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ENTROPY, HOMOLOGY AND SEMIALGEBRAIC GEOMETRY
[after Y. Yomdin]

by M. GROMOV

1. COMPUTATIONAL DEFINITION OF TOPOLOGICAL ENTROPY

1.1. The *entropy* of a partition Π of a set X into N subset is defined by

$$\text{ent}\,\Pi = \log N \quad .$$

The *intersection* of two partition say $\Pi_1 \cap \Pi_2$, is the partition of X into the pairwise intersections of the elements of Π_1 and Π_2.

For a map $g : Y \to X$ one obviously defines the pull-back partition of Y denoted Π_g for every partition Π of X. If f is a self mapping $X \to X$ one consideres the pull-backs of Π under the iterates $f^1 = f$, $f^2 = f \circ f \ldots f^i = f \circ f^{i-1}$ and set

$$\Pi^i = \Pi \cap \Pi_f \ \ldots \cap \Pi_{f^i}$$

and

$$\text{ent}(\Pi;f,i) = i^{-1}\,\text{ent}\ \Pi^i \ .$$

Similarly, if Y is mapped into X by g one defines

$$\text{ent}(\Pi|Y;f,i) = i^{-1}\,\text{ent}(\Pi^i)_g \quad .$$

1.2. Let X be a cubical polyhedron, that is a topological space divided into cubes $\square$, such that every two cubes meet at a common face. Denote by Π the partition of X into the open (i.e. taken without boundary but not necessarily open as subsets in X) cubes of the polyhedron X and let $\Pi(j)$ be the refinement of Π obtained by dividing every $\square$ into $j^{\dim\square}$ equal subcubes. Now define the *topological entropy* ent f of a map $f : X \to X$ as the lower bound of the numbers $h \geq 0$ with the following property :

(P) *There exists an arbitrarily large integer* $k \geq 0$ *(depending on* h*) such that*

$$\limsup_{i\to\infty} \text{ent}(\Pi(j)\ ;\ f^k,i) \leq hk$$

for all $j = 1,2,\ldots$.

In the same way one defines ent $f|Y$ for every space Y mapped into X.

This definition is justified by the following easy theorem.

1.3. Topological invariance of the entropy. *If X is compact and f is continuous then* ent f *does not depend on a choice of the (cubical) polyhedral structure on X. The same applies to* ent $f|Y$ *for compact spaces Y continuously mapped into X. Moreover, if X is finite dimensional and $Y \subset X$ is a compact subset invariant under f then* ent $f|Y$ *only depends on Y and $f : Y \to Y$ (but not an embedding $Y \to X$), provided the map f is continuous on Y.*

1.4. *Remark.* Consider *the standard partition* Π_{st} of $\mathbb{R}^n$ into unit cubes which are the faces of the integer translates of the cube $\{0 \leq x_i \leq 1 , i = 1...n\} \subset \mathbb{R}^n$. The entropy defined with this Π_{st} is not topologically invariant over all $\mathbb{R}^n$. Yet it is invariant on every compact subset Y, such that f is continuous on Y and $f(Y) \subset Y$. Thus one obtains an invariant entropy for a continuous selfmaps of an arbitrary finite dimensional compact space Y, since Y embeds into some $\mathbb{R}^n$.

1.4. *Examples.* (A) Take a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ and define *the spectral radius*

$$\mathrm{Rad}\, f = \lim \|f^i\|^{1/i}$$

for

$$\|f\| = \sup_{\|x\|=1} \|f(x)\| .$$

Let $\Lambda_* f = \Lambda_0 f \oplus \Lambda_1 f \oplus \ldots \oplus \Lambda_n f$ be the full exterior power of f. Then by an easy argument, the entropy (for the standard cubical partition of $\mathbb{R}^n$) satisfies,

$$\mathrm{ent}\, f|Y = \log \mathrm{Rad}\, \Lambda_* f$$

for every non-empty open bounded subset Y in $\mathbb{R}^n$.

(B) Let f be an endomorphism of the torus $T^n = \mathbb{R}/\mathbb{Z}^n$. It is easy to see that

$$\mathrm{ent}\, f = \mathrm{ent}\, \tilde{f}|Y$$

for the covering linear map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ and for every non-empty bounded open subset $Y \subset \mathbb{R}^n$. It follows with (A) that

$$\mathrm{ent}\, f = \log \mathrm{Rad}\, f_*$$

for the *induced endomorphism* f_* on the real homology $H_*(T^n)$.

(C) Every *holomorphic* map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has

$$\mathrm{ent}\, f = \log \mathrm{Rad}\, f_* . \qquad (*)$$

Furthermore, ent $f|Y$ = ent f for every subset $Y \subset \mathbb{C}P^n$ whose complement is non-dense and invariant under f. For example, if f on $\mathbb{C}P^1$ is given by a polyno-

mial f_o on $\mathbb{C}^1 \subset \mathbb{CP}^1$ of degree $d > 0$, then $\text{ent } f|Y = \log d$ for $Y = \{|z| \leq r\} \subset \mathbb{C}$, provided $|f(z)| \geq r$ for $z \geq r$.

Notice that $\text{Rad } f_*$ equals the topological degree $\deg f$ for every continuous selfmap f of $\mathbb{CP}^n$ with $\deg f > 0$.

The proof of (*) consists of showing that

(C1) $$\text{ent } f \geq \log \deg f$$

and

(C2) $$\text{ent } f \leq \log^+ \deg f ,$$

where $\log^+ t = \max(0, \log t)$, which takes care of $\deg = 0$.

The first inequality is an immediate corollary of the following theorem by Misiurewicz and Przyticki (see $[M\text{-}P]_1$).

1.5. Theorem. *Let* f *be a* C^1*-smooth self-mapping of a compact manifold* X, *such that the pull back* $f^{-1}(x)$ *contains at least* d *point for all* x *in a subset of full measure in* X. *Then* $\text{ent } f \geq \log d$.

The second inequality (C2) follows from the (obvious) bound

$$\text{Vol } \Gamma_{f^i} \leq \text{const } d^i$$

for the 2n-dimensional volumes of the graphs $\Gamma_{f^i} \subset \mathbb{CP}^n \times \mathbb{CP}^n$ of the iterates of f. (See 2.4.)

