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MINIMAL MODELS OF ALGEBRAIC THREEFOLDS: MORI'S PROGRAM

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The aim of Mori's program is to provide a rough classification of algebraic varieties in dimension three (and higher if possible). Before I explain the exact aims, let us engage in a rather slanted review of the case of curves and surfaces.

1. Curves and surfaces

1.1. Let C be a smooth proper algebraic curve over $\mathbb{C}$ (equivalently, a compact Riemann surface). It is well known that C can be endowed with a metric of constant curvature, and one has the following classification according to the sign of the curvature:

curvature	structure
positive	$\mathbb{P}^1$
zero	$\mathbb{C}/\mathbb{Z}^2$
negative	$\mathbb{H}/\pi_1(C)$

(Here $\mathbb{H}$ is the upper half plane.)

This should be considered only a partial classification. The positive curvature case is completely clear but in the negative case much remains to be done.

1.2. One can attempt to extend these results to higher complex dimensions in several ways. Considering the sectional or the holomorphic bisectional curvature turns out to be too restrictive. Instead we can consider the curvature of the determinant of the tangent bundle, which is essentially the same as the Ricci curvature of the tangent bundle. For historical reasons we also dualize and consider our basic object:

$$\boxed{(det\ T_M)^* \quad \overset{def}{=} \quad K_M \quad \overset{def}{=} \quad \omega_M} \tag{1.3}$$

This will be referred to as the **canonical bundle** or the **dualizing sheaf** (for reasons that are unimportant now). Dualizing changes the sign of the curvature, creating the possibility of confusion. One can easily prove the following:

1.4. PROPOSITION. *Let $M \subset \mathbb{P}^n$ be a smooth hypersurface of degree k. Then K_M admits a metric whose curvature is*

negative	*if*	$k < n+1$
zero	*if*	$k = n+1$
positive	*if*	$k > n+1$

This sounds very promising but even in complex dimension two life is more complicated. There are two very simple constructions that create "mixed curvature" surfaces.

Products. The product of a positively curved and a negatively curved manifold cannot have a metric with semidefinite curvature. This is not surprising.

Blowing up or down. Let M be a complex manifold of dimension n and pick a point $x \in M$. We can "enlarge" M by removing x and introducing a copy of $\mathbb{P}^{n-1}$ corresponding to the complex directions at x. This way we obtain a new manifold B_xM which admits a natural map $p : B_xM \to M$. This is called the blowing up or blowing down. The map p is an isomorphism over $M - \{x\}$ and $p^{-1}(x) \cong \mathbb{P}^{n-1}$. If $L \subset \mathbb{P}^{n-1} \subset B_xM$ is a straight line then an easy computation yields that $\frac{\sqrt{-1}}{2\pi}\int_L \Theta(K_{B_xM}) = -1$. Thus the canonical line bundle of B_xM cannot be positive semidefinite.

In complex dimension two these are the only sources of indefinite curvature:

1.5. THEOREM. *Let M be a smooth proper algebraic surface. Then K_M admits a metric whose curvature is semipositive unless one of the following holds:*

(1.5.1) there is another (smooth) surface M_1 and $x \in M_1$ such that $M \cong B_xM_1$;

(1.5.2) M is a $\mathbb{P}^1$ bundle over a curve C;

(1.5.3) $M \cong \mathbb{P}^2$.

In all three cases there is an embedded copy of $\mathbb{P}^1 \cong C \subset M$ such that $\frac{\sqrt{-1}}{2\pi}\int_C \Theta(K_M) < 0$.

These cases are very different in nature. (1.5.2-3) are very precise global structural statements. One can hardly wish for more. (1.5.1) merely identifies (and removes) a small part of M and gives no global information. On the other hand it introduces a new surface M_1 which is simpler than M since it has "fewer" curves. (In fact $dimH^2(M_1,\mathbb{Q}) + 1 = dimH^2(M,\mathbb{Q})$.) We can apply (1.5) to M_1 and continue if possible. This gives the following:

1.6. THEOREM. *Let M be a smooth proper algebraic surface. There is a sequence of blowing downs $M \to M_1 \to \cdots \to M_n = M'$ such that M' satisfies exactly one of the following conditions:*

(1.6.1) $K_{M'}$ admits a metric whose curvature is semipositive.

(1.6.2) M' is a $\mathbb{P}^1$-bundle over a curve C;
(1.6.3) $M' \cong \mathbb{P}^2$.

The aim of Mori's program is to find an analog of these results for higher dimensional varieties.

2. First steps in higher dimensions

Before continuing further we replace the curvature assumptions with something more algebraic. Let L be a line bundle on a complex manifold M with metric h and curvature Θ and let $C \subset M$ be any proper curve. Then

$$c_1(L) \cap C = \frac{\sqrt{-1}}{2\pi} \int_C \Theta.$$

We will denote this number by $C \cdot L$. In particular, if Θ is semipositive then $C \cdot L \geq 0$ for every C.

2.1. Definition. We say that a line bundle L on a proper variety X is **nef** if $C \cdot L \geq 0$ for every compact curve $C \subset X$. (This replaces the earlier confusing terminology "numerically effective"). It is conjectured that for the canonical line bundle being nef is equivalent to admitting a metric with semipositive curvature.

As (1.5) suggests, we should try to understand those varieties X for which K_X is not nef. This means that there are curves $C \subset X$ such that $C \cdot K_X < 0$. First we would like to find the worst such curve C. To this end we consider

$$NE(X) = \begin{pmatrix} \text{The convex cone generated by the homology} \\ \text{classes of effective curves in } H_2(X, \mathbb{R}). \end{pmatrix}$$

and

$$\overline{NE}(X) = (\text{ The closure of } NE(X) \text{ in } H_2(X, \mathbb{R}).)$$

K_X gives a linear function on $H_2(X, \mathbb{R})$; thus the "worst" curves are on the boundary of the cone $\overline{NE}(X)$. More precisely, they should be extremal.

If M is a surface and K_M is not nef then by (1.5) we can always find an embedded copy of $\mathbb{P}^1$. In higher dimensions we will be able to guarantee only a nonconstant map $\mathbb{P}^1 \to X$. The image of such a map is called a **rational curve**. (In all examples known to me one can also find an embedded copy of $\mathbb{P}^1$.)

Mori's first major result is a partial description of the cone $\overline{NE}(X)$:

2.2. THEOREM. *(Mori [Mo1]) Let X be a smooth projective variety (any dimension). The extremal edges of the closed cone of curves $\overline{NE}(X)$ are discrete in the open halfspace $\{z \in H_2(X,\mathbb{R}) | z \cdot K_X < 0\}$. If $R \subset \overline{NE}(X)$ is such a negative extremal edge then there is a rational curve $C \subset X$ such that $[C] \in R$.*

Negative extremal edges are usually called **extremal rays**. Once an extremal edge is identified as the source of the trouble, one would like to use it to construct a map as in (1.5). In dimension three a complete description is known:

2.3. THEOREM. *(Mori [Mo1]) Let X be a smooth projective threefold over $\mathbb{C}$. Assume that K_X is not nef and let $R \subset \overline{NE}(X)$ be a negative extremal edge. Then there is a normal projective variety Y and a surjective map $f : X \to Y$ such that an irreducible curve $C \subset X$ is mapped to a point by f iff $[C] \in R$. One can choose Y such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and then Y and f are unique up to isomorphism. The following is a list of all the possibilities for f and Y.*

(2.3.1) Case 1: f is birational.

Let $E \subset X$ be the exceptional set of f. One has the following possibilities for E, Y and f:

(2.3.1.1) E is a smooth minimal ruled surface with typical fiber C and $C \cdot E = -1$. Y is smooth and f is the inverse of the blowing up of a smooth curve in Y.

