = (\pi_{c,h}(p_2^*X) + c\ \alpha(X, \Sigma_q, p_i))\ A_{c,h}(\Sigma_q, p_i) \tag{1.23}$$

as follows easily from (1.21). Here

$$p_i^* f\frac{\partial}{\partial z} = \left(\frac{dp_i}{d\theta}\right)^{-1} f \circ p_i\ \frac{d}{d\theta}$$

and

$$\alpha(X, \Sigma_q, p_i) = \alpha(p_{1*}X, D_1) + \alpha(D_{2*}p_2^*X, D_2^{-1})\ .$$

1. 4. 5. Take now any Riemann surface Σ of annular topology with analytic parametrizations p'_i of the boundary loops, p'_1 negative (i.e. with disagreement with the orientation of Σ) and p'_2 positive. (Σ, p'_i) may be mapped conformally to certain (Σ_q, p_i), uniquely up to rotations. We may define

$$A_{c,h}(\Sigma, p'_i) = A_{c,h}(\Sigma_q, p_i)\ .$$

Let us identify (Σ, p_i)'s related by conformal diffeomorphisms (preserving the parametrizations). The set $\mathcal{E}$ of classes of (Σ, p_i)'s forms a semigroup with the product defined by gluing two surfaces via identification of the positive boundary loop of one of them with the negative one of the other, according to their parametrizations. As stressed by G. Segal [75], $\mathcal{E}$ replaces a non-existent complexification of $Diff_+S^1$.

PROPOSITION 1.3. *$(A_{c,h}, W^-_{c,h})$ is a projective representation of $\mathcal{E}$.*

S k e t c h   o f   p r o o f .   The intertwining property (1.23) is consistent with the semigroup law. One can show that (1.23) fixes $A_{c,h}(\Sigma, p_i)v_0$ up to a factor by considering the holomorphic vector fields X on Σ extending analyticaly to a unit disc glued via p_1 to Σ and by an inductive analysis of the components of $A_{c,h}(\Sigma, p_i)v_0$ in eigenspaces of $\pi(L_0)^-$. With $A_{c,h}(\Sigma, p_i)v_0$ fixed, eq. (1.23) determines $A_{c,h}(\Sigma, p_i)$ on a dense set, hence everywhere.

2. UNITARY SERIES OF CONFORMAL FIELD THEORIES

2. 1. Axioms of real CFT's

2. 1. 1. The operators $A_{h,c}(\Sigma, p_i)$ constitute a tip of the iceberg of a richer CFT structure which assigns operators (often called *amplitudes*) to general Riemann surfaces with boundary. We shall describe this structure in an axiomatic way patterned on G. Segal's and D. Quillen's approach [75], see also [80,81,1].

2. 1. 2. Consider the compact, possibly disconnected, Riemann surfaces Σ with analytic parametrizations p_i, $i \in I$, of the boundary loops, negative (positive) for $i \in I_-(I_+)$, and with a Riemannian metric g agreeing with the conformal structure of Σ, trivial around the boundary (i.e. $p_i^* g = \frac{1}{2|z|^2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ under the analytic extension of p_i to a neighborhood of $|z| = 1$). Fix a Hilbert space H with an anti-unitary involution P.

DEFINITION 2.1. *A real CFT theory is an assignment*

$$(\Sigma, p_i, g) \longmapsto \mathbf{A}(\Sigma, p_i, g)$$

where

$$\mathbf{A}(\Sigma, p_i, g) : \bigotimes_{i \in I_-} H \longmapsto \bigotimes_{i \in I_+} H$$

are trace-class operators (empty tensor product of $H = \mathbf{C}$ by convention) s.t.

I.1. *if $(\Sigma, p_i, g) = \coprod_{\beta} (\Sigma^\beta, p^\beta_{i_\beta}, g^\beta)$ (disjoint union) then*

$$\mathbf{A}(\Sigma, p_i, g) = \bigotimes_{\beta} \mathbf{A}(\Sigma^\beta, p^\beta_{i_\beta}, g^\beta) \ ,$$

I.2. *if $p^\vee_{i_0}(e^{i\theta}) = p_{i_0}(e^{-i\theta})$ and $i_0 \in I_+$ then*

$$(\mathbf{A}(\Sigma, p^\vee_{i_0}, p_{i'}, g) x_{i_0} \otimes x, y) \ = \ (\mathbf{A}(\Sigma, p_i, g)x, P x_{i_0} \otimes y)$$

with $i' \neq i_0$, $x \in \bigotimes_{i \in I_-} H$, $y \in \bigotimes_{i' \in I_+} H$,

I.3. *if $D : \Sigma^1 \longrightarrow \Sigma^2$ is a conformal diffeomorphism and $p_i^2 = D \circ p_i^1$, $D^* g^2 = g^1$, then*

$$\mathbf{A}(\Sigma^1, p_i^1, g^1) = \mathbf{A}(\Sigma^2, p_i^2, g^2),$$

I.4. *if Σ' is obtained from Σ by identifying the boundary loops $i_1 \in I_-$ and $i_2 \in I_+$ then*

$$\mathbf{A}(\Sigma', p_{i'}, g) = Tr_{i_1 i_2}\, \mathbf{A}(\Sigma, p_i, g)\ ,$$

where $i' \neq i_1, i_2$ and $Tr_{i_1 i_2}$ is the trace of maps between factors i_1 and i_2 in the tensor products of H,

I.5. *if $\bar{\Sigma}$ is the complex conjugate of Σ* then

$$\mathbf{A}(\bar{\Sigma}, p_i, g) = \mathbf{A}(\Sigma, p_i, g)^*\ ,$$

I.6. *if σ is a real smooth function on Σ vanishing in a neighborhood of $\partial\Sigma$ then*

$$\mathbf{A}(\Sigma, p_i, e^{\sigma} g) \ = \ \exp[\frac{ci}{24\pi}\int_{\Sigma}(\frac{1}{2}\partial\sigma \wedge \bar{\partial}\sigma + R_g \sigma)]\ \mathbf{A}(\Sigma, p_i, g)$$

where R_g is the (imaginary) curvature form of g and c is a positive constant.