1.6. <u>Elementary properties of the entropy</u>.

The following list of facts (whose proofs are straightforward) gives some idea on the dynamical significance of the entropy.

(i) For any two subsets in X,

$$\text{ent } f|Y_1 \cup Y_2 = \max_{i=1,2} \text{ent } f|Y_i .$$

(ii) If $Y_1 \subset Y_2$ then $\text{ent } f|Y_1 \leq \text{ent } f|Y_2$.

(iii) Take two continuous selfmappings of compact spaces, say $f_i : X_i \to X_i$ for $i = 1,2$ and let $F : X_1 \to X_2$ be a continuous map commuting with f_i. If F is onto, then $\text{ent } f_1 \geq \text{ent } f_2$. If F is finite-to-one then, $\text{ent } f_1 \leq \text{ent } f_2$.

(iv) Suppose a continuous map $f : X \to X$ fixes a closed subset $X_o \subset X$ and *wanders* on the complement $\Omega = X \setminus X_o$. That is each point $x \in \Omega$ admits a neighborhood U such that $f^i(U)$ does not meet U for all sufficiently large i. Then $\text{ent } f = 0$, provided X is compact.

Examples. (a) Let f be a linear selfmapping of $\mathbb{R}^2$ with two real eigenvalues

$\neq \pm 1$. Such an f wonders outside the origin but $\operatorname{ent} f|Y$ may be positive on bounded subsets Y in $\mathbb{R}^2$ (see 1.4.A.). Next we extend f to a projective self-mapping $\bar{f}$ of the projective plane $P^2 \supset \mathbb{R}^2$. This $\bar{f}$ fixes, besides the origin in $\mathbb{R}^2$, two points on the projective line $P^1 = P^2 \setminus \mathbb{R}^2$ corresponding to the two eigenspaces (if the eigenvalues are equal $\bar{f}$ fixes P^1) and again $\bar{f}$ wanders outside the fixed point set. Since P^2 is compact, $\operatorname{ent} \bar{f} = 0$ by (iv) (compare (C.2) above. (Notice that $\operatorname{ent} f|Y \neq \operatorname{ent} \bar{f}|Y$ for $Y \subset \mathbb{R}^2 \subset P^2$ as the entropy in $\mathbb{R}^2$ defined with the standard cubical partition of $\mathbb{R}^2$ does depend on the partition and is not topologically invariant).

(b) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in the polar coordinates by $f : (\rho,\theta) \to (2\rho, d\theta)$ for some $\lambda > 1$ and an integer d . This f obviously extends to a continuous selfmap $\bar{f}$ of the one-point compactification of $\mathbb{R}^2$, that is $S^2 \supset \mathbb{R}^2$. The map $\bar{f}$ wanders outside the two (obvious) fixed points. Thus $\operatorname{ent} \bar{f} = 0$ and $\bar{f}$ violates the inequality $\operatorname{ent} \geq \log|\deg|$ for $|d| \geq 2$ (here $\deg \bar{f} = d$) as well as Theorem 1.5. This is due to the non-smoothness of f at the origin $0 \in \mathbb{R}^2$.

2. ENTROPY AND THE VOLUME GROWTH

2.1. Let X be a smooth Riemannian manifold (e.g. a submanifold in $\mathbb{R}^n$) and $f : X \to X$ a C^1-smooth maps. Take an ℓ-dimensional submanifold $Y \subset X$ and define

$$\operatorname{logvol}(f|Y) = \limsup_{i\to\infty} i^{-1} \log \operatorname{Vol}(\Gamma_{f^i}|Y)$$

where $\Gamma_{f^i}|Y \subset Y \times X$ stands for the graph of the i-th iterate of f on Y and Vol denotes the ℓ-dimensional Riemannian volume.

Notice that logvol can be bounded by the norm of the differential $Df : T(X) \to T(X)$,

$$\operatorname{logvol}(f|Y) \leq \log^+ ||Df||^{\ell} ,$$

where

$$||Df|| \overset{\text{def}}{=} \sup_x ||Df \mid T_x(X)|| .$$

The same estimate (obviously) holds with $\operatorname{Rad} Df$ instead of $||Df||$, where

$$\operatorname{Rad} Df \overset{\text{def}}{=} \limsup_{i\to\infty} ||Df^i||^{1/i} .$$

Observe that $\operatorname{Rad} Df \leq ||Df||$ and that $\operatorname{Rad} Df$ (unlike $||Df||$) does not depend on a choice of the Riemannian metric on X , provided X is compact.

2.2. YOMDIN THEOREM. *Let f be a C^r-smooth self-map of a compact C^∞-manifold X and let $Y \subset X$ be a compact C^r-submanifold. Then*

$$\log\mathrm{vol}(f|Y) \le \mathrm{ent}(f|Y) + \log^+(\mathrm{Rad}\, Df)^{\ell/r} . \qquad (*)$$

In particular, if f *and* Y *are* C^∞ *, then*

$$\log\mathrm{vol}(f|Y) \le \mathrm{ent}(f|Y) \le \mathrm{ent}\, f . \qquad (**)$$

2.3. COROLLAIRE. (*Solution of Shub entropy conjecture for* C^∞*-maps*). *If* f *is* C^∞*-smooth then*

$$\mathrm{ent}\, f \ge \log \mathrm{Rad}\, f_* \qquad (***)$$

for the spectral radius $\mathrm{Rad}\, f_*$ *of the induced endomorphism on the real homology,* $f_* : H_*(X) \to H_*(X)$.

PROOF. Consider pairs of closed forms w_1 and w_2 on $X \times X$ with $\deg w_1 + \deg w_2 = \dim X$ and observe that

$$\mathrm{Rad}\, f_* = \mathrm{Rad} f^* = \sup_{w_1, w_2} \limsup_{i\to\infty} \left| \int_{\Gamma_{f^i}} w_1 \wedge w_2 \right|^{1/i} \le \limsup_{i\to\infty} (\mathrm{vol}\, \Gamma_{f^i})^{1/i} .$$

Remark. The spectral radius of f_* on H_ℓ is obviously bounded by the volume growth of the ℓ-simplices of fixed triangulation of V under the iterates of f .

