A. VAN DE VEN

On the differentiable structure of certain algebraic surfaces

Astérisque, tome 145-146 (1987), Séminaire Bourbaki,
exp. nº 667, p. 299-312
<http://www.numdam.org/item?id=SB_1985-1986__28__299_0>

Séminaire BOURBAKI Juin 1986
38ème année, 1985-86, n° 667

ON THE DIFFERENTIABLE STRUCTURE OF CERTAIN ALGEBRAIC SURFACES

par A. VAN DE VEN

The past decades have seen great progress in the classification of the differentiable structures on a given topological manifold M, say compact, oriented and simply-connected.

Already in the early fifties Moise showed that if $\dim(M) \leq 3$, then M admits a unique differentiable structure. On the other hand, for higher-dimensional manifolds it was known since Milnor (1956) that in some cases there may be more than one differentiable structure, and since Kervaire (1960) that in other cases there may be none at all. Nowadays a well-developed theory tells us that for $\dim(M) \geq 5$ the differentiable structures on M can be obtained from obstruction theory. However, neither Moise's methods nor the methods of obstruction theory work in dimension 4, and until some years ago little was known for this case. But during the last five years S. Donaldson has obtained a number of striking results. These results are based on Yang - Mills theory, where analysis plays an essential role.

In this talk I want to focus on *one* of these achievements, namely the construction of a new invariant for a certain class of differentiable 4-manifolds. The invariant has a beautiful interpretation (again due to Donaldson) in the case of algebraic surfaces. This interpretation makes it possible to apply the methods of algebraic geometry to settle questions about the differentiable equivalence of some algebraic surfaces, questions which had been open for many years. In this way not only a counterexample to the h-cobordism conjecture in dimension 4 is obtained, but also an example of a topological 4-fold with an infinity of inequivalent differentiable structures. This contrasts with the fact that for $\dim(M) \geq 5$ there is only a finite number of differentiable manifolds M with given homotopy type and Pontrjagin classes.

1. A PROBLEM CONCERNING THE DIFFERENTIABLE STRUCTURE OF TWO ALGEBRAIC SURFACES

With Donaldson in [Do 3] and [Do 6] we start by comparing two algebraic surfaces (by which I shall mean here smooth, connected, 2-dimensional projective algebraic varieties) as differentiable manifolds.

Let $\mathbb{P}_n = \mathbb{P}_n(\xi) = \mathbb{P}_n(\xi_1 : \ldots : \xi_{n+1})$ be the n-dimensional complex projective space with homogeneous coordinates $(\xi) = (\xi_1 : \ldots : \xi_{n+1})$. Given any open set $U \subset \mathbb{C}^2(z_1, z_2)$ with $0 \in U$, we can consider on $U \times \mathbb{P}_1(\xi_1:\xi_2)$ the submanifold V given by $z_1\xi_2 - z_2\xi_1 = 0$. If $\pi : U \times \mathbb{P}_1 \longrightarrow U$ is the projection, then we can identify $V \setminus \pi^{-1}(0)$ with $U \setminus 0$ via π, whereas $\pi^{-1}(0) \cong \mathbb{P}_1$. We say that V is obtained from U by *blowing up* 0 ([BPV], p. 28). In this way we can blow up any point of a complex surface (2-dimensional complex manifold) A. If A is an algebraic surface, then so are all of its blow-ups ([BPV], p. 128). If you see A as a differentiable manifold, then blowing up a point in A is nothing but taking a connected sum $A \# \bar{\mathbb{P}}_2$, where $\bar{\mathbb{P}}_2$ is $\mathbb{P}_2$ with orientation reversed.

Our first surface will be $\mathbb{P}_2$, blown up in nine points. From the differentiable point of view it does not matter how these points are chosen, so we choose them in a way that suits our purposes, namely in the nine intersection points $x_1, \ldots, x_9$ of two general cubics $K_i(\xi) = 0$, $i = 1,2$. We define X to be the algebraic surface obtained by blowing up $\mathbb{P}_2$ in $x_1, \ldots, x_9$.

Blowing up does not change the fundamental group, so $\pi_1(X) = \pi_1(\mathbb{P}_2) = 1$. Let Q_X be the integral cup form on $H^2(X,\mathbb{Z})$. Since X is diffeomorphic to $\mathbb{P}_2 \# \bar{\mathbb{P}}_2^{(1)} \# \ldots \# \bar{\mathbb{P}}_2^{(9)}$, we see that $rk(Q_X) = 1 + 9 = 10$, whereas the index $\tau(Q_X) = 1 - 9 = -8$. Obviously Q_X is an odd form.

The rational function $\frac{K_1(\xi)}{K_2(\xi)}$ on $\mathbb{P}_2$ yields a holomorphic map $f' : \mathbb{P}_2 \setminus \bigcup_{i=1}^{9} x_i \longrightarrow \mathbb{P}_1$, which, after blowing up, can be extended to

$$f : X \longrightarrow \mathbb{P}_1 .$$

All but a finite number of the fibres of f are smooth elliptic curves ; thus f is an elliptic fibration with base $\mathbb{P}_1$, and X an elliptic surface.