REMARK 2.1. Properties I.1 - I.6 may be deduced from the physicists' intuitive representation of the amplitudes $\mathbf{A}$ by formal functional integrals. H is then viewed as a space of functions on the loop space LM of a finite dimensional space M. P consists of the inversion $^\vee$ on LM combined with the complex conjugation. Operators $\mathbf{A}$ are given by kernels represented as formal integrals over maps $f : \Sigma \longrightarrow M$ fixed on $\partial\Sigma$:

$$\mathbf{A}(\Sigma, p_i, g)(f_i) = \int\limits_{\substack{f:\Sigma \longrightarrow M \\ f\circ p_i = f_i}} e^{-S_\Sigma(f)} \prod_{\xi \in \Sigma} d\mu(f(\xi))$$

where S_Σ is a local, conformally invariant (i.e. under $g \mapsto e^{\sigma} g$) "action" functional of f and $d\mu$ is a measure on M. Sometimes, one can make rigorous sense out of such formal expressions. This is the case e.g. when M is a torus or its orbifold, $S_\Sigma(f) = \|df\|^2_{L_2}$ (plus eventually $i\int_\Sigma \omega$ where ω is a closed form on M) and $d\mu$ is the translation-invariant measure on M. Then the functional integral reduces to a sum over the connected components of the space of maps f and to infinite-dimensional Gaussian integrals. The Gaussian integrals lead to determinants which, when ζ-function regularized, see e.g. [70,10], exhibit a dependence on the conformal factor of the metric (conformal anomaly) as in I.6 and partly motivate this least intuitive but crucial axiom. The toroidal theories provide the simplest candidates for models of real CFT, where for some special cases the essential work towards checking the properties I.1 - I.6 has been done [1].

REMARK 2.2. Although not apparent in the formulation, a projective action of $Diff_+S^1 \times Diff_+S^1$ in H may be extracted from the axioms under weak regularity assumptions. Let Σ_q be as in 1.4.4 with the boundary parametrized by $p_{01}(e^{i\theta}) = qe^{i\theta}$ and $p_{02}(e^{i\theta}) = e^{i\theta}$. Let

$g_0 = \frac{1}{2|z|^2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ and X be a holomorphic vector field on Σ_q. Then the relation

$$(2.1) \qquad \mathbf{A}(\Sigma_q, p_{0i}, g_0)\ \pi(p_{01}^* X) \ = \ \pi(p_{02}^* X)\ \mathbf{A}(\Sigma_q, p_{0i}, g_0) \ = \oint X \rfloor \nabla_{g^{20}} \mathbf{A}(\Sigma_q, p_{0i}, g_0)$$

defines a representation π of the Virasoro algebra with the central charge c (the same as in I.6, see e.g. [21]) provided that $\mathbf{A}(\Sigma_q, p_{0i}, g_0)$ does not have zero eigenvectors. Above $\nabla_{g^{20}}$ denotes the (2,0) part of the g-derivate of $\mathbf{A}$ and the contour integral is over $|z|$ =const, $|q| < |z| < 1$. Similarly

$$(2.2) \qquad \mathbf{A}(\Sigma_q, p_{0i} g_0)\ \pi(p_{01}^* \bar{X}) \ = \ \pi(p_{02}^* \bar{X})\ \mathbf{A}(\Sigma_q, p_{0i}, g_0) \ = \oint \bar{X} \rfloor \nabla_{g^{02}} \mathbf{A}(\Sigma_q, p_{0i}, g_0)$$

defines a representation $\bar{\pi}$ of Vir, of the same central charge, commuting with π.

2. 2. Unitary series partition functions

2. 2. 1. It is natural to try to construct models of CFT out of the discrete series representations of $Diff_+ S^1$ of Theorem 1.1 (with fixed c), using their extensions to the semigroup $\mathcal{E}$, see 1.4.5, as first building blocks. Let us take

$$H = \bigoplus_{\substack{r,s \\ \bar{r},\bar{s}}} (\mathbf{C}^{N_{rs,\bar{r}\bar{s}}} \bigotimes W^-_{c_m, h_{rs}} \bigotimes \overline{W^-}_{c_m, h_{\bar{r}\bar{s}}})$$

with the natural action of $Vir \times Vir$ and P as in 1.4.3. ($\bar{W}$ denotes the complex conjugate space of W). We shall assume that the multiplicities $N_{rs,\bar{r}\bar{s}} \geq 0$ are symmetric and $N_{11,11} = 1$. Notice that the amplitudes $\mathbf{A}(\tau) = \mathbf{A}(\tau + 1) \equiv \mathbf{A}(\Sigma_q, p_{0i}, g_0)$ have to form a semigroup and $\mathbf{A}(\tau)^* = \mathbf{A}(-\bar{\tau})$. By comparing the intertwining properties (1.23) and (2.1), (2.2), we infer that the only possible (non-degenerate) choices for $\mathbf{A}(\tau)$ are

$$(2.3) \qquad \mathbf{A}(\tau) = \bigoplus_{\substack{r,s \\ \bar{r},\bar{s}}} (\exp[X_{rs,\bar{r}\bar{s}}\ \tau + \bar{X}_{rs,\bar{r}\bar{s}}\ \bar{\tau}] \bigotimes \exp[2\pi i \tau\ \pi_{c_m, h_{rs}}(L_0)^-] \bigotimes \exp[-2\pi i \bar{\tau}\ \bar{\pi}_{c_m, h_{\bar{r}\bar{s}}}(L_0)^-])$$

where $X_{rs,\bar{r}\bar{s}}$ and $\bar{X}_{rs,\bar{r}\bar{s}}$ is a pair of commuting skew-symmetric matrices in $\mathbf{C}^{N_{rs,\bar{r}\bar{s}}}$.

2. 2. 2. Properties I.4 together with I.3 and I.6 of 2.1.2 impose the condition of modular invariance on $Z(\tau) = Tr\ \mathbf{A}(\tau)$

$$(2.4) \qquad Z\Big(\frac{a\tau + b}{c\tau + d}\Big) = Z(\tau) \ \text{ for } \ \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in SL(2, \mathbf{Z})\ .$$

Eq. (2.4) expresses the fact that the amplitude $Z(\tau)$, called the *(toroidal) partition function* should depend only on the complex torus obtained from Σ_q by gluing the boundary loops. The strength of (2.4) was first realized by J. Cardy [15] (in fact (2.4) will also imply that $\mathbf{A}(\tau) = \mathbf{A}(\tau + 1)$). Trace of (2.3) gives

$$Z(\tau) = \sum_{\substack{r,s \\ \bar r,\bar s}} Tr \exp[X_{rs,\bar r\bar s}\,\tau + \bar X_{rs,\bar r\bar s}\,\bar\tau]\ \chi_{c_m,h_{rs}}(\tau)\ \overline{\chi_{c_m,h_{\bar r\bar s}}(\tau)}$$

where $\chi_{c_m,h_{rs}}$ are the characters of the Virosoro representations (1.18) with known transformation properties. It appears that the modified characters

$$\tilde\chi_{c_m,h_{rs}}(\tau) = e^{-\pi i\tau c_m/12}\chi_{c_m,h_{rs}}$$

transform under a unitary $\frac{1}{2}m(m-1)$-dimensional representation of $SL(2,\mathbf{Z})$ (which may be reduced to a projective representation of $SL(2,\mathbf{Z}_{4m(m+1)})$). This enforces $X_{rs,\bar r\bar s} = -\bar X_{rs,\bar r\bar s} = -\pi i c_m/12$ so that

$$Z(\tau) = \sum_{\substack{r,s \\ \bar r,\bar s}} N_{rs,\bar r\bar s}\ \tilde\chi_{c_m,h_{rs}}(\tau)\ \overline{\tilde\chi_{c_m,h_{\bar r\bar s}}(\tau)}\ . \tag{2.5}$$