2.3.A. *Example.* If f wanders outside the fixed point set of f (see 1.6. (iv)) then every eigenvalue λ of f_* on $H_*(x)$ satisfies $|\lambda| \le 1$.

2.4. <u>An upper bound for the entropy</u>

Several months prior to Yomdin's result, Sheldon Newhouse [N] found the following converse to (**) for C^2-selfmaps of compact manifolds,

$$\mathrm{ent}\, f \le \sup_Y \log\mathrm{vol}(f\, Y) \qquad (****)$$

over all compact C^∞-submanifolds $Y \subset X$. A similar inequality for diffeomorphisms was proven earlier by Felix Przyticki [P] .

2.5. <u>Semicontinuity of the entropy</u>

Using (****) and his main lemma (see 3.4) Yomdin shows that

$$\limsup_{\tau\to 0} \mathrm{ent}\, f_\tau \le \mathrm{ent}\, f_0$$

for every C^∞-continuous in $\tau \in [0,1]$ family of C^∞-maps $f_\tau : X \to X$ of a compact manifold X .

<u>Example of non-continuous entropy</u>

Map the unit disk in $\mathbb{C}$ into itself by $f_\tau : z \to (1-\tau)z^2$ for $\tau \in [0,1]$. Then $\mathrm{ent}\, f_0 = \log 2$ (see 1.4.C.) and $\mathrm{ent}\, f_\tau = 0$ for $\tau > 0$ as f_τ wanders outside the center of the disk for $\tau > 0$.

2.6. Yomdin's inequality (*) is sharp. To see this, let $Y \subset \mathbb{R}^2$ be the graph of the function $Y = x^{r+\varepsilon} \sin x^{-1}$ for $x \in [0,1]$ which is C^r-smooth for all r and $\varepsilon > 0$. Take the projective map f on $P^2 \supset \mathbb{R}^2$ given by the linear map $(x,y) \to (\frac{1}{2}x, 2y)$ of $\mathbb{R}^2$. Then the length of $f^i(Y)$ is about $2^{\frac{i}{r+\varepsilon}} = (\mathrm{Rad}\, Df)^{\frac{i}{r+\varepsilon}}$, while $\mathrm{ent}\, f = 0$. This makes $(*)$ sharp for $\varepsilon \to 0$. If one insists on a C^∞- smooth Y and a C^r-smooth f then one just appropriately changes the smooth structure on P^2.

2.7. Several historical remarks

The relation between entropy and topology was discovered by Dinaburg [D] who observed that the time one map f^1 of the geodedic flow of a compact Riemannian manifold V has $\mathrm{ent}\, f^1 > 0$ if the fundamental group $\pi_1(V)$ has exponential growth. This is seen by looking at the universal covering of V and applying the following simple fact (compare 1.4.B) to the associated covering of the tangent bundle of V,

(A) *Let* $\tilde{X} \to X$ *be a Galois covering of a finite (cubical) complex* X *and let a continuous map* $f : X \to X$ *lift to a continuous map* $\tilde{f} : \tilde{X} \to \tilde{X}$. *If a compact subset* $Y \subset \tilde{X}$ *projects onto* X, *then*

$$\mathrm{ent}\, \tilde{f}|Y = \mathrm{ent}\, f ,$$

where one computes $\mathrm{ent}\, \tilde{f}$ *for the induced cubical structure on* $\tilde{X}$.

Notice that Yomdin's inequality (**) also yields $\mathrm{ent}\, f^1 > 0$ for C^∞-*smooth* V (Dinaburg's proof only needs the continuity of the geodesic flow). In fact, the inequality $\mathrm{ent}\, f^1 > 0$ follows from (**) for all C^∞-smooth V, where every two generic point, are joined by at least C^λ geodesic segments of length $\leq \lambda$ for all $\lambda \geq 1$ and some $C > 1$. This lower bound on the number of geodesic segments is satisfied for example, by those *simply connected* manifolds V for which the Betti numbers b_i of the loop space of V grow exponentially in $i = 1,2,\ldots$ (see [G]).

(B) Manning [Ma] proved that the spectral radius of $f_* : H_1(X) \to H_1(X)$ provides the (lower) bound

$$\mathrm{ent}\, f \geq \log \mathrm{Rad}\, f_*|H_1(X)$$

for every *continuous* map f of a compact polyhedron X (to see this apply (A) to the maximal Abelian cover $X \to X$) and Misiurewicz and Przyticki [M-P$_2$] sharpened this inequality for X homotopy equivalent to the n-torus,

$$\mathrm{ent}\, f \geq \log \mathrm{Rad}\, f_* = \log \mathrm{Rad}\, \Lambda_* f_*|H_1(X) .$$

(C) Shub conjectured that $\text{ent } f \geq \log \text{Rad } f_*$ is satisfied by C^1-maps on all manifolds (see (b) in 1.6. for a C^0-counterexample). Now, this conjecture is settled (besides tori) for C^1-maps of the spheres S^n (by 1.5.) and for C^∞-maps on all X by Yomdin's (***) .

3. REDUCTION OF YOMDIN THEOREM TO AN ALGEBRAIC LEMMA

3.1. C^r-size of a submanifold

Fix an integer $\ell = 1,2,..$ and define the C^r-*size* of a subset $Y \subset \mathbb{R}^n$ as the lower bound of the numbers $s \geq 0$ for which there exists a C^r-map of the unit ℓ-cube into $\mathbb{R}^n$, say $h : [0,1]^\ell \to \mathbb{R}^n$, whose image contains Y and such that $\|D_r h\| \leq s$. Here $D_r h$ is the vector assembled of (the components of) the partial derivatives of h of orders $1,2,\ldots,r$ and the norm refers to the supremum over $x \in [0,1]^\ell$,

$$\|D_r h\| = \sup_x \|D_r h(x)\| \quad .$$

Remark. We could use instead of $[0,1]^\ell$ another standard ℓ-dimensional manifold (e.g. the unit ball in $\mathbb{R}^\ell$ or the sphere S^ℓ) which would give us an essentially equivalent notion of C^r-size.