Let $n \in \mathbb{Z}$, $n \geq 2$, and let $\Delta \subset \mathbb{C}(t)$ be the unit disk. Suppose we have a proper, connected surjective holomorphic map

$$g : A \longrightarrow \Delta ,$$

where A is a complex surface. Let us further assume that g is everywhere of maximal rank and that its fibres are elliptic curves. Then there exists another complex surface B and a proper, surjective holomorphic map $h : B \longrightarrow \Delta$, such that $A \setminus g^{-1}(0)$ is isomorphic to $B \setminus h^{-1}(0)$ in a fibre-preserving way, whereas $h^{-1}(0)$, though set-theoretically still a smooth elliptic curve, has multiplicity n (i. e. $f^*(dt)$ vanishes exactly to the order $n-1$ on $h^{-1}(0)$, and $nh^{-1}(0)$ is homologous to a general fibre). Given any elliptic fibration $S \longrightarrow C$ we can construct by this procedure another one $S' \longrightarrow C$, which is isomorphic to the first, but for the fact that a smooth fibre has been replaced by one of multiplicity n. This procedure is called a *logarithmic transformation of order* n along the fibre in question. Contrary to blowing up, a logarithmic transformation on an algebraic surface is an analytic, not an algebraic procedure. It can change an algebraic surface into a non-algebraic one. For details I have to refer to [BPV], p. 164 or [GH], p. 566.

This said, we apply two logarithmic transformations, one of order 2 and one of order 3 to two (different) smooth fibres of f. Let Y be the resulting complex surface, and $g : Y \longrightarrow \mathbb{P}_1$ the elliptic fibration inherited from f. The *differentiable* structure of Y does not depend on the choice of the smooth fibres which are replaced by multiple ones, so as long as we are interested in the differential structure we can speak of *the* surface Y.

A theorem of Dolgacev ([Dg], Ch. II) asserts that $\pi_1(Y) = 1$. (N.B. *In general* the fundamental group changes if a logarithmic transformation is applied). It can also be proved that Y is again algebraic. What about Q_Y ? Clearly we have for the Euler-Poincaré characteristic $e(Y) = e(X) = 12$, so $\operatorname{rank}(Q_Y) = 10$. Furthermore, the canonical bundle formula for elliptic fibrations ([BPV], p. 161) implies in our case :

$$c_1(Y) = f^*(\ldots) + f_2 + 2f_3 ,$$

where $f_i \in H^2(Y,\mathbb{Z})$ is dual to the homology class of the fibre F_i with multiplicity i. So $c_1^2(Y) = 0$, and the index formula of Thom-Hirzebruch ([BPV], p. 18) yields

$$\tau(Y) = \frac{1}{3}(c_1^2(Y) - 2c_2(Y)) = -8 .$$

Finally we observe that Q_Y is odd, because of a theorem of Rohlin ([FU], p. 23), saying in particular that the index of a simply connected smooth 4-fold V is divisible by 16 if Q_V is even.

Hence Q_X and Q_Y have the same rank, index and parity. Both being unimodular (and indefinite), it follows from number theory that they are isomorphic over $\mathbb{Z}$ ([Se], V, §2). In other words, X and Y have the same cohomology ring. Are they homeomorphic ? If so, are they diffeomorphic ?

These problems were first raised by Kodaira around 1965. At the time they could not be answered. The only thing which could be said was that, according to a theorem of J.H.C. Whitehead, X and Y have the same homotopy type. It would take fifteen years before the first question was answered by Freedman (see [Fr]). But his deep and admirable methods, though settling the topological problem in full generality for the simply-connected case, don't give any information about differentiable equivalence. As I have mentioned before, the question is of much interest also from a more general point of view.

2. THE DONALDSON INVARIANT

Let M be a (compact, connected) smooth 4-manifold, oriented, with $\pi_1(M) = 1$ and Q_M equivalent to $\langle +1\rangle \oplus \langle -1\rangle \oplus \ldots \oplus \langle -1\rangle$ *over* $\mathbb{R}$.

If g is any Riemannian metric on M, then its staroperator induces an involution

$$* : H^2(M,\mathbb{R}) \longrightarrow H^2(M,\mathbb{R})$$

with a 1-dimensional space of invariant classes. Identifying $H^2(M,\mathbb{R})$ with the space of harmonic 2-forms, we shall denote the $*$-invariant forms by ω_g. Clearly we have $\omega_g \wedge \omega_g > 0$ for $\omega_g \neq 0$.

We also need the following facts. In $H^2(M,\mathbb{R})$ the *positive cone* is defined by

$$\mathfrak{C} = \mathfrak{C}_M = \{h \in H^2(M,\mathbb{R}), h^2 > 0\} .$$

$\mathfrak{C}_M$ consists of two components. For any $a \in H^2(M,\mathbb{Z})$, $a^2 = -1$ we define the a-wall in $\mathfrak{C}_M$ by

$$W_a = \{h \in \mathfrak{C}_M, ha = 0\} .$$

The connected components of $\mathfrak{C}_M \setminus \cup W_a$ (a as above) are the *chambers* in $\mathfrak{C}_M \setminus \cup W_a$.

Now we are ready to describe the Donaldson invariant ([Do 3], [Do 6]).

THEOREM 2.1.- *Let* M *be a compact, connected oriented smooth* 4-*fold, with* $\pi_1(M) = 1$ *and* Q_M *equivalent to* $\langle +1\rangle \oplus \langle -1\rangle \oplus \ldots \oplus \langle -1\rangle$ *over* $\mathbb{R}$. *Then there exists a uniquely defined map*

$$\rho_M : \mathfrak{C}_M \setminus \cup W_a \longrightarrow H^2(X,\mathbb{Z}) ,$$

satisfying the following conditions :

(i) $\rho_M(-h) = -\rho_M(h)$;

(ii) *if* N *satisfies the same conditions as* M, *and* $f : M \longrightarrow N$ *is an orientation-preserving diffeomorphism, then* $\rho_M(f^*(j)) = f^*(\rho_N(j))$;

(iii) *if* $h_1, h_2 \in \mathfrak{C}_M \setminus \cup W_a$ *are contained in the same component of* $\mathfrak{C}_M$, *then*

$$\rho_M(h_2) = \rho_M(h_1) + 2\,\Sigma\, a_i ,$$

where the sum is taken over all $a_i \in H^2(M,\mathbb{Z})$, $a_i^2 = -1$, *with* $h_1 a_i < 0$ *and* $h_2 a_i > 0$; *in particular* ρ_M *is constant on every chamber ;*

(iv) *if* $h = \omega_g$, *where* g *is a generic Riemannian metric on* M, *then* $\rho_M(\omega_g) = d(g,\omega_g)$.