2. 2. 3. The problem of classification of *modular invariants* (2.5) proved to be equivalent, as first suggested by D. Gepner [39], to a similar but somewhat simpler one of classification of modular invariants

$$\mathcal{Z}(\tau) = \sum_{\mu,\bar\mu=0,1,\ldots,k} M^k_{\mu,\bar\mu}\ \tilde\chi^{A_1}_{k,\mu}(\tau)\ \overline{\tilde\chi^{A_1}_{k,\bar\mu}(\tau)} \tag{2.6}$$

for $M^k_{00} = 1$, $M^k_{\mu,\bar\mu} = M^k_{\bar\mu,\mu}$ non-negative integers and the modified affine characters of the algebra $\hat A_1$

$$\tilde\chi^{A_1}_{k,\mu}(\tau) = e^{-\pi i\tau c^{A_1}_k/12}\chi^{A_1}_{k,\mu}(\tau,0),$$

see (1.13) and (1.19). The correspondence is

$$N_{rs,\bar r\bar s} = \frac{1}{2}(M^{m-2}_{r-1,\bar r-1}M^{m-1}_{s-1,\bar s-1} + M^{m-2}_{r'-1,\bar r'-1}M^{m-1}_{s'-1,\bar s'-1}) \tag{2.7}$$

where $r' = m-r$, $s' = m+1-s$ and similarly for $\bar r', \bar s'$.

2. 3. A-D-E classification

2. 3. 1. The complete classification of modular invariants (2.6) was first conjectured by A. Cappelli, C. Itzykson and J.-B. Zuber in [13] and subsequently, after a contribution from D. Gepner and Z. Qiu [41], proven by the first authors in [14] and by A. Kato in [54]. At the same time, ref. [14,54] established the equivalence between the classifications of modular invariants (2.6) and (a generalization of) (2.5).

THEOREM 2.1. [14,54] The following is the exhaustive list of possible matrices $(M^k_{\mu,\bar\mu})$:

A_{k+1} . $M^k_{\mu,\bar\mu} = \delta_{\mu,\bar\mu}$,

$D_{\frac{k}{2}+2}$. $M^k_{\mu,\bar\mu} = \delta_{\mu,\bar\mu}1_{\mathrm{even}}(\mu) + \delta_{\mu,k-\bar\mu}1_{\mathrm{even}}(\mu-\frac{k}{2})$ for k even where 1_{even} is the characteristic function of even numbers,

E_6 . $\quad M^{10}_{\mu,\bar\mu} = (\delta_{\mu,0}+\delta_{\mu,6})(\delta_{0,\bar\mu}+\delta_{6,\bar\mu}) + (\delta_{\mu,3}+\delta_{\mu,7})(\delta_{3,\bar\mu}+\delta_{7,\bar\mu})$
$\quad +(\delta_{\mu,4}+\delta_{\mu,10})(\delta_{4,\bar\mu}+\delta_{10,\bar\mu})$,

E_7 . $\quad M^{16}_{\mu,\bar\mu} = (\delta_{\mu,0}+\delta_{\mu,16})(\delta_{0,\bar\mu}+\delta_{16,\bar\mu}) + (\delta_{\mu,4}+\delta_{\mu,12})(\delta_{4,\bar\mu}+\delta_{12,\bar\mu})$
$\quad +(\delta_{\mu,6}+\delta_{\mu,10})(\delta_{6,\bar\mu}+\delta_{10,\bar\mu}) + \delta_{\mu,8}\delta_{8,\bar\mu} + (\delta_{\mu,2}+\delta_{\mu,14})\delta_{8,\bar\mu}$
$\quad +\delta_{\mu,8}(\delta_{2,\bar\mu}+\delta_{14,\bar\mu})$,

E_8 . $\quad M^{28}_{\mu,\bar\mu} = (\delta_{\mu,0}+\delta_{\mu,10}+\delta_{\mu,18}+\delta_{\mu,28})(\delta_{0,\bar\mu}+\delta_{10,\bar\mu}+\delta_{18,\bar\mu}+\delta_{28,\bar\mu})$
$\quad +(\delta_{\mu,6}+\delta_{\mu,12}+\delta_{\mu,16}+\delta_{\mu,22})(\delta_{6,\bar\mu}+\delta_{12,\bar\mu}+\delta_{16,\bar\mu}+\delta_{22,\bar\mu})$.

2. 3. 2. The proof of this result is rather technical and we refer an interested reader to the original papers. It is based on a detailed study of the commutant of the modular group representation transforming the characters.

REMARK 2.3. The solutions listed above have been labelled by the simply-laced Lie algebras of Cartan's classification, with $k+2$ equal to the dual Coxeter number of the algebra. In [13] an observation was made that $M_{\mu-1,\mu-1}$ are the multiplicities of the exponents μ of the corresponding A-D-E algebras.

REMARK 2.4. The possible solutions for matrices N, corresponding to pairs of A-D-E algebras, are obtained from (2.7) by using the solutions for M listed above. This gives two solutions for each m plus additional ones for m=11, 12, 17, 18, 29 and 30.

2. 3. 3. The link between the $\hat A_1$ and the Virasoro modular invariants is reminiscent of the coset $A_1 \oplus A_1/\mathrm{diag}A_1$ construction of the discrete series of Virasoro representations out of those of $\hat A_1$ algebra. These, in fact, are two aspects of a more general relation which should extend to the level of complete CFT's.

In the language of formal functional integrals, one theory, the "WZW model" introduced by E. Witten [86], studied in [55,42], see also [24], integrates over maps from a Riemann surface to a compact Lie group G. This is the CFT which in recent Witten's work [88] was related to the three-dimensional topological gauge theory and to the Jones polynomials for knots (in the case $G = SU(N)$).

The other model, the "coset theory", is obtained by introducing additionally in the WZW model a gauge field with values in a subalgebra $\mathbf{h}$ of Lie algebra $\mathbf{g}$ of G [36,37]. More concretely, we have a formal expression

$$Z^G(\tau) = \int e^{-S^G_{T_\tau}(g)} \prod_{\xi \epsilon T_\tau} dg(\xi) \tag{2.8}$$

for the partition function of the WZW theory where $T_\tau = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ is the torus corresponding to the modular parameter τ and the action functional

$$S^G_{T_\tau}(g) = -\frac{i}{4\pi}\int_{T_\tau} < g^{-1}\partial g, g^{-1}\bar\partial g >_{\mathbf{g}} - \frac{i}{24\pi}\int_B < \tilde g^{-1}d\tilde g, [\tilde g^{-1}d\tilde g, \tilde g^{-1}d\tilde g] >_{\mathbf{g}}$$

with $< .,. >_{\mathbf{g}}$ being a Killing form on $\mathbf{g}^{\mathbf{C}}$ and $\tilde{g} : B \to G$ extending g to a three-dimensional manifold with boundary T_τ. For the coset model

$$Z^{G,\mathbf{h}}(\tau) = \int e^{-S^{G,\mathbf{h}}_{T_\tau}(g,A)} \prod_{\xi \epsilon T_\tau} dg(\xi) \prod_{\xi \epsilon T_\tau} dA(\xi) \tag{2.9}$$

where $A = A^{10} + A^{01}$ is an $\mathbf{h}$-valued 1-form and

$$S^{G,\mathbf{h}}_{T_\tau}(g,A) = S^G_{T_\tau}(g) + \frac{i}{2\pi} \int_{T_\tau} [< g\partial g^{-1}, A^{01} >_{\mathbf{g}} + < A^{10}, g^{-1}\bar{\partial} g >_{\mathbf{g}} \\ + < Ad_g A^{10}, A^{01} >_{\mathbf{g}} - < A^{10}, A^{01} >_{\mathbf{g}}].$$