3.2. It is obvious that the C^r-size is monotone increasing in r and in $Y \subset \mathbb{R}^n$ and that the C^1-size bounds the diameter and the ℓ-dimensional volume (i.e. the Hausdorff measure) of Y by

$$C^1\text{-size}(Y) \geq \max((\text{Vol } Y)^{1/\ell} \ , \ \ell^{-1/2} \text{ Diam } Y) \ .$$

In fact, if ℓ and r equal one and Y is a smooth ark in $\mathbb{R}^n$, then the C^r-size of Y equals the length of Y . The C^2-size of such a Y measures, in a way, the total curvature of Y but the precise geometric meaning of the C^r-size for $\max(\ell,r) \geq 2$ is rather obscure.

If a subset $Y \subset \mathbb{R}^n$ has C^r-size ≤ 1 and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^r-map , then by the chain rule the image Y' of f has

$$C^r\text{-size } Y' \leq \text{const } \|D_r f\| \ , \tag{1}$$

for some universal constant depending on r, ,m and n . In fact, (1) remains valid if f is defined on a neighbourhood $U \supset Y$ in $\mathbb{R}^n$ which contains the image of the implied map $h : [0,1]^\ell \to \mathbb{R}^n$. If $C^r\text{-size}(Y) \leq \varepsilon\ell^{-1/2}$, then the ε-neigbourhood U_ε of Y will do.

Every $Y \subset \mathbb{R}^n$ of C^r-size $\leq S$ can be subdivided into j^ℓ subsets of C^r-size $\leq S/j$ for all $j = 1,2\ldots$. This is done by dividing $[0,1]^\ell$ into

j^ℓ-cubes $[0,j^{-1}]^\ell$ and using the obvious scaling map $[0,1]^\ell \to [0,j^{-1}]^\ell$ for each cube of this subdivision.

3.3. ALGEBRAIC LEMMA. *Let* $Y \subset [0,1]^n \subset \mathbb{R}^n$ *be the zero set of (a system of some) polynomials* $p_1,\dots,p_k$ *on* $[0,1]^n$ *, such that* $\dim Y = \ell$ *. For each* $r = 1,2,\dots$ *there exists an integer* N_o *which only depends on* n,r *and* $\deg Y \overset{\text{def}}{=} \sum_{i=1}^{k} \deg p_i$ *, and* C^r*-maps* $h_\nu : [0,1]^\ell \to Y$ *for* $\nu = 1,\dots,N_o$ *, whose images cover all of* Y *and* $\|D_r h_\nu\| \le 1$ *for* $\nu = 1,\dots,N_o$ *.* Furthermore,

(i) *each* h_ν *is algebraic of degree* $\le d'$ *for same* d' *depending only on* r *,* $\deg Y$ *and* n *(i.e. the graph of* h *in* $[0,1]^\ell \times \mathbb{R}^n$ *is given by some polynomials of total degree* $\le d'$*) ;*

(ii) *each* h_ν *is a real analytic diffeomorphism of the interior of* $[0,1]^\ell$ *onto its image and these images only meet at the boundaries of the cubes. That is, if* $h_\nu(x) = h_{\nu'}(y)$ *, then* x *and* y *lie in the boundary of* $[0,1]^\ell$ *for all* ν *and* $\nu' = 1,\dots,N_o$ *.*

The proof is given in 4. To get some insight the reader may look at the hyperbola $xy = \varepsilon$ in the square $\{0 \le x \le 1 \ , \ 0 \le y \le 1\} \subset \mathbb{R}^2$ for small positive ε , say $\varepsilon = 0.0001$ and find h_ν for $r = 2$ and $N = 6$.

3.4. MAIN LEMMA. *Let* Y *be an arbitrary subset in the graph* $\Gamma_g \subset \mathbb{R}^{\ell+m} \supset [0,1]^\ell \times \mathbb{R}^m$ *of a* C^r*-map* $g : [0,1]^\ell \to \mathbb{R}^m$ *and take some positive number* $\varepsilon \le 1$ *. Then* Y *can be subdivided into* $N \le C\varepsilon^{-\ell}(1+\|\partial_r g\|)^{\ell/r}$ *subsets of* C^r*-size* $\le C\varepsilon \operatorname{Diam} Y$ *, where* $\partial_r g$ *denotes the vector assembled of the partial derivatives of* g *of order* r *and where* $C = C(\ell,m,r)$ *is a universal constant.*

PROOF. With a change $g(x) \longrightarrow ag(\lambda x)+b$ we can make $Y \subset [0,1]^\ell \times [1/3,2/3]^m$ and we also can assume $\operatorname{Diam} Y = 1$. Then, using subdivisions of subsets of C^r-size ≤ 1 to j^ℓ pieces of C^r-size $\le j^{-1}$, we reduce further to the case, where $\varepsilon = 1$. Now, fix a small $\delta > 0$, say $\delta = (m+\ell+r)^{-(m+\ell+r)}$ and let k be the first integer $\ge \delta^{-1}\|\partial_r g\|^{1/r}$. Then cover $[0,1]^\ell$ by k^ℓ images of affine maps $\lambda_\nu : [0,1]^\ell \to [0,1]^\ell$ of the form $\lambda_\nu(x) = k^{-1}x + a_\nu$ for $\nu = 1,\dots,k$. The composed maps $\lambda_\nu \circ g : [0,1]^\ell \to \mathbb{R}$ have $\|\partial_r(\lambda_\nu \circ g)\| \le k^r \|\partial_r g\|$. Using this we reduce the lemma to the case where , $\|\partial_r g\| \le \delta^r$. (Notice that exactly at this stage we gain a lot for large r) .