(2.3.1.2) $E \cong \mathbb{P}^2$ and its normal bundle is $\mathcal{O}(-1)$. Y is smooth and f is the inverse of the blowing up of a point in Y.

In the remaining subcases Y has exactly one singular point P and f is the inverse of the blowing up of P in Y. Let $\hat{\mathcal{O}}_{P,Y}$ be the completion of the local ring of $P \in Y$.

(2.3.1.3) $E \cong \mathbb{P}^2$ and its normal bundle is $\mathcal{O}(-2)$. $\hat{\mathcal{O}}_{P,Y} \cong k[[x^2, y^2, z^2, xy, yz, zx]]$.

(2.3.1.4) $E \cong Q$ where Q is a quadric cone in $\mathbb{P}^3$ and its normal bundle is $\mathcal{O}_{\mathbb{P}^3}(-1)|Q$. $\hat{\mathcal{O}}_{P,Y} \cong k[[x,y,z,t]]/(xy - z^2 - t^3)$.

(2.3.1.5) $E \cong Q$ where Q is a smooth quadric surface in $\mathbb{P}^3$, the two families of lines on Q are homologically equivalent in X and its normal bundle is $\mathcal{O}_{\mathbb{P}^3}(-1)|Q$. $\hat{\mathcal{O}}_{P,Y} \cong k[[x,y,z,t]]/(xy - zt)$.

(2.3.2) Case 2: f is not birational.

Then we have one of the following subcases:

(2.3.2.1) dim$Y$=2. Then Y is smooth and f is a conic bundle (i.e. every fiber is isomorphic to a conic in $\mathbb{P}^2$).

(2.3.2.2) dim$Y$ = 1. Then Y is a smooth curve and every fiber of f is an irreducible and reduced (possibly singular) Del Pezzo surface.

(2.3.2.3) dim$Y$ = 0. Then X is a Fano variety (i.e. K_X is negatively curved).

It is crucial to assume that X is projective. If we allow nonprojective but proper algebraic varieties then infinitely many new subcases of birational contractions will occur. These have not yet been classified.

I do not want to dwell on the second case; (2.3.2) provides a very satisfactory classification.

The main point of interest is the first case. Instead of having only one subcase (as for surfaces) there are five. The first two are as expected but in the last three subcases the space Y has isolated singularities, although fairly simple ones.

Examples show that we cannot hope to get an analog of 1.6 if we insist on considering smooth varieties only. For a long time this was a considerable stumbling block and even conjectural approaches were lacking.

A crucial conceptual step forward is to abandon smooth varieties. In retrospect, the signs were already clear in dimension two. If one considers families of surfaces, then it is frequently more convenient to allow certain mild singularities in all the surfaces. Before we decide which class of singularities to allow, let us formulate clearly what do we want.

2.4. Choice of Singularities.

(2.4.1) We want to investigate varieties X for which K_X is not nef. In order to do this, K_X should exist and being nef should make sense.

(2.4.2) The usual definition of K_X works over the smooth locus of X. If X is normal (a harmless assumption) then $codim(Sing X) \geq 2$, hence K_X has a well defined homology class in $H_{2dim X-2}(X, \mathbb{Z})$. However, because of the singularities there is no intersection product between $H_{2dim X-2}$ and H_2. Thus the symbol $C \cdot K_X$ makes no sense in general.

(2.4.3) If $K_{X-Sing X}$ extends to a line bundle over X then its first Chern class is in $H^2(X, \mathbb{Z})$, and we can take the intersection product with $[C] \in H_2(X, \mathbb{Z})$. For the singularity given in (2.3.1.3), this condition is not satisfied because of the group action. However, $K^{\otimes 2}_{X-Sing X}$ will extend to a line bundle over X. Thus we can still define a first Chern class $c_1(K_X) \in H^2(X, \mathbb{Q})$, and this is also satisfactory.

(2.4.4) Failure of any of the above conditions would result in the death of the program. It is however desirable to have further conditions that keep us from straying too far from smooth varieties. Earlier we were willing to put up with blowing up smooth subvarieties. Therefore, the ideal would be to have a class of singularities that has no effect on properties that are invariant under blowing up of smooth subvarieties. This "metacondition" is too strong but serves as very good guide.

One of the most important invariance properties under blowing up is the invariance of the plurigenera: $P_m(X) = dim \Gamma(X, K_X^{\otimes m}) \quad (m \geq 0)$. Singularities that do not affect the plurigenera form one of the important classes of singularities for our purposes. There is however a technical strengthening of this property that we will ultimately use.

(2.4.5) Finally, there is a condition whose role is less clear at the moment. We say that a variety X has $\mathbb{Q}$-factorial singularities if for every codimension one subvariety $V \subset X$ there is an integer m such that mV is locally definable by one equation. The main consequence of this is that every codimension one subvariety will have a cohomology class in $H^2(X, \mathbb{Q})$.

2.5. Definition. An algebraic variety X is said to have **canonical** resp. **terminal** singularities if the following three conditions are satisfied:

(2.5.1) X is normal;

(2.5.2) $K^{\otimes m}_{X-Sing X}$ extends to a line bundle over X for some $m > 0$; (This unique extension will be denoted by $K^{\otimes m}_X$.)

($2.5.3_{canonical}$) If $f : X' \to X$ is a resolution of singularities then $P_m(X) = P_m(X')$ for every $m \geq 0$. (To be precise, we require an appropriate local version.)

($2.5.3_{terminal}$) If $f : X' \to X$ is a resolution of singularities and $\sigma \in K^{\otimes m}_X$ is a local section then $f^*\sigma \in K^{\otimes m}_{X'}$ vanishes along any codimension one component of the exceptional locus. (This is a strengthening of ($2.5.3_{canonical}$).)

2.6. PROPOSITION. *(2.6.1) A two dimensional terminal singularity is smooth.*

(2.6.2) Two dimensional canonical singularities are exactly the **DuVal singularities**. *(They are also called* **rational double points**).

In dimension three there is a complete list of terminal singularities as a result of works by Reid [R1], Danilov [D], Mori [Mo2] and Morrison-Stevens [MS]:

2.7. PROPOSITION. *Three dimensional terminal singularities are isolated. They are all quotients of hypersurface singularities by cyclic groups. The typical three dimensional terminal singularity can be described as the quotient of the hypersurface singularity $(xy + f(z, u^n) = 0) \subset \mathbb{C}^4$ by the cyclic group action $(x, y, z, u) \mapsto (\zeta x, \zeta^{-1}y, z, \zeta^a u)$ where $\zeta^n = 1$ is primitive and $(a, n) = 1$. (A similar description is available for the remaining exceptional ones.)*

2.8. The aim of Mori's program is to find certain "elementary" birational transformations such that by a successive application of these transformations every threefold X can be transformed into a threefold X' such that

either: $K_{X'}$ is nef;

or: X' is similar to a projective space bundle.

The "elementary" birational transformations correspond to extremal rays, though in a more complicated way than (2.3) suggests. The extremal rays correspond to certain rational curves $C \subset X$. Thus one can claim that if K_X is not nef then some rational curve $C \subset X$ is responsible for this, and the program provides a way of getting rid of these "bad" rational curves.

The whole program may work in all dimensions. At the moment only the first step, corresponding to (2.2), is known in all dimensions, as is a partial result corresponding to (2.3). These results are due to Kawamata [Ka1][Ka2],Benveniste [B], Kollár [Ko], Reid [R2] and Shokurov [S2].

2.9. THEOREM. *Let X be a projective variety (any dimension) over $\mathbb{C}$ with only $\mathbb{Q}$-factorial terminal (resp. canonical) singularities.*

(2.9.1) The extremal edges of the closed cone of curves $\overline{NE}(X)$ are discrete in the open halfspace $\{z \in H_2(X, \mathbb{R}) | z \cdot K_X < 0\}$ and they have rational directions.