2. 3. 4. Integrals (2.8), (2.9) may be computed in a closed form by using their formal symmetry properties [7,37]. For $G = SU(2)$ or $SO(3)$ and the Killing form of level k (see 1.2.2), for k even in the case of $SO(3)$, one obtains from (2.8) partition functions (2.6):

$$Z^G(\tau) = \mathcal{Z}_{A_{k+1}}(\tau) \text{ or } \mathcal{Z}_{D_{\frac{k}{2}+2}}(\tau) \tag{2.10}$$

respectively where the subscript labels the matrix M used. Similarly, for

$$G = \begin{cases} SU(2) \times SU(2) \\ SO(3) \times SU(2) \\ (SU(2) \times SU(2))/\mathrm{diag}\mathbf{Z}_2 \end{cases}$$

at level $(m-2, 1)$ (for m even in the second case, odd in the third one) and for $\mathbf{h}=\mathrm{diag}A_1 \subset A_1 \oplus A_1 = \mathbf{g}$, expression (2.9) gives partition functions of (2.5)

$$Z^{G,\mathbf{h}}(\tau) = \begin{cases} \mathcal{Z}_{(A_{m-1}, A_m)}(\tau) \\ \mathcal{Z}_{(D_{\frac{m}{2}+1}, A_m)}(\tau) \\ \mathcal{Z}_{(A_{m-1}, D_{\frac{m}{2}+\frac{3}{2}})}(\tau) \ . \end{cases}$$

2. 3. 5. In general, the Hilbert space for the WZW models is built from spaces $W^-_{k,\lambda}$ of the unitary HW representations of affine algebra $\hat{\mathbf{g}}$ (see Proposition 1.1):

$$H^G = \bigoplus_{\lambda,\bar{\lambda}} (\mathbf{C}^{M_{\lambda,\bar{\lambda}}} \bigotimes W^-_{k,\lambda} \bigotimes \overline{W^-_{k,\bar{\lambda}}}) \ .$$

The WZW annular amplitudes are (in notation of 1.3.1)

$$\mathbf{A}^G(\tau) = \bigoplus_{\lambda,\bar{\lambda}} (1 \bigotimes \exp[2\pi i\tau(\pi_{\mathbf{g}}(L_0)^- - c^{\mathbf{g}}_k/24)] \bigotimes \exp[-2\pi i\bar{\tau}(\bar{\pi}_{\mathbf{g}}(L_0)^- - c^{\mathbf{g}}_k/24)])$$

so that the partition functions become sesquilinear combinations of (modified) affine characters, as in the special case (2.10).

2. 3. 6. The coset construction allows to ellucidate the A-D-E classification. Let $S = G/(H \times SU(2))$ be a *quaternionic symmetric space* (the possibilities are classified by $\mathbf{g}$ from the A-D-E list). Let $4k$ be the (real) dimension of S. $\mathbf{h} \bigoplus A_1$ can be naturally embedded into the Lie algebra of group $Spin(T_e S) \cong Spin(4k)$. As shown by W. Nahm [63,64],

$$Z^{Spin(4k),\mathbf{h}}(\tau) = 2((\text{rank } G)\mathcal{Z}_{A_{k+1}}(\tau) \pm \mathcal{Z}_{\mathbf{g}}(\tau)).$$

3. RATIONAL CONFORMAL FIELD THEORIES

3. 1. Green functions

3. 1. 1. Up to now, in our attempt to build a consistent real CFT from the discrete $c < 1$ series of representations of $Diff_+S^1 \times Diff_+S^1$, we have considered only the amplitudes assigned to the standard annuli and to tori. They can be extended to general annuli basically as in (1.22). For disc $D = \{z|\ |z| \leq 1\}$ with parametrization $p_0(e^{i\theta}) = e^{i\theta}$ of the boundary, one has to choose

$$\mathbf{A}(D, p_0, g) = \text{const. } v_0^{c_m,0} \bigotimes v_0^{c_m,0}$$

where $v^{c_m,0}$ is the HW vector in $W_{c_m,0}$ (the constant can be easily determined from the consistency of gluing (D, p_0) with standard annuli). Extension to the discs with arbitrary parametrization of the boundary may be done by gluing (D, p_0) to general annuli. The amplitude for $P\mathbf{C}^1$ results by gluing two discs.

Extension of the amplitude to more general Riemann surfaces is a highly non-trivial problem, not solved yet completely, although the crucial elements of the solution have been already obtained along the following lines.

3. 1. 2. Let $H^{HW} = \bigoplus(\mathbf{C}^{N_{r_s,r_t}} \otimes \mathbf{C}v_0^{c_m,h_{r_s}} \otimes \mathbf{C}v_0^{c_m,h_{r_t}}) \subset H$. Consider for $z_i \in \mathbf{C}$, $i = 1, ..., |I|$, $z_i \neq z_{i'}$, if $i \neq i'$, and for $0 \neq q_i \in \mathbf{C}$ sufficiently small, the Riemann surface $\Sigma(z_i, q_i) = \mathbf{C}P^1 \setminus \cup_i\{z|\ |z - z_i| < |q_i|\}$ with parametrizations $p_{0i}(e^{i\theta}) = z_i + q_i e^{i\theta}$ of the boundary. Let g be a metric on $\mathbf{C}P^1$, agreeing with its conformal structure, trivial around the boundary of $\Sigma(z_i, q_i)$ (see 2.1.2) and let g_i be the induced metrics on $\mathbf{C}P^1$ which for $|z| < 1$ ($|z| > 1$) are pullbacks of g under $z \mapsto z_i + q_i z$ ($z \mapsto z_i + q_i z^{-1}$). Fix vectors $v_i \in H^{HW}$.