Now, we invoke the Taylor polynomial of g of degree $r-1$ at some point $x_o \in [0,1]$. That is a polynomial map $p : [0,1]^\ell \to \mathbb{R}^m$ of degree (of each component of p) $r-1$ which satisfies, for $\|\partial_r g\| \le \delta^r$ and small δ , by Taylor remainder theorem,

$$\|\partial_i(p-g)\| \leq 1/3 \quad \text{for} \quad i = 0,1,\dots,r \ .$$

Then we apply Algebraic Lemma to the part Y_o of te graph of p lying in the unit cube $[0,1]^{\ell+m}$ and get N_o maps $h_\nu : [0,1]^\ell \to [0,1]^\ell \times [0,1]^m$ with $\|D_r h_\nu\| \leq 1$ which cover Y_o. Denote by $\bar{h}_\nu$ and $\tilde{h}_\nu$ the $[0,1]^\ell$ - and $[0,1]^m$-components of h_ν correspondingly and observe that $\tilde{h}_\nu = p \circ \bar{h}$ for $\mathrm{Im} h_\nu \subset \Gamma_p$. Then we replace $h_\nu = (\bar{h}_\nu, p \circ \bar{h}_\nu)$ by $h'_\nu = (\bar{h}_\nu, g \circ \bar{h}_\nu)$. Since $\|p-g\| \leq 1/3$, the images of h' contain our Y. Finally, we estimate $D_r h'_\nu$ by

$$\|D_r h'_\nu\| \leq \|D_r h_\nu\| + \|D_r(h_\nu - h'_\nu)\| \leq$$

$$\leq 1 + \|D_r((p-g) \circ \bar{h}_\nu)\| \ .$$

Since $\|D_r \bar{h}_\nu\| \leq 1$ and $\|D_r(p-g)\| \leq r/3$, we obtain with the chain rule,

$$\|D_r h'\| < C(\ell,m,r) \ ,$$

which is the required bound on the C^r-size of the images of h'_ν, $\nu = 1,\dots,N_o$, covering Y. Q.E.D.

3.5. MAIN COROLLARY. *Take an open subset* $U \subset \mathbb{R}^m$, *let* $f : U \to \mathbb{R}^m$ *be a* C^r*-map and let* $Y_o \subset U$ *be a subset of* C^r*-size* ≤ 1 *and such that* Y_o *is far from the boundary* ∂U *of* U. *Namely,* $\mathrm{dist}(Y_o, \partial U) \geq \sqrt{\ell}$. *Then the intersection* Y_1 *of the image* $f(Y_o) \subset \mathbb{R}^m$ *with every cube* $\square \subset \mathbb{R}^m$ *of unit size (i.e. with diameter* $\sqrt{m}$*) can be subdivided into* $N \leq C'\|D_r f\|^{\ell/r}+1$ *subsets of* C^r*-size* ≤ 1 *for some constant* $C' = C'(\ell,m,r)$.

PROOF. Let $h : [0,1]^\ell \to \mathbb{R}^m$ be the map with $\|D_r h\| \leq 1$ covering Y_o. By the chain rule, the composed map $g = f \circ h$ has $\|D_r g\| \leq C''(\ell,m,r)\|D_r f\|$ and the Main Lemma applies to $Y = \Gamma_g \cap ([0,1]^\ell \times \square) \subset [0,1]^\ell \times \mathbb{R}^m$. Since Y maps <u>onto</u> Y_1 under the projection $[0,1]^\ell \times \mathbb{R}^m \to \mathbb{R}^m$, the covering of Y by subsets of C^r-size ≤ 1 (unsured by the lemma) induces the required covering of Y_1. Q.E.D.

Remark. An important special case is that of a *linear* map f which, in fact, is sufficient for the proof of Yomdin theorem.

3.6. Suppose, the map f sends U into itself and such that $\mathrm{dist}(f(U), \partial U) \geq \sqrt{\ell}$. Then 3.5. also applies to the pieces of Y_1 of C^r-size ≤ 1 which are provided by 3.5. Then by induction on $i = 1,2,\dots$ we come to the following conclusion.

Let $\square_1,\dots,\square_i$ *be arbitrary unit cubes in* U, *let* $\square'_i$ *denote the pullback of* $\square_i$ *under the* i*-th iterate* f^i *of* f *and* Y_i *be the* f^i *image of the intersection* $Y_o \cap \square'_1 \cap \square'_2 \cap \dots \cap \square'_i$. *Then* Y_i *can be subdivided into* $N_i \leq (C'\|D_r f\|^{\ell/r}+1)^i$ *subsets of* C^r*-size* ≤ 1. *In particular*

$$\mathrm{Vol}\, Y_i \leq (C'\|D_r f\|^{\ell/r}+1)^i \ . \qquad (*)$$

3.7. *A bound for* $\mathrm{Vol}\, f^j(Y_o)$. Let Π be the restriction of the standard cubical partition Π_{st} of $\mathbb{R}$ to the above U . Then one has with $(*)$ and the notations in 1.1.,

$$i^{-1}\log \mathrm{Vol} f^i(Y_o) \leq \mathrm{ent}(\Pi|Y_o;f,i) + \ell/r \log\|D_r f\| + c \qquad (**)$$

for some $c = c(\ell,m,r)$.

3.8. <u>The proof of Yomdin theorem</u>

First observe that it suffices to consider the case of maps $f : U \to U$ satisfying the assumption in 3.6. because every manifolds X embeds into some $\mathbb{R}^m$ and every map $X \to X$ extends to the normal neighbourhood $U \subset \mathbb{R}^m$ of X with the normal projection $U \to X$. Furthermore, by scaling U to a larger set $\lambda_o U$ for some $\lambda_o \geq 1$ one can make $\mathrm{dist}(X,\partial U)$ as large as one wishes.

Next consider (rescaled) maps $f_j : jU \to jU$ for $j = 1,2,\ldots,$ defined by $f_j(x) = jf(jx)$ and notice that

(i) $\|\partial_r f_j\| = j^{-r}\|\partial_r f\|$;

(ii) the partition Π of jU into unit cubes corresponds to the partition $\Pi(j)$ of U into j^{-1}-cubes.

(iii) the set jY_o can be subdivided into j^ℓ subsets of C^r-size ≤ 1 .