(2.9.2) For every extremal edge R there is a contraction map $f : X \to Y$ such that a curve $C \in X$ is mapped to a point by f iff $[C] \in R$. One can always assume that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ and then f and Y are unique.

(2.9.3) We have the following possibilities for f and Y:

(2.9.3.1) f is birational and the exceptional set is an irreducible divisor. Then Y again has $\mathbb{Q}$-factorial terminal (resp. canonical) singularities. Such a contraction is called **divisorial**.

(2.9.3.2) f is birational and the exceptional set has codimension at least two in X. In this case $K_Y^{\otimes m}$ is **never** *a line bundle for $m > 0$. Such an f is called a* **small** *extremal contraction.*

(2.9.3.3) $dimY < dimX$. Then X is covered by rational curves. The general fiber F has negative canonical class. Such a contraction is called a **Fano** *contraction. Such an X should be considered "similar" to a projective space bundle.*

By (2.3) the small contraction case does not occur for smooth threefolds. Also, it leads us out of the required class of singularities since $K_Y^{\otimes m}$ is **never** a line bundle for $m > 0$. To see this assume that $K_Y^{\otimes m}$ is a line bundle. Then $K_X^{\otimes m}$ and $f^*K_Y^{\otimes m}$ are two line bundles on X and they are isomorphic outside the exceptional set. Since the exceptional set has codimension at least two, these line bundles are isomorphic. On the other hand, if $[C] \in R$ then

$$deg\ (K_X^{\otimes m}|C) < 0 = deg\ (f^*K_Y^{\otimes m}|C).$$

This is a contradiction.

Therefore (2.9.3.2) is an incorrect step in the program. Something new must be done; this new operation is called a **flip**.

3. Definition and examples of flips

From now on we restrict our attention to threefolds.

3.1 Definition. Let X be a threefold with terminal singularities and let $f : X \to Y$ be the contraction of an extremal ray. Assume that f is small. Let the exceptional set be $C \subset X$ and its image $Q \subset Y$. By the **flip** of f we mean a threefold X^+ together with a morphism $f^+ : X^+ \to Y$ which satisfies the following conditions:

(3.1.1) X^+ has terminal singularities;

(3.1.2) The exceptional set $C^+ \subset X^+$ is one dimensional and its image is again $Q \subset Y$. In particular, $X - C \cong X^+ - C^+$.

(3.1.3) K_{X^+} has positive intersection with any component of C^+.

The rational map $X --> X^+$ will also be called the flip of f.

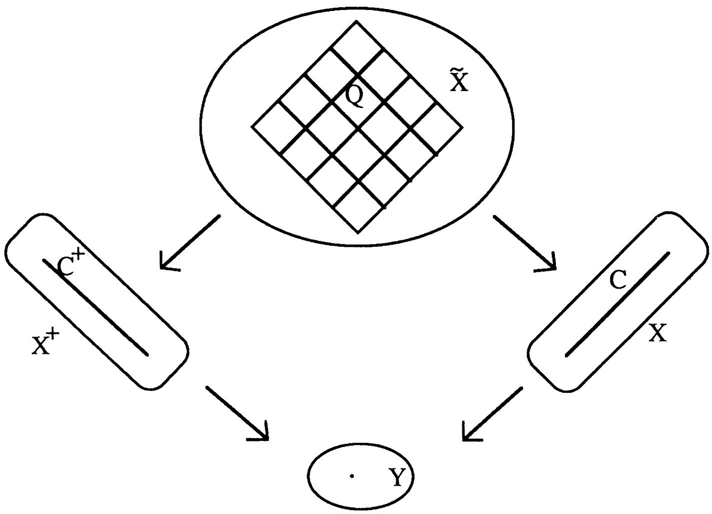
Heuristically speaking, a flip improves the situation because it replaces C (which has negative intersection with K_X) with C^+ (which has positive intersection with K_{X^+}). Unfortunately, it is not known how to attach a precise meaning to this remark.

It is not at all clear that flips exist; in fact, this is the hardest part of the whole program.

3.2 Examples of flips. The following is probably the simplest series of examples.

We start with an auxiliary construction.

Let us consider $Y = (xy - uv = 0) \subset \mathbb{C}^4$. This has an isolated singularity at the origin. If we blow it up, we get $\tilde{X} = B_0 Y$. The exceptional set $Q \subset \tilde{X}$ is the projective quadric $(xy - uv = 0) \subset \mathbb{P}^3$. This has two families of lines: $x = cv; y = c^{-1}u$ and $x = cu; y = c^{-1}v$. These two families can be blown down to smooth threefolds X resp. X^+. X resp. X^+ can also be obtained alternatively by blowing up the ideals (x, v) resp. (x, u). Let $C \subset X$, resp. $C^+ \subset X^+$ be the exceptional curves of $X \to Y$, resp. $X^+ \to Y$. Thus we have the following varieties and maps:



Consider the action of the cyclic group $\mathbb{Z}_n$: $(x, y, u, v) \mapsto (\zeta x, y, \zeta u, v)$ where ζ is a primitive n^{th} root of unity. This defines an action an all of the above varieties. The corresponding quotients are denoted by a subscript n.

The fixed point set of the action on Y is the 2-plane $(x = u = 0)$. On the projective quadric Q the action has two fixed lines: $(x = u = 0)$ corresponding to the above fixed 2-plane and $(y = v = 0)$ corresponding to the ζ-eigenspace. On X therefore the fixed point set has two components: the proper transform of the $(x = u = 0)$ plane and the image of the $(y = v = 0)$ line, this latter is an isolated fixed point. It is easy to see that $(x, v' = vx^{-1}, u)$ give local coordinates at the isolated fixed point. The group action is $(x, v', u) \mapsto (\zeta x, \zeta^{-1} v', \zeta u)$. In particular, the quotient is a terminal singularity.

On X^+ the fixed point set will have only one component and it contains the exceptional curve C^+. Thus X_n^+ is smooth.

It is not too hard to compute the intersection numbers of the canonical classes with the exceptional curves. We obtain that

$$C_n \cdot K_{X_n} = -\frac{n-1}{n} \quad \text{and} \quad C_n^+ \cdot K_{X_n^+} = n - 1.$$

Thus $X_n^+ \to Y_n$ is the flip of $X_n \to Y_n$ for $n \geq 2$.

Before going further let us note two special properties of this example. At the isolated fixed point on X we have coordinates (x, v', u) and the curve C is the v'-axis. A typical local $\mathbb{Z}_n$-invariant section of K_X^{-1} is given by $\sigma = (v'^{n-1} - x)(dx \wedge dv' \wedge du)^{-1}$, which has intersection number $(n-1)$ with C. Since this section is invariant, it descends to a local section σ_n of $K_{X_n}^{-1}$. Let $D_n = (\sigma_n = 0)$. By construction $D_n \cong \{(v', u) - plane\}/\mathbb{Z}_n$ which is a DuVal singularity (= rational double point) of type A_{n-1}. Since $C_n \cdot D_n = C_n \cdot K_{X_n}^{-1}$, one can easily see that even globally D_n is a member of $|K_{X_n}^{-1}|$.

Another simple way of getting a surface singularity out of the above construction is to consider the general hyperplane section H_n of Y_n. This is given as the quotient of an invariant section of Y. $v - u^n = 0$ is such a section whose zero set on Y is isomorphic to the the singularity $(xy - u^{n+1} = 0)$. This itself is a quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}_{n+1}$. Using this, H_n can be written as a quotient of $\mathbb{C}^2$ and we easily get that H_n is isomorphic to the singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$ where the action is $(z_1, z_2) \mapsto (\epsilon z_1, \epsilon z_2)$ and ϵ is a primitive $(n+1)^{st}$ root of unity.