The first thing to do is to explain what is meant by a "generic metric" and by $d(g,\omega_g)$. Let us for a short moment return to the earliest work of Donaldson in this direction, I mean

THEOREM 2.2.- *If* M *is a (compact, connected) oriented smooth* 4-*fold with* $\pi_1(M) = 1$ *and* Q_M *positive definite, then* Q_M *is equivalent to* $\langle +1\rangle \oplus \dots \oplus \langle +1\rangle$ *over* $\mathbb{Z}$.

Remark.- Donaldson has recently been able to drop the condition $\pi_1(M) = 1$ completely, see Theorem 4.3 below.

The proof of Theorem 2.2 can be found in [Do 1] and is also extensively treated in an important book by Freed and Uhlenbeck ([FU]). Since the construction of Donaldson's new invariant plays on the same stage, I recall the proof in two words.

Let $\mathcal{V}$ be a smooth SU(2)-bundle on M with $c_2(\mathcal{V}) = -1$, and let $\mathcal{A}$ be the affine space of all SU(2)-connections on $\mathcal{V}$. For technical reasons it is important not to take only smooth connections, but L_p^k-connections for some large enough p. Similarly, one has to consider C^k-metrics for some $k \geq 3$. The technical reason is simply that in the last case certain sets, occurring in the proof, have a Banach structure, which makes it possible to apply the implicit function-theorem and the Sard - Smale theorem. I shall take the liberty to ignore this type of technical points.

The group of bundle automorphisms $\mathcal{I}$ of $\mathcal{V}$ (the "group of gauge transformations") operates on $\mathcal{A}$. Let $\mathcal{B} = \mathcal{A}/\mathcal{I}$. A connection is *reducible* if it is the direct sum of connections on two U(1)-bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, with $\mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{V}$. If $2m$ is the number of elements $a \in H^2(M,\mathbb{Z})$ with $a^2 = 1$, then the subset of $\mathcal{B}$ coming from reducible connections consists of m (infinite-dimensional) components $\mathcal{R}_1, \dots, \mathcal{R}_m$, and $\mathcal{B}^* = \mathcal{B} \smallsetminus \coprod_{i=1}^{m} \mathcal{R}_i$ is a Banach manifold.

Each connection $A \in \mathcal{A}$ has a curvature form $F_A \in \Omega^2(\mathrm{ad}(\mathcal{V}))$ (i.e. a 2-form on M with coefficients in the adjoint bundle of $\mathcal{V}$). Any Riemannian metric g on M induces a $*$-operator on $\Omega^2(\mathrm{ad}(\mathcal{V}))$ with $** = \mathrm{id}$. The connection A is called *self-dual* (anti-self-dual) if $*F_A = F_A (*F_A = -F_A)$. Self-dual connections are transformed by $\mathcal{I}$ into self-dual connections and thus we obtain a moduli set $S_g \subset \mathcal{B}$ of (equivalence classes of) self-dual connections. Using the Sard - Smale theorem as well as the index theorem for elliptic complexes, Donaldson shows that, if g is general enough, then $S_g \cap \mathcal{B}^*$ is a 5-dimensional oriented smooth manifold, whereas $S_g \cap \mathcal{R}_i$ consists of one point p_i, $i = 1, \dots, m$. In the neighbourhood of each of these points S_g is modelled after a cone over $\mathbb{P}_2(\mathbb{C})$. Then Donaldson proves that there is a component of S_g, containing, say, $p_1, \dots, p_k$, whose unique end is modelled after $M \times (0,\delta)$. The essential tool here is the Taubes-construction, by which self-dual connections on M are obtained first locally, by perturbing a transplanted self-dual connection from S^4, and then extending it to all of M. In this way an oriented cobordism is obtained between M and the disjoint union of copies of $\mathbb{P}_2$ or $\bar{\mathbb{P}}_2$, together k in number. Finally, application of the following proposition completes the proof.

PROPOSITION 2.3.- *Let* Q *be a positive definite unimodular integral quadratic form of rank* k . *If* $2m$ *is the number of integral solutions to* $Q(x,x) = 1$, *then* $m \leq k$ *with equality if and only if* Q *is equivalent to* $\langle +1\rangle \oplus \ldots \oplus \langle +1\rangle$ *over* $\mathbb{Z}$.

Now we consider a similar situation for our case, i.e. the case where M is as in Theorem 2.1 above. Instead of $\mathfrak{V}$ we take the $SU(2)$-bundle $\mathfrak{W}$ with $c_2(\mathfrak{W}) = 1$. We give $\mathcal{B}$ the same meaning as before, and denote again by $\mathcal{B}^* \subset \mathcal{B}$ the space of irreducible connections. For any metric g on M we write A_g for the moduli space of *anti*-self-dual connections.

PROPOSITION 2.4.- *If* g *is a sufficiently general Riemannian metric, then* A_g *is a 2-dimensional orientable differentiable manifold, contained in* $\mathcal{B}^*$.

N.B. A_g need not be connected, and its components may or may not be compact.