Now, by the definition of $\mathrm{ent}\, f|Y_o$, for every $\varepsilon > 0$ there exist an integer k , such that

$$\mathrm{ent}(\Pi(j)|Y_o;f^k,i) \leq k\ \mathrm{ent}\, f|Y_o + k\varepsilon$$

for all j and all sufficiently large (depending on j^ℓ and k) i . This is equivalent to

$$\mathrm{ent}(\Pi|jY_o;f_j^k,i) \leq k\ \mathrm{ent}\, f|Y_o + k\varepsilon\ . \qquad (***)$$

Next, we choose j sufficiently large in order to make

$$\|D_r f_j^k\| \leq (1+\varepsilon)\|Df^k\|$$

which is possible by (i). Then we apply $(**)$ to f^k and the j^ℓ pieces of jY_o of C^r-size ≤ 1 (see (iii)) and conclude that

$$i^{-1}\log j^{-\ell}\mathrm{Vol} f^{ki}(Y_o) \leq k\ \mathrm{ent}\, f|Y_o + \ell/r \log\|Df^k\| + k\varepsilon(1+\tfrac{\ell}{r}) + c\ ,$$

for all sufficiently large i . We make $i \to \infty$ and observe that

$$\limsup_{i\to\infty} i^{-1}\log \mathrm{Vol}\, f^{ki}(Y) = k \limsup_{i\to\infty} i^{-1}\log \mathrm{Vol} f^i(Y)$$

for all compact submanifolds $Y \subset X$. Therefore

$$\limsup_{i\to\infty} i^{-1}\log \mathrm{Vol}\, f^i(Y_o) \le \mathrm{ent}\, f|Y_o + \frac{\ell}{kr}\log\|Df^k\| + \varepsilon(1+\frac{\ell}{r}) + c/k .$$

Then we let $k \to \infty$ and $\varepsilon \to 0$ and obtain,

$$\limsup_{i\to\infty} i^{-1}\log \mathrm{Vol}\, f^i(Y_o) \le \mathrm{ent}\, f\, Y_o + \ell/r \log^+ \mathrm{Rad}\, Df ,$$

for all subsets $Y_o \subset X$ with C^r-size ≤ 1 . Since every compact ℓ-dimensional submanifold Y can be covered by finitely many pieces with C^r-size ≤ 1 this inequality holds true for all Y .

Now, to prove Yomdin inequality (*) in 2.2. with the volume of the graphs $\Gamma_{f^i}|Y$ instead of the images $f^i(Y)$ (we used graphs rahter than images mainly to avoid the multiplicity problem for non-injective maps) we observe that $\Gamma_{f^i}|Y = F^i(\Gamma_{Id}|Y)$ for $F : (y,x) \to (Y,f(x))$ and that $\mathrm{ent}\, f|Y = \mathrm{ent}\, f|(\Gamma_{id}|Y)$. Hence, the above inequality for F in place of f yields Yomdin's (*) for f . Q.E.D.

3.9. C^r-entropy and semicontinuity

Let $g_o, g_1, \dots, g_i : [0,1]^\ell \to \mathbb{R}^m$ be C^r-maps . Then a collection of maps $h_1,\dots,h_N : [0,1]^\ell \to [0,1]^\ell$ whose images cover $[0,1]^\ell$ is called an ε-*cover* if $\|D_r h_\nu\| \le \varepsilon$ and $\|D_r(g_j \circ h_\nu)\| \le \varepsilon$ for all $j = 0,\dots,i$ and $\nu = 1,\dots,N$. Let $\mathrm{ent}_\varepsilon(g_o,\dots,g_i) = \log N$ for the minimal N for which an ε-cover exists. Observe that

$$\mathrm{ent}_\varepsilon \le \mathrm{ent}_\delta \le k^\ell \mathrm{ent}_\varepsilon$$

for $k^{-1}\varepsilon \le \delta \le \varepsilon$ and all $k = 1,2,\dots$.

Next, if $\{h_\nu\}$ is an ε-cover for $g_o \dots g_i$ and $\{h_{\mu\nu}\}$ is an ε-cover for the composed maps $g_j \circ h_\nu$ for $j = 1,\dots,i$, $\nu = 1,\dots,N$, then $\{h_\nu \circ h_{\mu\nu}\}$ also is an ε-cover for $g_o \dots g_i$, provided $\varepsilon \le \varepsilon_o$, where $\varepsilon_o = \varepsilon_o(\ell,m,r) > 0$ is a universal constant.

Now let $f : X \to X$ be a C^r-map of a smooth compact submanifold $X \subset \mathbb{R}^m$ and let $g : [0,1]^\ell \to X$ be C^r-smooth. Then the limit

$$\limsup_{i\to\infty} i^{-1}\, \mathrm{ent}_\varepsilon(g, f\circ g, \dots, f^i g)$$

does not depend on $\varepsilon > 0$ by the earlier discussion and is called C^r-*entropy* $\mathrm{ent}^r(f|g)$. Obviously,

$$\mathrm{ent}^r(f^k|g) = k\ \mathrm{ent}^r(f|g)$$

for all $k = 1,2,\dots$ and

$$\mathrm{ent}^r(f|g) \geq \mathrm{ent}\, f|g([0,1]^\ell) \quad ,$$

for all $r = 1,2,\ldots$

Then let

$$\mathrm{ent}_\varepsilon(f,i) = \sup i^{-1}\, \mathrm{ent}_\varepsilon(g, f\circ g, \ldots f^i\circ g)$$

over all g with $\|D_r g\| \leq 1$. If $\varepsilon \leq \varepsilon_o$ for the above $\varepsilon_o = \varepsilon_o(\ell,m,r)$, then obviously

$$\mathrm{ent}_\varepsilon(f,i+j) \leq (i+j)^{-1}(i\, \mathrm{ent}_\varepsilon(f,i) + j\, \mathrm{ent}_\varepsilon(f,j)) \quad .$$

for all $i,j = 1,2,\ldots$. Therefore, there exists a limit

$$\mathrm{ent}^{r,\ell}(f) = \lim_{i\to\infty} \mathrm{ent}_\varepsilon(f,i)$$

for $\varepsilon \leq \varepsilon_o$ which does not depend on ε and which is semicontinuous in f.

If f_τ *is* C^r*-continuous in* $\tau \in [0,1]$, *then*

$$\limsup_{\tau\to 0} \mathrm{ent}^{r,\ell} f_\tau \leq \mathrm{ent}^{r,\ell} f_o \; .$$

Also observe that

$$\mathrm{ent}^{r,\ell}(f) \geq \sup_g \mathrm{ent}^r(f|g)$$

over all C^r-maps $g : [0,1]^\ell \to X$.