4. Small contractions

In this section we will outline some steps toward the structure theory of small contractions in dimension three. A small extremal contraction can contract several irreducible curves simultaneously. If we pass to the analytic category then we can factor it into a series of morphisms, each contracting one irreducible curve only. These are the ones that we will consider for the most part.

4.1. Definition. Let $f : X \to Y$ be a proper bimeromorphic morphism of complex spaces which satisfies the following conditions:

(4.1.1) X has only terminal singularities;

(4.1.2) Y is normal with a distinguished point $Q \in Y$;

(4.1.3) $f^{-1}(Q)$ consists of a single irreducible curve $C \subset X$,

(4.1.4) the canonical class of X has negative intersection with C.

(4.1.5) $f : X - C \to Y - Q$ is an isomorphism.

In the above situation we say that $f : X \supset C \to Y \ni Q$ is an **extremal neighborhood**. We usually think of Y as being a germ around Q.

The ideal sheaf of the curve $C \subset X$ will be denoted by I.

By an appropriate version of Kodaira's vanishing theorem, $R^1 f_* \omega_X = 0$. By (4.1.4) we can say that $\mathcal{O}_X$ is "more positive" than ω_X and therefore $R^1 f_* \mathcal{O}_X = 0$. Let $J \subset \mathcal{O}_X$ be any ideal sheaf whose cosupport is C. Consider the sequences

$$\begin{aligned} 0 \to J \quad &\to \mathcal{O}_X \to \mathcal{O}_X/J \quad \to 0 \qquad \text{and} \\ 0 \to J\omega_X &\to \omega_X \to \omega_X/J\omega_X \to 0. \end{aligned}$$

Taking f_* we get long exact sequences. All the $R^2 f_*$'s are zero since the fibers of f are at most one dimensional. Since $R^1 f_* \mathcal{O}_X = 0$ and $R^1 f_* \omega_X = 0$ we obtain that

$$\boxed{H^1(\mathcal{O}_X/J) = 0 \quad \text{and} \quad H^1(\omega_X/J\omega_X) = 0}$$

As we will see, these are very restrictive conditions.

4.2. Notation. (4.2.1) For a sheaf F we define $gr^0 F = F \otimes \mathcal{O}_C/(torsion)$.

(4.2.2) If $i : X - SingX \hookrightarrow X$ is the natural injection then we define $\omega_X^{[k]} = i_*(\omega_{X-SingX}^{\otimes k})$.

4.3. Corollary. *With the above notation,* $C \cong \mathbb{P}^1$.

Proof. $H^1(\mathcal{O}_C) = H^1(\mathcal{O}_X/I) = 0$.

4.4. Corollary. *With the above notation,* $gr^0 \omega_X \cong \mathcal{O}_C(-1)$.

Proof: $\omega_X/I\omega_X$ is a generically rank one sheaf on $\mathbb{P}^1$. Therefore it is the direct sum of $gr^0\omega_X \cong \mathcal{O}(a)$ and a torsion sheaf. Since $H^1(\omega_X/I\omega_X) = 0$, we see that $a \geq -1$. Let $m > 0$ such that $\omega_X^{[m]}$ is locally free. We have a natural map

$$\mathcal{O}(ma) \to (\omega_X/I\omega_X)^{\otimes m} \xrightarrow{\beta} \omega_X^{[m]}|C.$$

By (4.1.4), $deg(\omega_X^{[m]}|C) = m(K_X \cdot C) < 0$, thus $ma < 0$. This gives that $a = -1$.

4.5. COROLLARY. *With the above notation, $gr^0 I \cong \mathcal{O}_C(a) + \mathcal{O}_C(b)$ and $a, b \geq -1$.*

Proof. Let us look at the long cohomology sequence of

$$0 \to I/I^2 \to \mathcal{O}_X/I^2 \to \mathcal{O}_X/I \to 0.$$

The map $H^0(\mathcal{O}_X/I^2) \to H^0(\mathcal{O}_X/I)$ is clearly surjective. $H^1(\mathcal{O}_X/I^2) = 0$, and therefore $H^1(I/I^2) = 0$. Now we get the result as in 4.3.

It is clear that one can continue in this way to obtain results about higher powers of I as well. The crucial point is however to get a handle on the singularities. Let $X \supset C \ni P$ be a singular point on X. By (2.7) X is locally the quotient of a hypersurface singularity $X^\sharp$. Let $C^\sharp \subset X^\sharp$ be the preimage of C. Although C is smooth, $C^\sharp$ can be quite complicated. For example, the quotient of the monomial curve

$$\mathbb{C} \xrightarrow{(t^a, t^{n-a}, t^{n+b})} \mathbb{C}^3$$

by the group action $(x, y, z) \mapsto (\zeta^a x, \zeta^{-a} y, \zeta^b z)$ is smooth. This curve singularity is fairly complicated if a is large. Also, in general $C^\sharp$ can be reducible.

To analyse the situation further, we will define certain local invariants of the triplet $X \supset C \ni P$ and then we use global inequalities to obtain bounds for them. There are two very important such invariants. Let

$$(4.6) \qquad \begin{aligned} w_P(X, C) &= \frac{1}{m} length_P \left(coker[(gr^0 \omega_X)^{\otimes m} \xrightarrow{\beta} \omega_X^{[m]} | C] \right) \quad \text{and} \\ i_P(X, C) &= \quad length_P \left(coker[\wedge^2 (I/I^2) \otimes \omega_C \xrightarrow{\alpha} gr^0 \omega_X] \right) \end{aligned}$$

where β is as in (4.4) and α comes from the natural map

$$\begin{array}{ccl} I/I^2 \times I/I^2 \times \omega_C & \to \omega_X | C & \to gr^0 \omega_X \\ x \times y \times z du & \to z dx \wedge dy \wedge du & \end{array}$$

Using the information we already have about the source and target of α and β we obtain:

4.7. COROLLARY. *With the above notation,*

$$\sum_P w_P(X, C) = 1 - C \cdot K_X \quad < 1;$$

$$\sum_P i_P(X, C) = 1 - deg(gr^0 I) \leq 3$$

4.8. COROLLARY. *With the above notation,* $-1 \leq C \cdot K_X < 0$ *and* $gr^0 I \cong \mathcal{O}(a) + \mathcal{O}(b)$ *where* $-1 \leq a, b$ *and* $a + b \leq 1$.

The usefulness of these inequalities hinges on our ability to compute these invariants. To simplify notation we assume that $C^\sharp$ is irreducible. The following result is very helpful:

4.9. LEMMA. *Let* $D \subset \mathbb{C}^k$ *be an irreducible curve singularity. Assume that the cyclic group* $\mathbb{Z}_n$ *acts on* $D \subset \mathbb{C}^k$ *and that* $D/\mathbb{Z}_n$ *is smooth. Then after an equivariant local analytic coordinate change* D *becomes monomial, i.e.*

$$D = image[\mathbb{C} \xrightarrow{(t^{a_1}, \dots, t^{a_k})} \mathbb{C}^k]$$

Proof. Let $p : \overline{D} \to D \subset \mathbb{C}^k$ be the normalization of D. Let t be a local parameter on $\overline{D}$ which is a $\mathbb{Z}_n$-eigenfunction. Let x_i be the coordinate functions on $\mathbb{C}^k$. We can write $p^* x_i = t^{a_i} g_i(t)$ where $g_i(t)$ is $\mathbb{Z}_n$-invariant with nonzero constant term. Since $D/\mathbb{Z}_n$ is smooth, $\overline{D}/\mathbb{Z}_n = D/\mathbb{Z}_n$. Therefore g_i is a regular function on D. Hence it extends to a regular function h_i on $\mathbb{C}^k$. Now introduce new coordinates $x_i' = x_i h_i^{-1}$. D is clearly monomial in this new coordinate system.