DEFINITION 3.1.

$$\mathbf{G}(z_i, v_i) = \mathbf{A}(\Sigma(z_i, q_i), p_{0i}, g)\ (\bigotimes_i q_i^{-\pi(L_0)^-}\, \bar{q}_i^{-\bar{\pi}(L_0)^-}\, v_i) \cdot (\mathbf{A}(\mathbf{C}P^1, g) \prod_i \mathbf{A}(\mathbf{C}P^1, g_i)^{-1/2})^{-1} \tag{3.1}$$

is called the Green function (of z_i).

REMARK 3.1. Roughly speaking, $\mathbf{G}(z_i, v_i)$ is the value of the amplitude for $(\Sigma(z_i, q_i), p_{0i})$ on vector $\bigotimes_i v_i$. The details are chosen so that $\mathbf{G}(z_i, v_i)$ depends neither on q_i nor on the metric g.

3\. 1. 3. The knowledge of the Green functions and of the amplitude for $\Sigma = \mathbf{C}P^1$ allows to reconstruct the value of $\mathbf{A}(\Sigma(z_i, q_i), p_{0i}, g)$ on vectors $\bigotimes w_i$ where w_i are obtained by action of $\prod_{\alpha_i} \pi(X_{\alpha_i}) \prod_{\beta_i} \bar{\pi}(X_{\beta_i})$ on vectors $v_i \epsilon H^{HW}$. This can be done by multiple use of (2.1) and (2.2). Since w_i span a dense set in H, this should determine $\mathbf{A}(\sum(z_i, q_i), p_{0i}, g)$ and consequently $\mathbf{A}(\Sigma(z_i, q_i), p_i, g)$ with arbitrary parametrizations. The latter would give the rest of the amplitudes by the gluing procedure. Hence giving Green functions is another possible presentation of CFT.

3\. 1. 4. In fact, it would be enough to know the three-point Green function only, since out of the amplitudes for $P\mathbf{C}^1 \setminus 3$ discs (called often *vertex operators*) (+ annuli + disc) one can glue the amplitude for any Σ. On the other hand, the three-point function being covariant under the *Möbius group* $SL(2, \mathbf{C})$ (due to property I.3 in 2.1.2), it is determined by its value at three fixed points in $\mathbf{C}P^1$ i.e. by an element in the dual space of $H^{HW} \otimes H^{HW} \otimes H^{HW}$ called the tensor of *operator product coefficients.*

3\. 1. 5. An important observation by A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [6], which was at the origin of the recent progress in two-dimensional CFT, was that the Green functions for the models discussed here obey differential equations. For the four-point function, these become a pair of complex conjugate ordinary differential equations in the complex variable (of the moduli space of $\mathbf{C}P^1$ with four punctures). Each of the equations has a finite number of multivalued analytic (antianalytic) solutions $\{f_\alpha(z_1, ..., z_4)\}$, $\{\overline{f_\beta(z_1, ..., z_4)}\}$, called in [6] conformal blocks.

$$\mathbf{G}(z_1, ..., z_4) = \sum_{\alpha,\beta} h_{\alpha\beta} f_\alpha(z_1, ..., z_4) \overline{f_\beta(z_1, ..., z_4)} \tag{3.2}$$

where the matrix of coefficients $h_{\alpha\beta}$ may be found by demanding univaluedness and symmetry properties of $\mathbf{G}$.

3\. 1. 6. Following a somewhat different, but closely related procedure, V.S. Dotsenko and V.A. Fateev [19,20] have computed the four-point Green functions for the (A, A) series of models built from the unitary $c < 1$ representations of $Diff_+S^1 \times Diff_+S^1$. The simplest of these models, corresponding to $m = 3$ $(c = \frac{1}{2})$ has been identified as describing the scaling limit of the Ising model. In this case

$$H = (W^-_{\frac{1}{2},0} \bigotimes \overline{W^-_{\frac{1}{2},0}}) \bigoplus (W^-_{\frac{1}{2},\frac{1}{2}} \bigotimes \overline{W^-_{\frac{1}{2},\frac{1}{2}}}) \bigoplus (W^-_{\frac{1}{2},\frac{1}{16}} \bigotimes \overline{W^-_{\frac{1}{2},\frac{1}{16}}})$$

and the Green functions corresponding to vectors $v_0^{\frac{1}{2},\frac{1}{16}} \otimes v_0^{\frac{1}{2},\frac{1}{16}}$ coincide with the ones of formula (0.1).

3. 2. Modular functors

3\. 2. 1. The four-point functions determine the three-point ones and hence, in principle, all the other amplitudes. The crucial question to be solved remains whether this determination is self-consistent. In order to see this question, which has received much attention recently, in the right perspective, let us remind that both the amplitudes for the torus and the four-point functions studied above were sesquilinear combinations of finite number of multivalued holomorphic functions (Virasoro characters, conformal blocks) on the relevant moduli space. That leads to a link of CFT with *modular geometry* studying analytic vector bundles over the moduli spaces of complex curves.

The first comprehensive proposal to reformulate CFT in terms of modular geometry was due to D. Friedan and S. Shenker [31,32]. Their papers have triggered further research in this direction. Friedan and Shenker considered the amplitudes for closed Riemann surfaces as basic objects and imposed on them a consistency condition of factorization at degenerate surfaces. In the approach which we sketch here, the amplitudes for surfaces with boundary are treated on equal footing, in the spirit closer to the traditional operator techniques of quantum field theory. The modular geometry appears in an attempt to construct a real CFT with the Hilbert space $H = \bigoplus_{(\alpha,\bar{\alpha})}(H_\alpha \otimes \bar{H}_{\bar{\alpha}})$, each H_α carrying a representation of $Diff_+S^1$, out of its "holomorphic square root". Our presentation will be close to and largely inspired by G. Segal's one [76]. The reader should be warned that what follows is not a report on a complete work but more a sketch of a program and a glimpse into the subject in constant developement. It will also be more loosely worded falling in this respect somewhat behind the Bourbaki standard.

3. 2. 2. Suppose that we are given

i/. a finite set $\mathcal{A}$ with a distinguished element α^0 and an involution $\alpha \mapsto \alpha^\vee$, $\alpha^{0\vee} = \alpha^0$,

ii/. finite dimensional vector spaces $V(\Sigma, p_i, \alpha_i)$, $\alpha_i \epsilon \mathcal{A}$, together with isomorphisms between them denoted by $\cong$, s.t.