Remark. There is the following topological version of $\mathrm{ent}^{r\ell}$. Take all $Y \subset X$ with C^r-size ≤ 1 , set

$$s(j;k,i) = \sup_Y \mathrm{ent}(\Pi(j)|Y; f^k, i)$$

(compare 1.2) and define

$$\mathrm{top}_r^\ell f = \liminf_{k\to\infty} \lim_{j\to\infty} \limsup_{i\to\infty} s(j;k,i) \; .$$

Clearly

$$\mathrm{top}_r^\ell f \geq \sup_Y \mathrm{ent}\, f|Y$$

over all C^r-submanifolds Y of dimension ℓ in X and

$$\mathrm{top}_r^\ell f \leq \mathrm{top}_r^n f = \mathrm{ent}\, f$$

for all $r = 1,\ldots,$ and $\ell \leq n = \dim X$.

Now by applying the argument in sections 3.4-3.8 to the C^r-entropy directly (without passing to volumes) one sees that

$$\mathrm{ent}^r(f|g) \leq \mathrm{ent}(f|g([0,1]^\ell) + \frac{\ell}{r} \log^+ \mathrm{RadDf}$$

for all C^r-maps $g : [0,1]^\ell \to X$ and

$$\mathrm{ent}^{r,\ell} f \leq \mathrm{top}_r^{\ell} f + \frac{\ell}{r} \log^+ \mathrm{Rad}\, Df$$

In particular, if f is C^∞-smooth, then

$$\mathrm{ent}\, f = \lim_{r\to\infty} \mathrm{ent}^{r,n} X$$

for $n = \dim X$ and the semicontinuity of $\mathrm{ent}^{r,n}$ implies that of $\mathrm{ent}\, f$.

4. THE PROOF OF ALGEBRAIC LEMMA

4.1. First we prove the lemma for algebraic curves Y in the (x,y)-plane such that the projection of Y to the x-axes is finite-to-one. Such a Y can be obviously divided into $N \leq d^4$ segments whose projections to the x-axes are one-to-one. Thus we reduce to the case where Y is the graph of a single valued function $y = y(x)$ for $x \in [0,1]$, such that $\|y(x)\| = \sup_x |y(x)| \leq 1$.

Next, we subdivide $[0,1]$ into smaller segments by the points where the derivative y' of y equals ± 1 . We switch the roles of x and y at those segments where $|y'| \geq 1$ and reduce the lemma to the case of functions $y = y(x)$, such that $\|y'\| \leq 1$. This proves the Lemma for $r=1$ since the map $x \mapsto (x,y(x))$ sends $[0,1]$ into Y with $\|D_1\| \leq \sqrt{2}$ and an obvious subdivision into two 1/2-subintervals makes $\|D_1\| \leq 1$.

Now, for $r \geq 2$, we assume,

$$\|y'\| \leq 1 \ , \ \|y''\| \leq 1, \ldots, \|y^{(r-1)}\| \leq 1$$

and divide $[0,1]$ by the zero points of the derivative $y^{(r+1)}(x)$. Then $y^{(r)}(x)$ is monotone on every subinterval (where $y^{(r+1)}$ does not change sign) and the problem obviously reduces to the case where $y^{(r)}(x)$ is positive and monotone decreasing on $[0,1]$. This monotonicity and the bound $\|y^{(r-1)}\| \leq 1$ imply that $y^{(r)}(x) \leq 2x^{-1}$ for all $x \in [0,1]$. Then a straightforward computation shows that the function $z(x) = y(x^2)$ has

$$\|z^{(i)}\| \leq 10^r \quad \text{for} \quad i = 1,\ldots,r \ ,$$

and the map $x \to (x,z(x))$ with an additional subdivision into 10^r segments provides the proof of Algebraic lemma for plane curves Y .

4.2. Now, let Y be a curve in $[0,1]^n \subset \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. We may assume that the projection of Y to $\mathbb{R}$ is finite to one. Then Y is the graph of $(n-1)$ algebraic functions $y_1(x)$, $y_2(x)$ ... $y_{n-1}(x)$. We assume, by induction, that the functions $y_1,\ldots,y_{n-2}$ have bounded derivatives of orders $\leq r$ and use the above change of variable, $x \rightsquigarrow x(t)$ to make the derivatives of y_{n-1} also bounded. Then all functions $z_i(t) = y_i(x(t))$, $i = 1,\ldots,n-1$ have bounded derivatives (on some subintervals) which obviously yields Algebraic Lemma for Y .

4.3. Consider a smooth vector valued algebraic function in ℓ variables, say $y = y(x_1,\dots,x_\ell)$, such that the components of the partial derivatives of orders $\leq r$ in the first $\ell-1$ variables are bounded in absolute values by one and let us make a change in the variable x_ℓ in order to achieve a similar bound for all partial derivatives. We assume by induction on r that the partial derivatives of orders $\leq s \leq r$ in x_ℓ are bounded. We denote by $\tilde{y} = \tilde{y}(x_1,\dots,x_\ell)$ the vector valued function whose components are the partial derivatives of the orders $\leq i_j$ in x_j, where

$$\sum_{j=1}^{\ell} i_j = r \quad \text{and} \quad i_\ell \leq s .$$

Let $\tilde{y}_1,\dots,\tilde{y}_N$ be the components of $\tilde{y}$ and assume by induction on the number of components that

$$\left\|\frac{\partial \tilde{y}}{\partial x_\ell}\right\| \leq 1 \quad \text{for} \quad \nu = 1,\dots,M-1 < N .$$

Then, for every fixed value of $x_\ell \in [0,1]$ we consider the maximum set $S(x_\ell) \subset x_\ell \times [0,1]^{\ell-1} = [0,1]^{\ell-1}$ of the function $\left|\frac{\partial \tilde{y}_M}{\partial x_\ell}\right|$ in the variables $x_1,\dots,x_{\ell-1}$. Then there obviously exists a subdivision of $[0,1]$ into subintervals, say I_k, and single valued algebraic functions $s_k : I_k \to [0,1]^{\ell-1}$, such that

(a) the number of the subintervals and $\deg s_k$ are bounded in terms of $\deg \tilde{y}_M$;

(b) $s_k(x_\ell) \in S(x_\ell)$ for all k and all $x_\ell \in I_k$.