The main advantage of this lemma is that generators of I, ω_X etc. can be easily written down in terms of monomials. This makes the combinatorics manageable. One can prove the following:

4.10. THEOREM. *(Mori [Mo4,3.1]) Let* $X \ni P$ *be a three dimensional terminal singularity and let* $X \supset C \ni P$ *be the germ of a smooth curve. Assume that*

$$w_P(X, C) < 1 \quad \text{and} \quad i_P(X, C) \leq 3.$$

Then either $C^\sharp \subset X^\sharp$ *is a planar curve singularity or* $C^\sharp$ *has multiplicity 3.*

Considering a third invariant will show that if $X \supset C \ni P$ is on an extremal neighborhood then in fact $C^\sharp$ is always planar. This result is in some sense best possible since planar singularities of arbitrary multiplicity can occur as $C^\sharp$.

Considering a whole series of new invariants Mori develops a complete local classification of the possible triplets $X \supset C \ni P$. See [Mo4, Appendix A] for a list.

5. How to flip

The previous chapter explained the approach to describing small contractions in dimension three. Now we turn to examine various strategies for flipping.

At the moment there is no direct proof of the existence of flips. All the proofs rely on the detailed structure theory developed in [Mo4]. Once we have a good classification of small contractions one can check various sufficient conditions for the existence of flips. In some cases this is easy, in some cases it is still very hard.

5.1. Backtracking Method. This method has probably the simplest underlying principle. The idea is that we essentially do not want to flip. In the program we start with a smooth threefold X and construct a sequence of contractions $X \to X_1 \to \cdots \to X_k \xrightarrow{f} Y$ where f is a small contraction. In the intermediate steps we usually have a choice of which extremal ray to contract. It is not too difficult to find an example that shows that even if we choose the rays with care we can not avoid flips in the process. However one still might hope that if we also allow blow-ups in the sequence then flips can be avoided. This approach has failed so far in general, partly because there seems to be an enormous amount of bookkeeping involved. In important special cases this was used by Tsunoda [T] and Shokurov [S1]. Even in these cases the process is very complicated.

There are however two variants of this method that work well. Both rely on the observation that a certain flip-like operation - called a flop - is much simpler than a flip. There are two known ways of using flops to flip.

5.2. Definition. Let X be a threefold with canonical singularities and let $f : X \to Y$ be a morphism. Assume that the exceptional set is a curve $C \subset X$ and its image is $Q \subset Y$. Assume that K_X has **zero** intersection with every component of C. Furthermore let $D \in H^2(X,\mathbb{R})$ be an algebraic cohomology class such that D has negative intersection with every component of C.

By the **flop** of f we mean a threefold X^+ together with a morphism $f^+ : X^+ \to Y$ which satisfies the following conditions:

(5.2.1) X^+ has canonical singularities;

(5.2.2) The exceptional set $C^+ \subset X^+$ is one dimensional and its image is again $Q \subset Y$. In particular, $X - C \cong X^+ - C^+$.

(5.2.3) There is a (necessarily unique) class $D^+ \in H^2(X^+,\mathbb{R})$ such that $D^+|X^+ - C^+ = D|X - C$ and D^+ has positive intersection with every component of C^+.

The rational map $X - -> X^+$ will also be called the flop of f.

The extra datum D is not really important; it is mainly there to avoid the possibility that $X = X^+$.

The main difference between flips and flops is that in the latter $C \cdot K_X = C^+ \cdot K_{X^+} = 0$.

The reason that flops are easier to work with is that Y has nicer singularities. Indeed, since $K_X \cdot C = 0$ it is reasonable to hope (and indeed it is true) that K_X descends to Y to give K_Y. Now it is easy to see that Y has canonical singularities. Since we understand three dimensional canonical singularities quite well, we can hope to prove the existence of flops. In the case in which the canonical singularities are actually terminal, this was done by Reid [R3]. The general canonical case can be reduced to the terminal case by a variant of the backtracking method. This was done by Kawamata [Ka3].

There are two methods of reducing flips to flops.

5.3. Double Covering Method. Let $f : X \supset C \to Y \ni Q$ be a small extremal contraction. Let $D \subset X$ be a general member of $|-2K_X|$.Construct the double cover $\pi : X^d \to X$ ramified along D. Thus we have $f^d : X^d \supset C^d \to Y^d \ni Q^d$. Note that

$$C^d \cdot K_{X^d} = C^d \cdot \pi^*(K_X + \frac{1}{2}[D]) = C^d \cdot \pi^*(K_X + \frac{1}{2}[-2K_X]) = 0.$$

Therefore, if X^d has canonical singularities, then f^d corresponds to a flop. Thus we can realize a flip as a quotient of a flop by an involution. This method was first used by Kawamata [Ka3].

The big question is of course to ensure that X^d has canonical singularities. This requires an understanding of the general member of $|-2K_X|$.

It is fairly easy to prove that if the the general member of $|-K_X|$ has only DuVal singularities, then the general member of $|-2K_X|$ satisfies the above requirement.

5.4. One Step Back Method. Let $f : X \supset C \to Y \ni Q$ be a small extremal contraction. We try to choose a modification at a singular point $g : X' \to X$ carefully enough such that X' has terminal singularities and that if C' denotes the proper transform of C on X' then $C' \cdot K_{X'} \leq 0$. If we have strict inequality then the contraction of C' results in a flip and we are done by some sort of induction. If $C' \cdot K_{X'} = 0$, then we are really happy since the contraction of C' results in a flop. The details of this approach (due to Kawamata) have not appeared yet, but it also requires a substantial portion of the information obtained in [Mo4].

5.5. Families of Surfaces Method. Let $f : X \supset C \to Y \ni Q$ be a small extremal contraction. Let $t \in \Gamma(\mathcal{O}_Y)$ be a general section such that $t(Q) = 0$. Let $H = (t = 0) \subset Y$. Using the map $t : Y \to \mathbb{C}$ we can view Y as a one parameter family of surfaces. If we understand H and its deformations well, then we can hope to understand X^+. In the example of section 3, H (denoted there by H_n) is the quotient singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$, where the action is given by $(z_1, z_2) \mapsto (\epsilon z_1, \epsilon z_2)$ with ϵ a primitive $(n+1)^{st}$ root of unity.

It is known that if $n \neq 3$ then the deformation space of this singularity has exactly one component, the so-called Artin component. Moreover, if $H^+ \to H$ is the minimal resolution of H, then every deformation of H can be lifted back to a deformation of H^+. It is easy to see that X^+ is exactly the total space of the corresponding deformation of H^+.

For this method to work, we need to understand H. Then we need a description of the deformations of H analogous to the above description of the Artin component. For quotient singularities this was done in [KSB]. It is interesting to note that the proof itself uses some results of Mori's program.

This leads to one of the drawbacks of this approach. One must do much of the work twice to avoid a vicious cycle. The other drawback is that it is much more difficult to control the general section of $\mathcal{O}_X$ than the general member of $|-K_X|$.

The advantage is that this approach works well for families of threefolds, see [KM].

5.6. Complete Intersection Method. Let $f : X \supset C \to Y \ni Q$ be a small extremal contraction and let $X^+ \supset C^+$ be the flip. In X^+ we take two surface germs D_1^+ and D_2^+ that are disjoint and intersect C^+ transversally. Let their proper transforms on X be D_1 and D_2. Clearly both contain C and they are disjoint outside C. Thus, set-theoretically, $D_1 \cap D_2 = C$. Conversely, it is easy to see that if we find two such surface germs along C then X^+ is essentially given by the linear system $\langle D_1, D_2 \rangle$. One can compute that $D_i \cdot C < 0$, thus one expects that the D_i will be members of $|jK_X|$ for some positive j.