If A_g is compact and can be given some canonical orientation, then we can take the image $h \in H_2(\mathcal{B}^*,\mathbb{Z})$ of the fundamental class of A_g , and try to use it to find an invariant for M . In fact, on $M \times \mathcal{B}^*$ there is a universal $U(2)$-bundle for the connections on $\mathfrak{W}$. More precisely, there is a $U(2)$-bundle $\mathcal{U}$ on $M \times \mathcal{B}^*$ with a partial connection P in the M-direction, such that $\mathcal{U}|M \times b$ is isomorphic to $\mathfrak{W}$ for every $b \in \mathcal{B}^*$ and $P|M \times b$ is a connection in the equivalence class represented by b . So on $M \times \mathcal{B}^*$ we have the Chern class $c_2(\mathcal{U})$, and using the *slant product* (see [Sp], p. 351)

$$H_2(\mathcal{B}^*,\mathbb{Z}) \times H^4(M \times \mathcal{B}^*,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{Z})$$
$$(b,\omega) \longmapsto b \backslash \omega$$

we could try as an invariant $h \backslash c_2(\mathcal{U}) \in H^2(X,\mathbb{Z})$.

Of course, in this primitive form the idea can't possibly work, since A_g is in general not compact. Nevertheless, the basic thought is right and I shall now explain how it can be made to work. What we are going to do is to construct a kind of "compactifying tail", though only homologically and not for A_g , but for $2A_g$.

Let $\mathfrak{T}_M^*$ be the cotangent bundle of M . Then $\overset{2}{\Lambda}\mathfrak{T}_M^* = \overset{2}{\Lambda}_+ \oplus \overset{2}{\Lambda}_-$, where the 3-bundles $\overset{2}{\Lambda}_+$ and $\overset{2}{\Lambda}_-$ correspond to the eigen values $+1$ and -1 of the induced operator $* : \overset{2}{\Lambda}\mathfrak{T}_M^* \longrightarrow \overset{2}{\Lambda}\mathfrak{T}_M^*$. If we consider ω_g as a section of $\overset{2}{\Lambda}_+$, the following proposition becomes plausible.

PROPOSITION 2.5.- *If* g *is sufficiently general, then* $\omega_g (\neq 0)$ *vanishes transversally on a smooth curve* $C_g \subset M$.

N.B. C_g need not be connected, and it may be empty.

We now fix a sufficiently general metric g , and choose $\omega_g \neq 0$. (The choice of ω_g is not important, but its direction is.) Starting from this choice of ω_g , Donaldson provides $\overset{2}{\Lambda}_+$ with a specific orientation, and also associates to this choice of ω_g an orientation of A_g (such that the choice of $-\omega_g$ corresponds to the reverse orientation of A_g). In particular we obtain an orientation for C_g.

Outside of C_g the bundle $\overset{2}{\Lambda}_+$ has a trivial subbundle $\mathcal{L}$ of rank 1. Let $\tilde{e}_g \in H^2(M \setminus C_g;\mathbb{Z})$ be the Euler-Poincaré class of the quotient bundle $\overset{2}{\Lambda}_+/\mathcal{L}$. The Lefschetz dual $\bar{e} \in H_2(M,C_g;\mathbb{Z})$ of $\tilde{e}$ has the property that it maps to *two* times the fundamental class of $H_1(C_g;\mathbb{Z})$. This 2 comes from the following. If you take a tubular neighbourhood of C_g in M, and restrict $\overset{2}{\Lambda}_+/\mathcal{L}$ to its boundary, then this restriction is the tangent bundle to the fibres. But these fibres are 2-spheres with Euler-Poincaré number 2.

There is a map $f : M \times (0,\delta) \longrightarrow \mathcal{B}^*$, mapping $C_g \times (0,\delta)$ diffeomorphically onto an open subset of A_g, such that $A_g \setminus f(C_g \times (0,\delta))$ is compact. In other words, f models the ends of A_g after $C_g \times (0,\delta)$. The map f is a small perturbation of the Taubes map.

Let $0 < \delta_0 < \delta$, let $A = A_g \setminus f(C_g \times (0,\delta_0))$ and let $c \in H_2(A,f(C_g \times \delta_0));\mathbb{Z})$ be the fundamental class. If we set $f_{\delta_0} = f|M\times\delta_0$, then it follows from the exact homology sequence

$$0 \longrightarrow H_2(\mathcal{B}^*,\mathbb{Z}) \longrightarrow H_2(\mathcal{B}^*,f(C_g \times \delta_0);\mathbb{Z}) \longrightarrow H_1(f(C_g \times \delta_0);\mathbb{Z}) \longrightarrow \ldots$$

and the choice of orientations that $2c - f_{\delta_0 *}(\bar{e})$ uniquely determines an element $e \in H_2(\mathcal{B}^*,\mathbb{Z})$. This element is independent of δ_0 and of the way the Taubes map has been perturbed. Of course, e still depends on g and ω_g.

Finally, denoting by $\psi : H_2(\mathcal{B}^*;\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$ the homomorphism

$$b \longmapsto b \setminus c_2(\mathcal{U}) ,$$

we put $d(g,\omega_g) = \psi(e)$.

Starting from $d(g,\omega_g)$ for a fixed general metric g we can define ρ on all chambers in a formal way. Then (i) is a consequence of the fact that if we use $-\omega_g$ instead of ω_g, the associated orientation of A_g becomes the opposite one. Property (ii) follows from the construction, and a moment of reflexion shows that to obtain properties (iii) and (iv) it suffices to prove the following claim.

PROPOSITION 2.6.- *Let* g_0,g_1 *be generic metrics on* M *with corresponding harmonic forms* ω_0,ω_1 *contained in the same component of* $\mathfrak{C}$. *Then*

$$d(g_1,\omega_1) = d(g_0,\omega_0) + 2\,\Sigma\, a_i ,$$

where the sum is taken over all $a_i \in H^2(M;\mathbb{Z})$ *with* $a_i^2 = -1$, $a_i[\omega_0] < 0 < a_i[\omega_1]$, *with* $[\omega_i]$ *denoting the cohomology class of* ω_i.