II.0. $V(D, p_0, \alpha) = \delta_{\alpha,\alpha^0}\mathbf{C}$,

II.1. $V(\Sigma, p_i, \alpha_i) \cong \bigotimes_\beta V(\Sigma^\beta, p^\beta_{i_\beta}, \alpha^\beta_{i_\beta})$ for $(\Sigma, p_i, \alpha_i) = \coprod_\beta V(\Sigma^\beta, p^\beta_{i_\beta}, \alpha^\beta_{i_\beta})$,

II.2. $V(\Sigma, p^\vee_{i_0}, p_{i'}, \alpha^\vee_{i_0}, \alpha_{i'}) \cong V(\Sigma, p_i, \alpha_i)$,

II.3. If $D : (\Sigma^1, p^1_i) \to (\Sigma^2, p^2_i)$ is a conformal diffeomorphism then $V(\Sigma^1, p^1_i, \alpha_i) \cong V(\Sigma^2, p^2_i, \alpha_i)$,

II.4. $V(\Sigma', p_{i'}, \alpha_{i'}) \cong \bigoplus_{\alpha \epsilon A} V(\Sigma, p_i, \alpha_i) \mid_{\alpha_{i_1} = \alpha_{i_2} = \alpha}$ if Σ' is obtained from Σ by gluing boundary loops i_1 and i_2,

II.5. $V(\bar{\Sigma}, p_i, \alpha_i) \cong \overline{V(\Sigma, p_i, \alpha_i)}$,

II.6. $V(\Sigma, p_i, \alpha_i)$ depend holomorphically on (Σ, p_i) (they arise from holomorphic bundles over the moduli spaces of (Σ, p_i) [49])

II.7. The isomorphisms II.1 to II.5 satisfy natural commutativity and associativity relations.

REMARK 3.2. The above is a version of what G. Segal calls in [76] a *modular functor*. Somewhat abusively, we shall use this name here.

3. 2. 3. The simplest non-trivial example of such a structure (with $\mathcal{A} = \{\alpha^0\}$) is given by the *determinant bundles* [69,10]. Each (Σ, p_i) determines a compact Riemann surface Σ^c (of genus h) obtained by gluing copies of the unit discs (D^i, p_0^i) along the boundary loops of Σ parametrized by p_i or $p_i^\vee$. Put

$$V_n(\Sigma, p_i) = (\wedge^h H^0(\omega_{\Sigma^c}))^{\otimes n} \tag{3.3}$$

where $H^0(\omega_{\Sigma^c})$ is the space of holomorphic forms on Σ^c (ω_{Σ^c} denotes the canonical bundle of Σ^c). Only the gluing isomorphism is not obvious. It is given by a version of the *Krichever construction* [57]. Let us describe it briefly.

Denote by $E_\pm$ the projections in $L^2(S^1)$ on the subspaces with positive (negative) Fourier coefficients and let $L_0 = (E_+ + E_-)L^2(S^1)$. Let Γ_0 be the space of L^2 1-forms on S^1 with vanishing integral. L_0 and Γ_0 are in natural duality defined by the integral over S^1. If surface Σ' is obtained from Σ by gluing two boundary loops via $p_{i_2} \circ p_{i_1}^{-1}$, it may have genus $h' = h$ or $h+1$ depending on whether the loops are in different or in the same connected component of Σ. We may embed $H^0(\omega_{\Sigma^c})$ and $H_0^0(\omega_{\Sigma'^c}) = \{\omega' \epsilon H^0(\omega_{\Sigma'^c}) \mid \int_{S^1} p_{i_1}^* \omega' = 0\}$ into Γ_0 by

$$H^0(\omega_{\Sigma^c}) \ni \omega \mapsto p_{i_1}^*\omega + p_{i_2}^*\omega \in \Gamma_0$$

and

$$H_0^0(\omega_{\Sigma'^c}) \ni \omega' \mapsto p_{i_1}^*\omega' \in \Gamma_0$$

respectively. Define now

$$W_1 = \{(E_+ + E_-)(f \circ p_{i_1} - f \circ p_{i_2}) \mid f \text{ holomorphic on } \Sigma^c \setminus (D^{i_1} \cup D^{i_2})\}$$
$$\text{with } L^2 \text{ boundary values}$$

$$W_2 = \{E_-(f \circ p_{i_1}) - E_+(f \circ p_{i_2}) \mid f \text{ as above }\}.$$

W_i are subspaces of codimension h in L_0^2. Let I be an isomorphism of L_0^2 extending the map from W_1 to W_2 identifying the elements arising from the same f. Then the dual $I' : \Gamma_0 \hookleftarrow$ maps isomorphicly $H^0(\omega_{\Sigma^c})$ onto $H_0^0(\omega_{\Sigma'^c})$ and is of the form: identity plus trace-class operator. We put

$$\iota = (\det I')^{-1}(\omega_0' \wedge) \wedge^h (I' \mid_{H^0(\omega_{\Sigma^c})})$$

where $(\omega_0'\wedge)$ factor appears only of $h' = h + 1$, ω_0' being a holomorphic form on Σ'^c such that $\int_{S^1} p_{i_1}^*\omega' = 1$. $\iota : \bigwedge^h H^0(\omega_{\Sigma_c}) \to \bigwedge^{h'} H^0(\omega_{\Sigma'^c})$ is defined intrinsically. It can be also verified that the consistency conditions II.7 are satisfied for n in (3.3) even.

3. 2. 4. The determinant bundles carry natural hermitian metrics associated to a metric on the surface, known as *Quillen metrics* [69,10]. Let g be a metric on (Σ, p_i) (agreeing with the

conformal structure of Σ, trivial around $\partial\Sigma$). Choosing a similar metric on (D, p_0), we obtain, by gluing, metrics g^c on Σ^c and g^0 on $\mathbf{C}P^1$. For $\omega \in \bigwedge^h H^0(\omega_{\Sigma^c})$, we put

$$||\omega||_Q = ||\omega||_{L^2}\left(\frac{\det{}'\Delta^{g^c}_{\Sigma^c}}{\text{Area }\Sigma^c}\right)^{-1/2}\left(\frac{\det{}'\Delta^{g^0}_{\mathbf{C}P^1}}{\text{Area }\mathbf{C}P^1}\right)^{|I|/4}$$

where $||\omega||_{L^2}$ is the norm induced from the natural L^2 norm on the sections of ω_{Σ^c} $\cdot$ $\Delta^{g^c}_{\Sigma^c}$ is the Laplacian corresponding to metric g^c, $\det'$ is the determinant in the subspace orthogonal to the zero modes, ζ-function regularized [70,10], and $|I|$ is the number of components of $\partial\Sigma$. The $\mathbf{C}P^1$ contribution is designed to remove the dependence on the metric on D. If g depends smootly on the surface, so does the Quillen metric. Each smooth metric on a holomorphic vector bundle induces naturally a connection. In the case of the Quillen metric, the curvature Ω of this metric is given by the local Riemann-Roch-Grothendieck theorem [10].

3. 2. 5. The metric structure on V_n agrees with isomorphisms II.1 - II.5 in 3.2.2, the only non-trivial point being

PROPOSITION 3.1. [16] *ι preserves the Quillen metric.*

3. 2. 6. In general, if V is a modular functor, we shall require that

II.8. given metric g on (Σ, p_i), there exists a non-degenerate hermitian form $< .,. >_V$ on $V(\Sigma, p_i) = \bigoplus_{(\alpha_i)} V(\Sigma, p_i, \alpha_i)$ smoothly depending on (Σ, p_i) if g does; $< .,. >_V$ is preserved by isomorphisms II.1 - II.5 and it induces a connection of curvature $\frac{c}{2}\Omega$.