This is Mori's original approach [Mo3]. Its disadvantage is that finding these D_i is very hard; in fact, it has not been done in all cases. On the other hand, this approach gives the most precise results about X^+.

6. Sections of $\omega_X^{[k]}$.

We saw in the previous section that various methods of flipping are based on finding sections of the sheaves $\omega_X^{[k]}$ for $k = -2, -1, 0, \ldots$. Let us observe first that if $k \ll 0$ then $\omega_X^{[k]}$ is f-very ample; thus, there are plenty of sections in general position. In particular, finding nice sections should be easier for $k = -2$ than for $k = -1$, and so on. On the other hand, we can use a section of $\omega_X^{[k-1]}$ to get some information about the sections of $\omega_X^{[k]}$. If $\mathcal{O}_X \to \omega_X^{[k-1]}$ is a section with zero set D, then consider the sequence

$$0 \to \omega_X \to \omega_X^{[k]} \to \omega_X^{[k]}|D \to 0.$$

As we already noted, $R^1 f_* \omega_X = 0$. In particular, any section of $\omega_X^{[k]}|D$ lifts to a section of $\omega_X^{[k]}$. Thus we have two tasks ahead of us. First we have to find a section for k negative and then get a section for k larger and larger step-by-step. While it is very easy to start with $k \ll 0$, the rest of the process becomes very difficult. Therefore we start with $k = -2$ or $k = -1$ depending on the extremal neighborhood.

6.1. $k = -2$ **case.** Since $C \cdot \omega_X^{[-2]} > 0$ we have a chance to find a section which does not contain C. In many cases this indeed will be possible. Consider a singularity of the extremal neighborhood $X \supset C \ni P$. Let $P \in D \subset X$ be any divisor which does not contain C and assume that some multiple of D is Cartier. Let us consider the hypersurface cover $X^\sharp \supset C^\sharp \ni P^\sharp$. Let $m_P = deg(X^\sharp/X)$. Let $P^\sharp \in D^\sharp \subset X^\sharp$ be the pull-back of D. Since $D^\sharp$ is a Cartier divisor,

$$C \cdot D = \frac{1}{m_P} C^\sharp \cdot D^\sharp \geq \frac{1}{m_P} mult_{P^\sharp} C^\sharp.$$

Assume that D is a global section of $\omega_X^{[-2]}$ not containing C. Then D must pass through all the points of C where $\omega_X^{[-2]}$ is not locally free. In particular we get that

$$C \cdot \omega_X^{[-2]} \geq \sum_{\{P \mid \omega_X^{[-2]} \text{ is not locally free}\}} \frac{1}{m_P} mult_{P^\sharp} C_P^\sharp.$$

Therefore we can have such a section D only if $C \cdot \omega_X^{[-2]}$ is large compared to the multiplicities of $C^\sharp$.

From the list in [Mo4,Appendix B], one can easily see that there is a section of $\omega_X^{[-2]}$ not containing C if there is at most one singular point of index > 2 along C. There are, however, examples when there are two singular points of indices m and $m+1$ such that

$$C \cdot \omega_X^{[-2]} = \frac{2}{m(m+1)} < \frac{1}{m} + \frac{1}{m+1}.$$

In this case every section of $\omega_X^{[-2]}$ must contain C, and it turns out to be easier to find a section of $\omega_X^{[-1]}$ directly.

6.2. $k = -1$ **case.** We consider the case when there are (at least) two singular points of index > 2 along C. One can prove that these are the only singular points along C. Finding a section of $\omega_X^{[-1]}$ is the same as finding a homomorphism $h : \omega_X \to \mathcal{O}_X$. We want the section to vanish along C. Thus the image of h lies in I (the ideal of C). We want to build this h via infinitesimal methods. The first approximation is a map

$$gr^0 h : gr^0 \omega_X \to gr^0 I.$$

By (4.4) $gr^0 \omega_X \cong \mathcal{O}_C(-1)$. There are two singular points along C; both have $i_P = 1$. Thus by (4.7) $deg(gr^0 I) = -1$. In fact one can see that $gr^0 I \cong \mathcal{O}_C(-1) + \mathcal{O}_C$. Thus we are searching for a map

$$gr^0 h : \mathcal{O}_C(-1) \to \mathcal{O}_C(-1) + \mathcal{O}_C.$$

Such maps certainly exist, our problem is that there are too many of these. One can see from examples that in general $gr^0 h$ is independent of h. Therefore somehow we have to find a unique way of picking such a map.

If we had started more systematically, the same problem would have come up earlier. Namely, as a first step we would have a map

$$\mathcal{O}_C(-1) \cong gr^0\omega_X \to gr^0\mathcal{O}_X \cong \mathcal{O}_C.$$

There are many such maps but we know that in general they cannot be extended to higher order neighborhoods. Here we can use an ad hoc argument which goes as follows: h cannot be an isomorphism at the points where ω_X is not locally free. If there are two such points then this implies that the induced map $\mathcal{O}_C(-1) \to \mathcal{O}_C$ is not an isomorphism at two points. Therefore it is the zero map.

In general it is unlikely that such ad hoc arguments will solve all our problems. In fact one can see that similar problems do exist at all levels of the extension process. Therefore we need to find a systematic way to provide extra rigidity to the sheaves and maps. This will be discussed next.

6.3. ℓ-structures. Consider a singularity $X \supset C \ni P$, and let its hypersurface cover be $X^\sharp \supset C^\sharp \ni P^\sharp$. $p : X^\sharp \to X$ is an étale $\mathbb{Z}_m$-cover outside P. Let F and G be reflexive sheaves on X (e.g. ω_X, I or $\mathcal{O}_X$.) Let $p^{[*]}F$ etc. denote the reflexive hull of the pull-back. Then any homomorphism $h : F \to G$ induces a homomorphism $p^{[*]}h : p^{[*]}F \to p^{[*]}G$. Moreover there is a natural $\mathbb{Z}_m$-action on the pulled-back objects and

$$(h : F \to G) = \mathbb{Z}_m - \text{invariant part of } \left(p^{[*]}h : p^{[*]}F \to p^{[*]}G\right).$$

For a sheaf H on $X^\sharp$ let $gr^\sharp(H) = H \otimes \mathcal{O}_{C^\sharp}/(\text{torsion})$. Then we have the following relationship:

$$gr^0 F = \mathbb{Z}_m - \text{invariant part of } \left(gr^\sharp(p^{[*]}F)\right).$$

Moreover, $gr^0 h$ lifts to a map

$$gr^\sharp h : gr^\sharp(p^{[*]}F) \to gr^\sharp(p^{[*]}G).$$

6.4. Definition. Let C be a smooth curve and let $P_j \in C$ be closed points. Let $P_j \in C_j \subset C$ be an analytic (or formal) neighborhood of P_j for every j. Assume that we specify germs of curves $C_j^\sharp$ with a $\mathbb{Z}_{m_j}$ action such that $C_j^\sharp/\mathbb{Z}_{m_j} \cong C_j$.

Let finally F^0 be a torsion free sheaf on C.

(6.4.1) An l-**structure** on F^0 is:

(6.4.1.1) a collection of torsion free sheaves F_j^0 on $C_j^\sharp$ with a $\mathbb{Z}_{m_j}$ action and

(6.4.1.2) a collection of isomorphisms

$$F^0|C_j \cong \mathbb{Z}_{m_j} - \text{invariant part of } F_j^0.$$

As we explained in (6.3), if F is a reflexive sheaf on X then $F^0 = gr^0 F$ comes endowed with an l-structure .

(6.4.2) An l-**homomorphism** between two sheaves with l-structures F^0 and G^0 is a homomorphism h^0 such that there exist homomorphisms $h_j^0 : F_j^0 \to G_j^0$ which induce h^0 on the invariant parts. If $h : F \to G$ is a homomorphism between reflexive sheaves on X then $gr^0 h$ is an l-homomorphism.