For the proof I have to refer to [Do 6], p. 17.

If C_g is empty, i.e. if ω_g nowhere vanishes, then A_g is compact, and

$d(g,\omega_g)$ consists of two distinct contributions, namely $2\psi(c)$ and $-\psi(f_{\delta_0 *}$(dual class of $c_1(\mathbb{Q}))$, where $\mathbb{Q}$ is the quotient of $\overset{2}{\Lambda}_+$ by the trivial rank-1 subbundle, given by ω_g . This quotient is an $SO(2)$-bundle, i.e. a $U(1)$-bundle in a natural way. It turns out that $\psi \circ f_{\delta_0 *}$ is nothing but Poincaré duality. If $\mathcal{U}'$ is the restriction of $\mathcal{U}$ to $M \times A_g$, and $\pi : M \times A_g \longrightarrow M$ the projection, it follows that

$$d(g,\omega_g) = 2\pi_*(c_2(\mathcal{U}')) - c_1(\mathbb{Q}) .$$

This situation arises in particular if M is a complex surface V and g a Kähler metric. In this case it is easy to see that $c_1(\mathbb{Q}) = c_1(V)$.

PROPOSITION 2.7.- *Let* V *be a compact, simply-connected complex surface with* $Q_V = \langle +1\rangle \oplus \langle -1\rangle \oplus \dots \oplus \langle -1\rangle$ *over* $\mathbb{R}$. *If* g *is a Kähler metric on* V , *which is sufficiently general as a Riemannian metric, then*

$$d(g,\omega_g) = 2\pi_*(c_2(\mathcal{U}')) - c_1(V) ,$$

where $\mathcal{U}'$ *is the restriction of a universal connection bundle to* $V \times A_g$, *and* $\pi : V \times A_g \longrightarrow V$ *the projection.*

Final remark.- At first sight it might appear that Donaldson's construction is rather arbitrary. However, the construction is only possible if you provide (a multiple of) S_g with a compact tail in some *universal* way, otherwise it is difficult to obtain "deformation invariance". This leaves few choices. Then the dependence on the metric forces you to consider an invariant of Donaldson's type.

3. AN INTERPRETATION OF DONALDSON'S INVARIANT FOR ALGEBRAIC SURFACES AND THE SOLUTION OF OUR PROBLEM

We shall need the concept of a stable vector bundle (in the sense of Mumford and Takemoto), but only for the special case of a rank-2 bundle on a surface. In this case the general definition simplifies to the one below.

If $V \hookrightarrow \mathbb{P}_N$ is an embedded algebraic surface, then we shall call the restriction of the natural generator of $H^2(\mathbb{P}_N,\mathbb{Z})$ the *hyperplane class*. It is of course Poincaré-dual to the homology class of a hyperplane section. Furthermore, by the *Hodge metric* on V we mean the restriction to V of the standard Fubini-Study metric on $\mathbb{P}_N$.

DEFINITION.- *Let* $V \hookrightarrow \mathbb{P}_N$ *be an embedded algebraic surface, and* h *its hyperplane class. A rank-2 algebraic vector bundle* $\mathfrak{V}$ *on* V *is* h-*stable if for every line bundle (rank-1 algebraic vector bundle)* $\mathfrak{M}$ *on* V *which admits a non-trivial homomorphism into* $\mathfrak{V}$, *the inequality* $c_1(\mathfrak{M})h < \frac{1}{2} c_1(\mathfrak{V})h$ *holds.*

Example.- A direct sum of two line bundles is never stable, but the tangent bundle to $\mathbb{P}_2$ is stable with respect to any hyperplane class.

Stable vector bundles have been studied intensively during the last decade, in particular on $\mathbb{P}_3$, where some of them appear as instanton bundles.

One of the main features of stable bundles is that, contrary to vector bundles in general, they have good moduli spaces. Again, we shall only need a very special result, namely the following (see [Ma], esp. p. 598 ff). The h-stable rank-2 vector bundles on an algebraic surface X with fixed Chern classes have a moduli scheme T. Furthermore, if $\chi(X) = \frac{c_1^2(X) + c_2(X)}{12} = 1$, then there exists a universal family on $X \times T$. The condition $\chi(X) = 1$ is satisfied in our case and in the other cases we are going to consider.

We come to the link between anti-self-dual connections and stable bundles. This will allow us to solve the problem of section 1.

THEOREM 3.1 (Donaldson, see [Do 2]).- *Let* $V \hookrightarrow \mathbb{P}_N$ *be an embedded algebraic surface with hyperplane class* h, *and let* $\mathbb{D}$ *be a smooth* $SU(2)$ *rank-2 vector bundle on* V. *Suppose, there are no reducible anti-self-dual connections with respect to the Hodge metric on* V. *Then there is a natural* $1-1$ *correspondence between the equivalence classes of anti-self-dual connections on* $\mathbb{D}$ *(with respect to the Hodge metric on* V *) and the isomorphism classes of* h*-stable rank-2 vector bundles on* V, *differentiably equivalent to* $\mathbb{D}$, *which have algebraically trivial determinant bundle.*

We consider again the case where M is as in Theorem 2.1, and $c_2(\mathbb{D}) = 1$. Let us assume that $h \not\in \cup W_a$, and $\dim H^2(\mathcal{E}nd_0(E)) = 0$ (traceless endomorphisms) for all stable bundle structures E on $\mathbb{D}$. This last condition implies that T is a smooth algebraic curve. Under these circumstances it can be proved that the correspondence of Theorem 3.1 is an orientation-preserving diffeomorphism between A_g and T. Also, a universal bundle on $V \times A_g$ corresponds to a universal bundle on $V \times T$.