3. 3. Holomorphic CFT's

3. 3. 1. Now we are ready to describe a possible set of axioms for a "holomorphic square root" of a real CFT.

Suppose that we are given a modular functor V and Hilbert spaces H_α , $\alpha \in \mathcal{A}$, with anti-unitary involution $P : H_\alpha \to H_{\alpha^\vee}$. A *holomorphic CFT* will be specified by giving for each $(\sum, p_i, \alpha_i)$ trace-class amplitudes

$$A(\Sigma, p_i, \alpha_i) : \bigotimes_{i\in I_-} H_{\alpha_i} \to (\bigotimes_{i\in I_+} H_{\alpha_i}) \bigotimes V(\Sigma, p_i, \alpha_i)$$

depending holomorphically on (Σ, p_i) and s.t.

III.1. if $(\Sigma, p_i, \alpha_i) = \coprod_\beta (\Sigma^\beta, p^\beta_{i_\beta}, \alpha^\beta_{i_\beta})$ then $A(\Sigma, p_i, \alpha_i) \cong \bigotimes_\beta A(\Sigma^\beta, p^\beta_{i_\beta}, \alpha^\beta_{i_\beta})$,

III.2. $(A(\Sigma, p^\vee_{i_0}, p_{i'}, \alpha^\vee_{i_0}, \alpha_{i'})x_{i_0} \otimes x, y) \cong (A(\Sigma, p_i, \alpha_i)x, Px_{i_0} \otimes y)$,

III.3. If $D : (\Sigma^1, p^1_i) \to (\Sigma^2, p^2_i)$ is a conformal diffeomorphism then $A(\Sigma^1, p^1_i, \alpha_i) \cong A(\Sigma^2, p^2_i, \alpha_i)$,

III.4. $A(\Sigma', p_{i'}, \alpha_{i'}) \cong \bigoplus_{\alpha\in A} Tr_{i_1 i_2} A(\Sigma, p_i, \alpha_i) \mid_{\alpha_{i_1}=\alpha_{i_2}=\alpha}$ if Σ' arises from gluing two boundary loops of Σ, III.5. $A(\bar{\Sigma}, p_i, \alpha_i) \cong A(\Sigma, p_i, \alpha_i)^*$.

3. 3. 2. Given a holomorphic CFT, we can immediately construct a real CFT. This may be done by taking $H = \bigoplus_{(\alpha,\bar{\alpha})} (H_\alpha \otimes \bar{H}_{\bar{\alpha}})$ where $(\alpha, \bar{\alpha})$ run through the set of indices s.t.

$$< V(\Sigma_q, p_{0i}, \alpha, \alpha),\ V(\Sigma_q, p_{0i}, \bar{\alpha}, \bar{\alpha}) >_V \neq 0,$$

with the standard annuli (Σ_q, p_{0i}) as in Remark 2.2, and by setting

$$\mathbf{A}(\Sigma, p_i, g) = \bigoplus_{(\alpha_i, \bar{\alpha}_i)} < A(\Sigma, p_i, \alpha_i),\ A(\Sigma, p_i, \bar{\alpha}_i) >_V \ .$$

The conformal covariance property I.6 of 2.1.2 follows now from the condition in II.8 of 3.2.6 for the curvature of bundles V as in [10, Proposition 2.3].

REMARK 3.4. The real CFT's which are obtained this way from holomorphic CFT are known under the name of *rational theories* (they have to correspond to rational values of the central charge c and of $\pi(L_0), \bar{\pi}(L_0)$ eigenvalues [2]).

3. 3. 3. The fundamental problem of CFT is a classification of possible models. The central idea behind the Friedan-Shenker program [31,32] was that a translation into the language of modular geometry will help solving this question at least for the subclass of rational CFT's.

3. 3. 4. The simplest rational CFT's are the factorizable ones with the determinant modular functors $V = V_{c/2}$ of 3.2.3. The candidate examples here include the theories obtained by a functional integral over fields f taking values in $\mathbf{R}^c/\Gamma$ where Γ is an even, self-dual lattice in $\mathbf{R}^c$, i.e. $\Gamma = \sum_{a=1}^{c} \mathbf{Z} e_a$ where $e_a \cdot e_b \in \mathbf{Z}$, $e_a^2 \in 2\mathbf{Z}$ and $\det(e_a \cdot e_b) = 1$. The toroidal holomorphic amplitudes for those theories are given by the θ functions [77]

$$A(T_\tau) = \theta_\Gamma(\tau) dz^{c/2} = \sum_{\gamma \in \Gamma} e^{\pi i \tau \gamma^2} (dz)^{c/2}$$

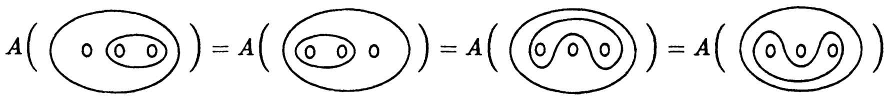
defining modular forms of weight c. c must be here a multiple of 8, the c=8 case corresponding to Γ_{E_8}, the root lattice of E_8 Lie algebra (the respective CFT is equivalent to level 1 WZW model based on group E_8, see 2.3.3.). For c=16, we have $\Gamma = \Gamma_{E_8} \bigoplus \Gamma_{E_8}$ or $\Gamma = \Gamma_{Spin(32)/\mathbf{Z}_2}$, the weight lattice of group $Spin(32)/\mathbf{Z}_2$. Both of the corresponding CFT's played an important role in string theory providing building blocks for the so called *heterotic string* [48].

For c=24 there are 24 so called Niemeier even self-dual lattices, the most important being the Leech lattice, the only one without vectors of length squared 2. In this case, by integrating over fields taking values in $\mathbf{R}^{24}/\Gamma_{Leech}/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts by multiplication by ± 1, one obtains a (candidate) theory with holomorphic toroidal amplitude given by

$$(j(\tau) - 744)\ \eta(\tau)^{24}\ (dz)^{12}$$

where $j(\tau) = \frac{1}{q} + 744 + 196884q + \ldots$ is the modular function of weight zero and $\eta(\tau)$ is the Dedekind function [77]. This is the theory which has the Monster group of B. Fischer and R. Griess as the symmetry group and Griess's algebra realized in the Hilbert subspace corresponding to the eigenvalue 2 of $\pi(L_0)$, see [27,79].