6.5. Example. Let C be a smooth rational curve. Let $p : C^\sharp = C_1^\sharp \to C$ be a smooth local double cover, ramified at a fixed $P \in C$. If z is a local prameter at P then $\sqrt{z}$ is a local prameter on $C^\sharp$. $\mathbb{Z}_2$ acts via $\tau(\sqrt{z}) = -\sqrt{z}$. If L is a line bundle on C there can be two different l-structures on L.

Trivial: $L_P \cong \mathcal{O}_{C^\sharp}$ with $\mathbb{Z}_2$-action $\tau(1^{\text{trivial}}) = 1^{\text{trivial}}$. In this case the invariant part is $1^{\text{trivial}} \cdot \mathcal{O}_C$. We denote this by L^{+0}.

Twisted: $L_P \cong \mathcal{O}_{C^\sharp}$ with $\mathbb{Z}_2$-action $\tau(1^{\text{twisted}}) = -1^{\text{twisted}}$. In this case the invariant part is $1^{\text{twisted}} \cdot \sqrt{z} \cdot \mathcal{O}_C$. We denote this by $L^{+\frac{1}{2}}$.

I claim that there are no l-homomorphisms from $L^{+\frac{1}{2}}$ to L^{+0}. Indeed, if σ is a local generator of L at P then we can assume that $h(\sigma) = \sigma$. In $L^{+\frac{1}{2}}$ we can write $\sigma = 1^{\text{twisted}} \cdot \sqrt{z}$. In L^{+0} we can write $\sigma = 1^{\text{trivial}}$.

The corresponding homomorphism $h_1 : \mathcal{O}_{C^\sharp} \to \mathcal{O}_{C^\sharp}$ should satisfy the equality $h_1(1^{\text{twisted}}) = z^{-\frac{1}{2}} 1^{\text{trivial}}$, which is impossible.

As suggested by this example, if $C^\sharp$ is smooth and $C^\sharp \to C$ has ramification index m at P then putting an l-structure on a line bundle L at P is essentially the same as specifying some negative fractional power $z^{-\frac{a}{m}}$ (of course $a < m$) as the hypothetical generator of L_P. To be precise, let $\sqrt[m]{z}$ be a local parameter on $C^\sharp$ and let the group action be given by $\sqrt[m]{z} \mapsto \zeta \sqrt[m]{z}$. The action on $\mathcal{O}_{C^\sharp}$ given by $1 \mapsto \zeta^{m-a} \cdot 1$ gives an l-structure on L. We denote this l-structure by $L^{+\frac{a}{m}}$. It is clear that there is a nonzero l-homomorphism between $L^{+\frac{a}{m}}$ and $L^{+\frac{a'}{m}}$ iff $a \leq a'$.

The same principle applies if there are several points P_i; we just have to keep track of them independently. However if $C^\sharp$ is singular or we consider higher rank sheaves then l-structures can even have moduli.

6.6. Now we return to our problem to see how l-structures help. First of all consider the possible maps

$$\mathcal{O}_C(-1) \cong gr^0 \omega_X \to gr^0 \mathcal{O}_X \cong \mathcal{O}_C.$$

The l-structure on $gr^0\mathcal{O}_X$ is trivial at both singular points. What about the l-structure on $gr^0\omega_X$? One can prove that at the two singularities P_1 and P_2 in suitable coordinates we have

$$X_i^\sharp = (xy + f_i(z, u^{m_i}) = 0) \subset \mathbb{C}^4,$$

and $C^\sharp$ is the x-axis. The group action is $(x, y, z, u) \mapsto (\zeta x, \zeta^{-1}y, z, \zeta^{a_i}u)$ where $\zeta^{m_i} = 1$ is primitive and $(a_i, m_i) = 1$. Thus the generator of $gr^0(\omega_X)$ at P_i is

$$x^{m_i - a_i}\frac{dy \wedge dz \wedge du}{y}.$$

Therefore, in the above notation we can write

$$gr^0(\omega_X) = \mathcal{O}_C(-1)^{+\frac{m_1-a_1}{m_1}, +\frac{m_2-a_2}{m_2}}.$$

Thus the zero map is the only l-homomorphism $gr^0(\omega_X) \to gr^0\mathcal{O}_X \cong \mathcal{O}_C^{+0,+0}$.

Let us turn our attention to $gr^0h : gr^0\omega_X \to gr^0I$. By the previous discussion, we are likely to have fewer l-homomorphisms than ordinary homomorphisms. In fact the new problem will be to prove that there are any l-homomorphisms at all. This requires the computation of the l-structure of gr^0I. This is far from trivial, since l-structures on higher rank sheaves are more complicated than on line bundles. In fact, very delicate arguments are needed to show that certain cases cannot occur.

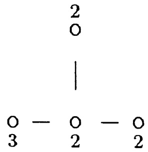
The step-by-step extension of the homomorphism requires a very thorough knowledge of how the two singularities behave with respect to higher order neighborhoods of C. The required arguments are long and delicate. I can only urge everyone to study [Mo4, Chapters 8-9].

7. Statement of the results

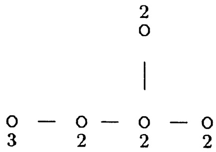
The description of extremal neighborhoods developed in [Mo4] and [KM] is too long to be given here. The following two results are the ones needed for various results concerning flips:

7.1. THEOREM. *(Mori [Mo4], Kollár-Mori [KM]) (Reid's conjecture about general elephants) Let $f : X \supset C \to Y \ni Q$ be an extremal neighborhood. Then the general member of $|-K_X|$ and the general member of $|-K_Y|$ have only DuVal singularities.*

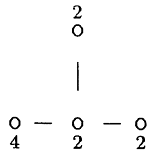
7.2. THEOREM. *(Kollár-Mori [KM]) Let $f : X \supset C \to Y \ni Q$ be an extremal neighborhood. Let $t \in I_Q \subset \mathcal{O}_Y$ be a general element of the ideal of Q and let $H = (t = 0)$. Then H is either a cyclic quotient singularity or one of the following singularities described by the dual graph of their minimal resolution:*



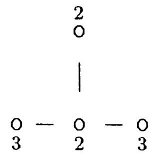
(triple point / dihedral quotient)



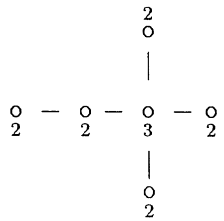
(triple point / dihedral quotient)



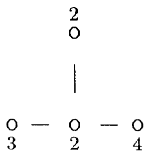
(quadruple point / dihedral quotient)



(quadruple point / tetrahedral quotient)



(quadruple point / nonquotient)



(quintuple point / icosahedral quotient)

As I explained in section 4 these results imply the existence of flips. Now we are ready to complete Mori's program in dimension three as follows:

7.3. MAIN THEOREM. *(Mori [Mo4]) Let X be a smooth projective three dimensional algebraic variety. A succession of divisorial contractions and flips transforms X into a variety X' which has the following properties:*

(7.3.1) X' and X are birationally equivalent;

(7.3.2) X' has only $\mathbb{Q}$-factorial terminal singularities;

(7.3.3) X' satisfies exactly one of the following alternatives:

either: $K_{X'}$ is nef (i.e. it has nonnegative intersection with any compact curve C in X');

or: there is a morphism $g : X' \to Z$ onto a lower dimensional variety such that $K_{X'}$ has negative intersection with every curve contained in a fiber of g.

This X' is not unique, but only one of the alternatives can occur. Moreover, if $K_{X'}$ is nef then it is well understood how the different choices of X' are related to each other.