We return to the surfaces X and Y from section 1. Let $p : X \longrightarrow \mathbb{P}_2$ be the projection, g' the natural generator of $H^2(\mathbb{P}_2,\mathbb{Z})$ and $g = p^*(g')$. Furthermore, let $e_i \in H^2(X,\mathbb{Z})$ be dual to the exceptional curve $E_i = p^{-1}(x_i)$, $i = 1, \dots, 9$. Then $\{g, e_1, \dots, e_9\}$ is a base for $H^2(X,\mathbb{Z})$ and it is easily verified that $ng - \Sigma_{i=1}^{9} e_i$ is the hyperplane class of a (unique) embedding of X, provided n is large enough. We fix such an n, thus fixing at the same time $h_1 \in H^2(X,\mathbb{Z})$. Obviously $h_1 \in \mathcal{C}_X$.

Next it is shown that

(i) h_1 is not contained in any wall ;

(ii) there are no h_1-stable rank-2 vector bundles $\mathbb{D}$ on X with $c_1(\mathbb{D}) = 0$, $c_2(\mathbb{D}) = 1$.

Let us prove (ii). First a remark. It follows from the Dolbeault isomorphism and the exact exponential cohomology sequence ([BPV], p. 21) that on the surfaces X and Y every continuous line bundle carries exactly one algebraic structure. Now let $\mathcal{V}$ be an algebraic rank-2 vector bundle with $c_1 = 0$, $c_2 = 1$. We set $\dim H^i(X,\mathcal{V}) = h^i(\mathcal{V})$ and denote the canonical line bundle $\overset{2}{\Lambda}\mathcal{T}^*_X$ by K_X. Applying Riemann-Roch and Serre duality yields

$$h^o(\mathcal{V}) + h^o(\mathcal{V}^* \otimes K_X) \geq 1 .$$

But $\mathcal{W}^* \cong \mathcal{W} \otimes (\overset{2}{\Lambda}\mathcal{W}^*)$ for every rank-2 bundle, so $\mathcal{V}^* \cong \mathcal{V}$, since $c_1(\overset{2}{\Lambda}\mathcal{V}^*) = -c_1(\mathcal{V}) = 0$, and we have

$$h^o(\mathcal{V}) + h^o(\mathcal{V} \otimes K_X) \geq 1 .$$

It is easy to see that $h^o(K^*_X) \neq 0$, and we conclude that in any case $h^o(\mathcal{V}) \neq 0$. This implies that $\mathcal{V}$ can't be stable, since $h^o(\mathcal{V}) \neq 0$ means that there is a non-trivial homomorphism from the trivial line bundle into $\mathcal{V}$.

N.B. There are many non-stable bundles $\mathcal{V}$ with $c_1 = 0$, $c_2 = 1$, e.g. $\mathcal{L}_i \oplus \mathcal{L}^*_i$, where $c_1(\mathcal{L}_i) = e_i$.

If we consider X as an oriented smooth manifold, then it follows from Theorem 2.7, Theorem 3.1, (i) and (ii), that

$$\rho_X(h_1) = -c_1(X) = -3g + \sum_{i=1}^{9} e_i .$$

We turn to the surface Y. Again we choose a suitable embedding $Y \hookrightarrow \mathbb{P}_M$, namely one with hyperplane class

$$h_2 = h_2^{(o)} - nc_1(Y) ,$$

where $h_2^{(o)}$ is the hyperplane class of some embedding, and n large. Again it is easy to verify that $h_2 \notin \cup W_a$. However, in this case there *are* h_2-stable rank-2 vector bundles with $c_1 = 0$, $c_2 = 1$. Denoting as before the unique fibre of $g : Y \longrightarrow \mathbb{P}_1$ of multiplicity 2 by F_2, we find

PROPOSITION 3.2.- *The* h_2-*stable rank-2 vector bundles on* Y *with* $c_1 = 0$, $c_2 = 1$ *have a moduli space, which is isomorphic to* F_2.

Sketch of proof.- Let $\mathcal{V}$ be a stable, rank-2 vector bundle on Y with $c_1 = 0$, $c_2 = 1$. Exactly like in the case of X we find that $h^o(\mathcal{V} \otimes K_Y) \geq 1$. Let $s \in H^o(Y,\mathcal{V} \otimes K_Y)$, $s \neq 0$. There is a divisor D on Y, effective or 0, such that $\mathcal{V} \otimes K_Y \otimes \mathcal{D}^*$ has a section, vanishing at a finite number of points. Here $\mathcal{D}$ is the line bundle with $c_1(\mathcal{D})$ dual to the homology class of D. In other words, there is a non-trivial homomorphism $\mathcal{D} \otimes K^*_Y \longrightarrow \mathcal{V}$. Using the canonical bundle formula and the stability of $\mathcal{V}$ it is not difficult to show that $D = 0$. Since $c_2(\mathcal{V} \otimes K_Y) = 1$, we see that s vanishes transversally in exactly one point $p \in Y$. We claim : $p \in F_2$. In fact, let $(\tilde{Y},E)$ be obtained from (Y,p) by blowing up p , and let $g : \tilde{Y} \longrightarrow Y$ be the projection. On $\tilde{Y}$ we have an exact sequence of *bundles* :

$$0 \longrightarrow \mathcal{E} \longrightarrow g^*(\mathcal{D} \otimes K_Y) \longrightarrow \mathcal{E}^* \otimes g^*(K_Y^{\otimes 2}) \longrightarrow 0 ,$$

with $c_1(\mathcal{E}) = E$. So there is an extension of $\mathcal{E}^* \otimes g^*(K_Y^{\otimes 2})$ by $\mathcal{E}$, which, restricted to E , yields the trivial bundle of rank 2 . Now $\deg(\mathcal{E}|E) = -1$, hence $h^1(\mathcal{E}^{\otimes 2}|E) = 1$. It follows that