3. 3. 5. Any amplitude $A(\Sigma, p_i, \alpha_i)$ of a holomorphic CFT can be obtained from $A(D, p_0, \alpha^0) = v_0 \in H_{\alpha^0}$, $A(\Sigma_q, p_i, \alpha_i)$ (i.e. essentially a projective representation of $Diff_+S^1$ in each H_α) and $A(\Sigma(z_i, q_i), p_{0i}, \alpha_i)$, $i = 1,2,3,$ in the notation of 3.1.2 (*holomorphic vertex operators*). I.B. Frenkel announced the result [26] that for the factorizable case (i.e. for $V = V_{c/2}$), if we are given the above elementary amplitudes which, besides the obvious mutual consistency, define consistently amplitudes for $\mathbf{C}P^1$ minus 4 discs and for the tori minus one disc, i.e. if, symbolically,



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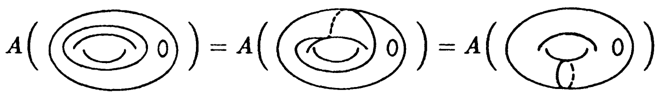


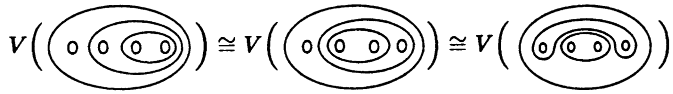
then, by subsequent gluing, one can unambiguously obtain the complete holomorphic CFT.

3. 3. 6. Since the bundles of a modular functor are projectively flat (flat for genus $\leq$1), see condition II.8 in 3.2.6, they are essentially determined by their holonomy defining a projective representation of the modular group $Diff(\Sigma, p_i)/Diff_0(\Sigma, p_i)$ of diffeomorphisms not contractible to identity. In particular, for $\Sigma = \mathbf{C}P^1$ without $|I|$ discs, the modular group is a central extension of the fundamental group of $\{(z_1, \ldots, z_{|I|}) \in (\mathbf{C}P^1)^{|I|} \mid z_i \neq z_{i'}, \text{ for } i \neq i'\}/S_{|I|}$ i.e. of the *braid group* [3] ($S_{|I|}$ denotes the permutation group of $|I|$ elements). Similar algebraic structure has been noticed in CFT in different but closely related contexts [74,56,33]. Recently, a systematic attempt at a construction of CFT models starting from braid group representations has been developed [72,73,34]. Such representations may in turn be constructed from the solutions of the so called *Yang-Baxter equation* [5], used to build exactly soluble lattice systems of statistical mechanics and related to *quantum groups* [84]. This motivated a search for even stronger connections between CFT and lattice models of statistical mechanics [50,66].

3. 3. 7. As was realized by E. Verlinde [85], the consistency of modular geometry imposes strong conditions on dimensions $N_{\alpha_1\alpha_2\alpha_3}$ of spaces $V(\Sigma(z_i, q_i), p_{0i}, \alpha_i)$, $i = 1,2,3$. Verlinde conjectured that $N_{\alpha_1\alpha_2\alpha_3}$ can be simply read off the holonomy matrix S corresponding to the transformations $\tau \mapsto -\frac{1}{\tau}$ on the torus bundle. His work has inspired an active research which may be viewed as aimed at classification of possible modular functors [82,12,60,61,62]. In particular, ref. [60,62]

which gave a proof of Verlinde's conjecture, were an attempt to write a complete list of consistency conditions on the bundles for the elementary surfaces which would guarantee the existence of the complete modular functor. The conclusion was that this could be done by studying the bundles over CP^1 without up to five discs and over tori without up to two ones.

To see an example of a consistency constraint, consider two chains of isomorphisms (with surfaces cut as indicated):



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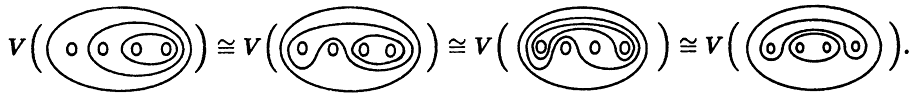
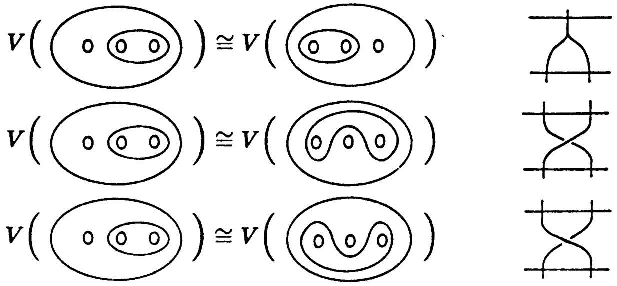
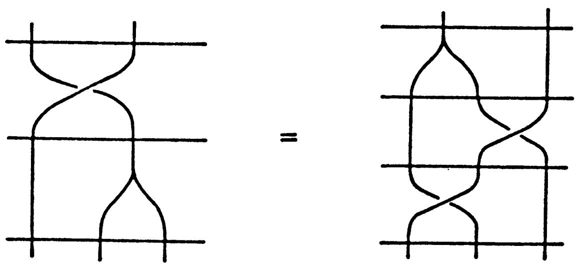


Upon the introduction of diagrams for the elementary *"fusing"* and *"braiding"* isomorphisms:



the equality of the isomorphisms between the constituent bundles induced by two chains leads to the *"pentagonal"* consistency relation [60,62,11]



Further consistency conditions can be obtained using also diffeomorphisms or (equivalently) holonomy operators which must be consistent with cutting and gluing of surfaces.

3. 3. 8. All these themes have returned recently in a quite surprising context of a three-dimensional topological gauge theory in the paper by E. Witten [88]. It would be out of place here

to discuss the main lines of this beautiful work, mostly devoted to a path-integral construction of knot invariants: the Jones polynomials [51]. Let us only mention that it also sketches a construction of the modular functor for the WZW model of CFT, see 2.3.3, as composed of (duals of) spaces of (fixed time) states for the three-dimensional non-abelian gauge theory with the action functional given by the integral of the *Chern-Simons* 3-form. Spaces $V(\Sigma, p_i)'$ become (sub)sets of holomorphic sections of a line bundle over the space of group G connections on Σ modulo complexified local gauge transformations fixed on the boundary. In particular, if $\partial\Sigma = 0$, the line bundles are based on the moduli space of holomorphic $G^{\mathbf{C}}$ vector bundles over Σ [4,65,17,18]. Vectors in $V(\Sigma, p_i)'$ may be formally constructed by a functional integral in the gauge theory over a three-dimensional manifold with Σ as (a part of) the boundary. This three-dimensional point of view allowed a better understanding of the monodromy properties of the WZW modular functor and a few lines proof of Verlinde's conjecture in this case. It should be possible to find a three-dimensional structure in other rational CFT's, in first turn in the coset models of 2.3.3. In [87] E. Witten suggested that for the unitary series of models, the three-dimensional counterpart might be quantum gravity.

3. 3. 9. Above, we have concentrated on the rational CFT's, still not completely explored (see e.g. recent papers [8,9]). Non-rational CFT's promise to be equally interesting. The preliminary results [40] reveal a rich geometry in spaces of other CFT models. It would be also interesting, especially from the statistical-mechanical and the "stringy" points of view, to understand fully the place occupied by CFT's among other two-dimensional quantum field theories [89,83].

It is clear that the last word in CFT has not been pronounced yet.

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