Proof. Starting with a smooth threefold X we define inductively a series of threefolds as follows. Let $X_0 = X$. If X_i is already defined, we consider K_{X_i}. If it is nef then let $X' = X_i$. If it is not nef then we contract an extremal ray. If the contraction gives a nontrivial fiberspace structure $X_i \to Z$ then again we set $X' = X_i$. If the contraction $f_i : X_i \to Y_i$ is divisorial then we set $X_{i+1} = Y_i$. If the contraction is small then we set $X_{i+1} = X_i^+$ (the flip). All that remains is to prove that the process will terminate.

A divisorial contraction decreases $dim H_2$ by one, so we can have only finitely many of these. A flip leaves $dim H_2$ unchanged. Shokurov [S2] proved that a flip "improves" the singularities and this easily implies that any sequence of flips is finite. This completes the proof.

8. Consequences

The Main Theorem (7.3) is only the starting point of the structure theory of threefolds. Most of the work still lies ahead. There are already several results that can be formulated (and originally were conjectured) independently of Mori's program, but whose proofs rely on the program in an essential way.

8.1. THEOREM. *(Miyaoka [Mi], Miyaoka-Mori [MM]) ($\kappa = -\infty$ characterization) Let X be a smooth projective threefold. Then the following two statements are equivalent:*

(8.1.1) There is a rational curve through every point of X.

(8.1.2) $H^0(X, \omega_X^{\otimes i}) = 0$ for every $i > 0$.

8.2. THEOREM. *(Kawamata [Ka1], Benveniste [B], Fujita [F]) (Finite generation of the canonical ring) Let X be a smooth projective threefold. Then the canonical ring*

$$\sum_{i=0}^{\infty} H^0(X, \omega_X^{\otimes i})$$

is finitely generated.

8.3. THEOREM. *(Kollár-Mori [KM]) (Deformation invariance of plurigenera) Let $\{X_t : t \in T\}$ be a flat family of smooth projective threefolds. Assume that T is connected. Assume that for some $0 \in T$ and for some $m > 0$ we have $h^0(X_0, \omega_{X_0}^{\otimes m}) \geq 2$.*

Then $h^0(X_t, \omega_{X_t}^{\otimes n})$ is independent of $t \in T$ for every n.

8.4. THEOREM. *(Kollár-Mori [KM]) (Moduli space for threefolds of general type) Let $\mathcal{M}$ be the functor "families of threefolds of general type modulo birational equivalence", i.e.*

$$\mathcal{M}(S) = \left\{ \begin{array}{l} \textit{Flat projective families } X/S \textit{ such that every fiber is a three-} \\ \textit{fold of general type. Two families } X^1/S \textit{ and } X^2/S \textit{ are} \\ \textit{equivalent if there is a rational map } f : X^1/S -\!-\!> X^2/S \\ \textit{which induces a birational equivalence on each fiber.} \end{array} \right\}$$

Then there is a separated algebraic space $\mathbf{M}$ which coarsely represents $\mathcal{M}$. Every connected component of $\mathbf{M}$ is of finite type.

Finally, I would like to mention two easy applications of the above results to nonprojective threefolds. These results are aside from the main direction, but they illustrate the scope of applications.

8.5. THEOREM. *(Peternell [P]) Let X be a compact complex threefold which is bimeromorphic to a projective threefold but not itself projective. Then X contains at least one rational curve.*

(When Peternell proved this result, (7.3) was still a conjecture. His proof - using only (2.3) - is quite delicate. Using (7.3) the proof is easy.)

8.6. THEOREM. *(Kollár-Mori [KM]) Let $g : X \to S$ be a proper smooth map of complex spaces. Assume that the fiber X_s is a projective threefold of general type for some $s \in S$. Then there is an open neighborhood $s \in U \subset S$ such that X_u is projective for every $u \in U$. (In general g is not projective over U.)*

References

Survey articles

H. Clemens, J. Kollár and S. Mori, "Higher Dimensional Complex Geometry," Asterisque 166, 1988
This booklet contains the simplest known proofs of (2.9) and (4.10). It also contains a lot of background material..

Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the Minimal Model Problem*, in "Algebraic Geometry, Sendai," Adv. Stud. Pure Math. vol 10. T. Oda ed., Kinokuniya - North-Holland, 1987, pp. 283-360.
The most complete discussion of (2.9) and related questions.

J. Kollár, *The structure of algebraic threefolds - an introduction to Mori's program*, Bull. AMS **17** (1987), 211-273.
A leisurely introduction, aimed at all mathematicians.

M. Reid, *Young person's guide to canonical singularities*, in "Algebraic Geometry Bowdoin 1985," Proc. Symp. Pure Math. vol. 46, 1987, pp. 345-416.
A nice treatment of the relevant singularities.

P.M.H. Wilson, *Toward a birational classification of algebraic varieties*, Bull. London Math. Soc. **19** (1987), 1-48.
An overview aimed at algebraic geometers, written before [Mo4] appeared.

Research articles

[B] X. Benveniste, *Sur l'anneau canonique de certaines variétés de dimension 3*, Inv. Math. **73** (1983), 157-164.

[D] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97-154.

[F] T. Fujita, *Zariski decomposition and canonical rings of elliptic threefolds*, J. Math. Soc. Japan **38** (1986), 19-37.

[Ka1] Y. Kawamata, *On the finiteness of generators of the pluri-canonical ring for a threefold of general type*, Amer. J. Math. **106** (1984), 1503-1512.

[Ka2] Y. Kawamata, *The cone of curves of algebraic varieties*, Ann. of Math. **119** (1984), 603-633.

[Ka3] Y. Kawamata, *The crepant blowing-up of 3-dimensional canonical singularities and its application to the degeneration of surfaces*, Ann. of Math **127** (1988), 93-163.

[Kol] J. Kollár, *The Cone Theorem*, Ann. of Math. **120** (1984), 1-5.

[KM] J. Kollár and S. Mori, *soon to be written up*.

[KSB] J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Inv. Math. **91** (1988), 299-338.
[Mi] Y. Miyaoka, *On the Kodaira dimension of minimal threefolds*, Math. Ann. **281** (1988), 325-332.
[MM] Y. Miyaoka and S. Mori, *A numerical criterion of uniruledness*, Ann. of Math **124** (1986), 65-69.
[Mo1] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. **116** (1982), 133-176..
[Mo2] S. Mori, *On 3-dimensional terminal singularities*, Nagoya Math. J. **98** (1985), 43-66.
[Mo3] S. Mori, *Minimal models for semistable degenerations of surfaces*, Lectures at Columbia University (1985), unpublished.
[Mo4] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, Journal AMS **1** (1988), 117-253.
[MS] D. Morrison and G. Stevens, *Terminal quotient singularities in dimension three and four*, Proc. AMS **90** (1984), 15-20.
[P] T. Peternell, *Rational curves on Moishezon threefolds*, in "Complex Analysis and Algebraic Geometry," Springer LN. 1194, 1986, pp. 133-144.
[R1] M. Reid, *Canonical Threefolds*, in "Géometrie Algébrique Angers," A. Beauville ed., Sijthoff & Noordhoff, 1980, pp. 273-310.
[R2] M. Reid, *Projective morphisms according to Kawamata*, preprint, Univ. of Warwick (1983).
[R3] M. Reid, *Minimal models of canonical threefolds*, in "Algebraic Varieties and Analytic Varieties," Adv. Stud. Pure Math. vol 1. S. Iitaka ed., Kinokuniya and North-Holland, 1983, pp. 131-180.
[S1] V. V. Shokurov, *letter to M. Reid (1985).*
[S2] V. V. Shokurov, *Theorem on nonvanishing*, Math. USSR Izv. **26** (1986), 591-604.
[T] S. Tsunoda, *Degenerations of Surfaces*, in "Algebraic Geometry, Sendai," Adv. Stud. Pure Math. vol 10. T. Oda ed., Kinokuniya - North-Holland, 1987, pp. 755-764.

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