$$H^1(\mathcal{E}^{\otimes 2} \otimes g^*(K_Y^{\otimes(-2)})) \longrightarrow H^1(\mathcal{E}^{\otimes 2} \otimes g^*(K_Y^{\otimes(-1)}|E))$$

must be surjective. Consequently,

$$H^2(\mathcal{E} \otimes g^*(K_Y^{\otimes(-2)})) \longrightarrow H^2(\mathcal{E}^{\otimes 2} \otimes g^*(K_Y^{\otimes(-2)}))$$

is injective, or, by Serre duality, since $K_{\tilde{Y}} = g^*(K_Y) \otimes \mathcal{E}$, the homomorphism

$$H^0(\mathcal{E}^* \otimes g^*(K_Y^{\otimes 3})) \longrightarrow H^0(g^*(K_Y^{\otimes 3}))$$

must be surjective. On Y this means that every section of $K_Y^{\otimes 3}$ vanishes at p . But the canonical bundle formula readily yields $c_1(K_X^{\otimes 3}) = f_2$, hence $p \in F_2$. Thus we associate to s a point of F_2 . Next it is shown that $h^0(\mathcal{D} \otimes K_Y) = 1$, so we can associate to $\mathcal{D}$ a *unique* point of F_2 . The proof is completed by reversing the above procedure.

A universal bundle can be constructed on $X \times F_2$, and its Chern class calculated. Using again Theorem 3.1, Theorem 2.7 and also the canonical bundle formula for elliptic fibrations, we find :

$$\rho_Y(h_2) = -7\, c_1(Y) .$$

Suppose, there was a diffeomorphism $\sigma : X \longrightarrow Y$. Then, in general, $\sigma^*(\rho_Y(h_2))$ and $\rho_X(h_1)$ would be in different chambers, and there is little we can conclude. Concerning X , however, there is a theorem of Wall, saying that given any two chambers in $\mathcal{C}_X \setminus \cup W_a$, there is an orientation-preserving diffeomorphism from X onto itself, carrying the first chamber into $\pm$ the second one ([Wa]). Thus we may assume that h_1 and $\sigma^*(h_2)$ are in the same or opposite chambers. By properties (ii) and (i) of Donaldson's invariant, we have $\rho_X(h_1) = \pm\sigma^*(\rho_Y(h_2))$. But the first class is primitive, whereas the second is not. We conclude that X and Y are not diffeomorphic.

4. FURTHER DEVELOPMENTS

If, more generally, we replace on X two smooth fibres by fibres of any multiplicity p and q , with $p < q$ and G.C.D.$(p,q) = 1$, then Dolgacev's result is still valid, and exactly as before it follows that the surface $X_{p,q}$ thus obtained is homeomorphic to X . Are all of these surfaces mutually distinct from the differentiable point of view ? This question has not yet been completely answered, since for $p \neq 2$ the moduli space of stable bundles (with respect to a suitable embedding) is *not reduced*. It is possible to determine the (smooth) reduction, but not the multiplicities of the different components. Even if these are known, Donaldson's method can't be applied without modification. However, the moduli space *is* reduced and smooth if $p = 2$. In fact, if h_2 is as before, we have

PROPOSITION 4.1.- *The moduli space of h_2-stable rank-2 vector bundles on $X_{2,q}$ with $c_1 = 0$, $c_2 = 1$ is smooth and consists of $\frac{q-1}{2}$ copies of F_2.*

This result was proved independently by Friedman and Morgan ([FM]) on the one hand and Okonek and myself on the other ([OV]). Once you have this proposition, you can find $\rho_{X_{p,q}}(h_2)$ exactly as in the case of $Y = X_{2,3}$. It follows that none of the $X_{2,q}$ is diffeomorphic to X. However it can *not* be concluded in the same way as in the case of X and Y that $X_{2,q}$ and $X_{2,q'}$ with $q \neq q'$ are distinct as smooth 4-folds, for Wall's theorem does not apply to these surfaces. Friedman and Morgan overcome this difficulty by studying extensively the group of diffeomorphisms from a surface $X_{p,q}$ onto itself. Their method (yielding the *first* proof of Theorem 4.2 below) is no doubt the most beautiful and most general one, but if you want only a proof for the fact that $X_{2,q}$ and $X_{2,q'}$ are not diffeomorphic for $q \neq q'$, I can recommend our proof. In fact we observe that, as a consequence of Proposition 4.1 and property (iii) of the Donaldson invariant, given any diffeomorphism between $X_{2,q}$ and $X_{2,q'}$, the element $h_2^{(0)} - nc_1(X_{2,q})$ and the pull-back of the corresponding element on $X_{2,q'}$ are in the same chamber, provided n is large enough. After this we can proceed as in the case of X and Y, though the divisibility argument becomes slightly more complicated.

THEOREM 4.2.- *The surfaces $X_{2,q}$ and $X_{2,q'}$ are diffeomorphic if and only if $q = q'$.*

So there are infinitely many differentiable structures on the topological 4-fold $X_{top} = \mathbb{P}_2 \# \bar{\mathbb{P}}_2^{(1)} \# \dots \# \bar{\mathbb{P}}_2^{(9)}$. In all dimensions but 4 there is only a *finite* number of smooth manifolds with given homotopy type and Pontrjagin classes. Friedman and Morgan show that $X_{2,q}$ and $X_{2,q'}$, both blown up in k points, are not diffeomorphic, so we also obtain an infinity of differentiable structures on $\mathbb{P}_2 \# \bar{\mathbb{P}}_2^{(1)} \# \dots \# \bar{\mathbb{P}}_2^{(k)}$, $k \geq 10$.