Define $\tilde{s}_k : I_k \to [0,1]^{\ell-1} \times [-1,1]$ by $\tilde{s}_k : x_\ell \mapsto (s_k(x_\ell), \tilde{y}_M(x(x_\ell)))$ and apply the construction of the previous section to each function $\tilde{s}_k(x_\ell)$. This makes the derivatives $\dfrac{d^i \tilde{s}_k(x_\ell)}{d^i x_\ell}$ bounded for $i = 1,\dots,r$ and all k which easily implies a bound on $\dfrac{\partial \tilde{y}_M}{\partial x_\ell}$.

4.4. Now we prove Algebraic lemma by induction on $\ell = \dim Y$ for an algebraic set $Y \subset [0,1]^n$. We view this Y as the graph of an algebraic map $y : [0,1]^\ell \to [0,1]^{n-\ell}$ and we assume, for every fixed $x_\ell \in [0,1]$, that there exists some change of variables $x_1,\dots,x_{\ell-1}$ providing a universal bound for the partial derivatives of every branch of y in the changed variables $x_1,\dots,x_{\ell-1}$. We asssume, moreover, this change of variables be the piece-wise algebraic in x_ℓ and thus come to the situation of the previous section. Since the constructions we use in 4.1. are piece-wise algebraic for families of functions algebraicly depending on parameters, this

induction does go through and the Algebraic Lemma is proven.

4.5. The above argument provides a (semi) algebraic cell decomposition of an arbitrary semi-algebraic set Y and the cells can be (obviously) subdivided into simplices without loosing the control over the partial derivatives, such that the number of the simplices is bounded in terms of $\deg Y$.

Recall, that a subset $Y \subset \mathbb{R}^n$ is called *semialgebraic* if it is a finite union of pairwise non-intersecting subsets $Y_1, \dots Y_k$ in Y where each Y_i is a connected component of the difference of algebraic sets, $Y_i \subset A_i \setminus B_i$. The sum of the degrees of the polynomials defining all A_i and B_i is called the *degree* of Y.

Now we give a precise version of the previous remark.

TRIANGULATION LEMMA. *There exists a constant* $C = C(n,r)$, *such that every compact semialgebraic subset* $Y \subset \mathbb{R}^n$ *can be triangulated into* $N \leq (\operatorname{diam} Y)^n (\deg Y+1)^C$ *simplices, where for every closed* k-*simplex* $\Delta \subset Y$ *there exists a homeomorphism* h_Δ *of the regular simplex* $\Delta^k \subset \mathbb{R}^k$ *with the unit edge length onto* Δ *such that* h_Δ *is algebraic of degree* $\leq (\deg Y+1)^C$ *(i.e. the graph of* h_Δ *is a subset in an algebraic set of dimension* k *and degree* $\leq (\deg Y+1)^C$*) and regular real analytic in the interior of each face of* Δ. *("Regular" means non-vanishing of the differential of* h_Δ *on non-zero vectors). Furthermore,* $\|D_r h_\Delta\| \leq 1$ *for all* Δ. *(Of course, just this inequality makes the triangulation truly interesting).*

Using this lemma and the argument in §3 we arrive at the following Corollary.

TRIANGULATION THEOREM. *Let* f *be a* C^r-*selfmap of an open subset* $U \subset \mathbb{R}^n$, *such that* $\|D_r f\| < \infty$ and let $Y \subset U$ *be a compact semialgebraic subset. Then there exists a sequence of triangulation* T_i *of* Y *where* T_{i+1} *is a refinement of* T_{i+1} *for all* $i = 1,2\dots$, *and such that*

(a) *The number* N_i *of simplices of* T_i *satisfies*

$$\limsup_{i\to\infty} i^{-1} \log N_i \leq \operatorname{ent} f|Y + \frac{\ell}{r} \log^+ \operatorname{Rad} Df ,$$

for $\ell = \dim Y$. *(If* Y *is invariant under* f, *then this inequality obviously implies*

$$\log \operatorname{Rad} f_* H_*(Y) \leq \operatorname{ent} f|Y + \frac{\ell}{r} \log^+ \operatorname{Rad} Df) .$$

(b) *For every* k-*simplex* Δ *of* T_i *there exists an algebraic homeomorphism* $h : \Delta^k \to \Delta$ *which has degree* $\leq d_i$ *and satisfies* $\|D_r(f^j \circ h)\| \leq \varepsilon_i$ *for all* $j \leq i$, *where*

$$i^{-1}\log d_i \to 0 \quad \text{for} \quad i \to \infty$$

and $\varepsilon_i \to 0$. (*This and* (a) *sharpen* (**) *in* 2.2. .

BIBLIOGRAPHIE

[1] E.I. DINABURG, 1971 - *On the relation between various entropy characteristics of dynamical systems*, Math. U.S.S.R.-Isvestia, 5:2 , 337-378.

[2] M. GROMOV, 1978 - *Homotopical effects of dilitation*, J. of Differential Geometry 13, 303-310.

[3] A. MANING, 1975 - *Topological entropy and frist homology group*, in Dynam. Syst. Warwick.

[4] M. MISIUREWICZ, F. PRZYTYCKI, 1977 - *Topological entropy and degree of smooth mappings*, Bull, Ac. Pol. Sci., Ser. Math. 25:6, 573-574.

[5] M. MISIUREWICZ, F. PRZYTYCKI, 1977 - *Entropy conjecture for tori*, Ibid., 575-578.

[6] S. NEWHOUSE, 1985 - *Entropy and volume*, preprint.

[7] F. PRZYTYCKI, 1980 - *An upper estimate for topological entropy of smooth diffeomorphisms*, Inv. Math. 59:3, 205-213.

[8] Y. YOMDIN, 1986 - *Volume growth and entropy*, preprint IHES.

[9] Y. YOMDIN, 1986 - Addendum : C^k-*resolution of semialgebraic mappings*, preprint Univ. of Negev, Beer-Sheva, Israël.

[10] M. SHUB - *Dynamical systems filtration and entropy*, BAMS, 80:1, 1974, 27-41.

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