It is expected that the remaining $X_{p,q}$ will yield still more differentiable structures on X_{top}. And there are yet other algebraic surfaces, homeomorphic to X, namely the Barlow surfaces, blown up in one point ([Bl]). It does not seem to be easy to describe the stable rank-2 vector bundles with $c_1 = 0$, $c_2 = 1$ on these surfaces.

To the best of my knowledge there is not a single example of two algebraic surfaces, which are diffeomorphic, but have different Kodaira dimension.

To finish I would like to mention some of Donaldson's most recent results.

First of all, he has extended Theorem 2.1 to the case of arbitrary fundamental group.

THEOREM 4.3.- *If* M *is a (compact, connected) oriented differentiable 4-fold with* Q_M *positive definite, then* $Q_M \cong \langle +1 \rangle \oplus \ldots \oplus \langle +1 \rangle$ *over* $\mathbb{Z}$.

Donaldson has also shown

THEOREM 4.4.- *If* M *is a (compact, connected) oriented differentiable 4-fold with no 2-torsion in* $H^1(M,Z)$ *, with even* Q_M *, and* Q_M *equivalent over* $\mathbb{R}$ *to either* $\langle +1 \rangle \oplus \langle -1 \rangle \oplus \ldots \oplus \langle -1 \rangle$ *or* $\langle +1 \rangle \oplus \langle +1 \rangle \oplus \langle -1 \rangle \oplus \ldots \oplus \langle -1 \rangle$ *, then* Q_M *is equivalent to* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *over* $\mathbb{Z}$.

The proof of the preceding two results can be found in [Do 4]. And finally there is

THEOREM 4.5.- *Let* X *be a simply connected algebraic surface. If* X *is diffeomorphic to a connected sum* $M_1 \# M_2$ *, then either* Q_{M_1} *or* Q_{M_2} *is equivalent to* $\langle -1 \rangle \oplus \ldots \oplus \langle -1 \rangle$ *over* $\mathbb{Z}$.

Even the case of quintics in $\mathbb{P}_3$ wasn't known before, though it has been considered for a long time.

Combining Theorem 4.5 with the fact that a complex surface is a deformation of an algebraic one if and only if the first Betti number is even (see [Mi]), we obtain the solution of another old problem :

COROLLARY 4.6.- *The differentiable 4-fold*

$$X^{(k,\ell)} = \mathbb{P}_2^{(1)} \# \ldots \# \mathbb{P}_2^{(k)} \# \bar{\mathbb{P}}_2^{(1)} \# \ldots \# \bar{\mathbb{P}}_2^{(\ell)}$$

carries a complex structure if and only if $k = 1$.

REFERENCES

[Bl] R. BARLOW - *A simply connected surface of general type with* $p_g = 0$, Inv. Math. 79, 293-301 (1985).

[BPV] W. BARTH, C. PETERS, A. VAN DE VEN - *Compact complex surfaces*, Erg. Math. (3) 4, Springer (1984).

[Dg] I. DOLGACEV - *Algebraic surfaces with* $p_g = q = 0$, in Algebraic Surfaces, C.I.M.E., 97-115 (1977).

[Do 1] S.K. DONALDSON - *An application of gauge theory to the topology of 4-manifolds*, J. Diff. Geom. 18, 279-315 (1983).

[Do 2] S.K. DONALDSON - *Anti-self-dual Yang - Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. Lond. Math. Soc. 50, 1-26 (1985).

[Do 3] S.K. DONALDSON - *La topologie différentielle des surfaces complexes*, C.R. Acad. Sc. Paris (1), 6, 317-320 (1985).

[Do 4] S.K. DONALDSON - *Connections, cohomology and the intersection form of 4-manifolds*, Preprint Oxford (1985).

[Do 5] S.K. DONALDSON - *The orientation of Yang - Mills moduli spaces and 4-manifold topology*, Preprint Oxford (1986).

[Do 6] S.K. DONALDSON - *Irrationality and the* h-*cobordism conjecture*, Preprint Oxford (1986).

[FU] D.S. FREED, K.K. UHLENBECK - *Instantons and four-manifolds*, Math. Sciences Research Institute Publications, v. 1, Springer (1984).

[Fr] M. FREEDMAN - *The topology of four-dimensional manifolds*, J. Diff. Geom., 17, 357-454 (1982).

[FM] R. FRIEDMAN, J. MORGAN - *On the diffeomorphism types of certain algebraic surfaces*, Preprint Columbia U., New York (1986).

[GH] Ph. GRIFFITHS, J. HARRIS - *Principles of algebraic geometry*, J. Wiley and Sons, New York (1978).

[Ma] M. MARUYAMA - *Moduli of stable sheaves II*, J. Math. Kyoto University 18, 557-614 (1978).

[Mi] Y. MIYAOKA - *Kähler metrics on elliptic surfaces*, Proc. Japan Acad. 50, 533-536 (1974).

[Mo] E. MOISE - *Affine Structures in 3-manifolds I - V*, Annals of Math. 54 (1951), 55 (1952).

[OV] C. OKONEK, A. VAN DE VEN - *Stable vector bundles and differentiable structures on certain elliptic surfaces*, to appear in Inv. Math.

[Se] J.-P. SERRE - *Cours d'arithmétique*, Presses Univ. de France, Paris (1970).

[Sp] E.H. SPANIER - *Algebraic Topology*, McGraw-Hill (1966).

[Su] D. SULLIVAN - *Infinitesimal computations in topology*, Publ. Math. I.H.E.S. 47, 267-332 (1977).

[Wa] C.T.C. WALL - *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. 39, 131-140 (1964).

A. VAN DE VEN
University of Leiden
Department of Mathematics
Postbus 9512
NL-2300 RA LEIDEN