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INTEGRAL QUADRATIC FORMS: APPLICATIONS TO ALGEBRAIC GEOMETRY

[after V.Nikulin]

by Igor Dolgachev

As is well-known, integral quadratic forms appear in topology as the intersection forms on the middle-dimensional homology of manifolds of dimension 4k . In algebraic geometry they also appear as the intersection forms on the group of middle-dimensional algebraic cycles on a nonsingular algebraic variety of dimension 2k. In particular, the automorphism group of an algebraic variety is represented naturally by orthogonal transformations of integral quadratic forms. This simple observation was one of the motivations for V.Nikulin in his work on the arithmetic of integral quadratic forms and arithmetic crystallographic groups [30-35] . The purpose of this talk is to give an account of a part of this work and some of its applications to surface singularities and automorphisms of algebraic K3-surfaces. For lack of time we leave completely the part of Nikulin's work which concerns the topology of real algebraic varieties [30,32,35] . We refer to [8,10,25, 26,29,42] for other closely related works which are not mentioned here.

1. The arithmetic of integral quadratic forms.

The classical problems in the theory of quadratic forms are the problem of classification and the problem of representation of one form by another. In the case of quadratic forms over algebraic number fields they were solved in the works of H.Minkowski, H.Hasse, M.Kneser, M.Eichler and others. In the case of unimodular indefinite integral forms they are solved by results of J.Milnor. Developping the technique of discriminant forms introduced in works of M.Kneser, C.T.C.Wall, A.Durfee and others (see [12]), V.Nikulin solves the above problems for non-unimodular indefinite integral quadratic forms.

1.1. Notation and definitions.

An integral symmetric bilinear form (or a lattice) of rank r is a free $\mathbb{Z}$-module S of rank r together with a symmetric bilinear form $S \times S \to \mathbb{Z}$, $(x,x') \to x \cdot x'$. Tensoring by $\mathbb{R}$ we obtain the real bilinear form $S_{\mathbb{R}}$ associated to S . All the usual terminology for the latter applies to S. Thus, we may speak about the signature sign(S) = $= (t_+,t_-,t_0)$, positive or negative definite, semi-definite, indefinite, non-degenerate lattices. If S is a non-degenerate lattice, we write sign(S) = (t_+,t_-) . An indefinite lattice of signature $(1,t_-)$ or $(t_+,1)$ is called a hyperbolic lattice.

A homomorphism of lattices $f:S \to S'$ is a homomorphism of the abelian groups such that $f(x) \cdot f(y) = x \cdot y$ for all $x,y \in S$. An injective (resp. bijective) homomorphism of latticies is called an embedding (resp. an isometry) .The group of isometries of a lattice S into itself is denoted by O(S) and is called the orthogonal group of S . An embedding $i:S \to S'$ is called primitive if $S'/i(S)$ is a free group. A sublattice of a lattice S is a subgroup of S equipped with the induced bilinear form. A sublattice S' of a lattice S is called primitive if the identity map $S' \to S$ is a primitive embedding. Two embeddings $i:S \to S'$ and $i':S \to S'$ are called isomorphic if there exists an isometry $\sigma \in O(S')$ such that $i' = \sigma \circ i$. We say that a lattice S' represents a lattice S if there exists an embedding $i:S \to S'$. Two lattices are called isomorphic if there exists an isometry from one to another.

By $S_1 \oplus S_2$ we denote the orthogonal sum of two lattices defined in the standard way. We write S^n for the orthogonal sum of n copies of a lattice S . The orthogonal complement of a sublattice S of a lattice S' is defined in the usual way and is denoted by $(S)^{\perp}_{S'}$.

If $\underline{e} = \{e_1,\ldots,e_r\}$ is a $\mathbb{Z}$-basis of a lattice S, then the matrix $G(\underline{e}) = (e_i \cdot e_j)$ is called the Gram matrix of S with respect to $\underline{e}$. It is clear that the structure of a lattice is determined by the Gram matrix with respect to some basis .

For every integer m we denote by S(m) the lattice obtained from a lattice S by multiplying the values of its bilinear form by m .

A lattice S is called even if $x^2 := x \cdot x$ is even for all x from S . In this case the map $x \to x^2$ is a quadratic form on S, i.e. a map $Q:S \to \mathbb{Z}$ such that i) $Q(nx) = n^2 Q(x)$ for all $n \in \mathbb{Z}$ and $x \in S$ and ii) the map $S \times S \to \mathbb{Z}$, $(x,y) \to \frac{1}{2}(Q(x+y)-Q(x)-Q(y))$ is the symmetric bilinear

form on S .

By A_n, D_n, E_n we denote the even negative definite lattices defined by the Gram matrix equal to (-1)x the Cartan matrix of the root system of type A_n, D_n, E_n respectively ([7], Chap.VI) . By U we denote the indefinite lattice of rank 2 defined by the Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (called the hyperbolic plane). For any integer n we denote by <n> the lattice $\mathbb{Z}e$, where $e^2 = n$.

1.2. Discriminant forms.

Let S be a lattice and $S^* = \mathrm{Hom}_{\mathbb{Z}}(S,\mathbb{Z})$ be the dual abelian group. We denote by i_S the homomorphism of abelian groups $S \to S^*$ given by $i_S(x)(y) = x \cdot y$ for every $x,y \in S$. The cokernel $S^*/i_S(S)$ is denoted by D_S and is called the discriminant group of S . This group is finite if and only if S is non-degenerate. In the latter case its order is equal to $|\det(G(\underline{e}))|$ for any basis $\underline{e}$ of S . A lattice S is called unimodular if D_S is trivial and p-elementary if D_S is a p-elementary abelian group. By l(A) we denote the minimal number of generators of an abelian group A of finite type, we put $l(S) = l(D_S)$.

Let D_S be the discriminant group of a non-degenerate lattice S . The bilinear form of S extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on S^* and induces a symmetric bilinear form $b_S: D_S \times D_S \to \mathbb{Q}/\mathbb{Z}$. If S is even, then b_S is the symmetric bilinear form associated to the quadratic form $q_S: D_S \to \mathbb{Q}/2\mathbb{Z}$ defined by $q_S(x+i_S(S)) = x^2+2\mathbb{Z}$. The latter means that $q_S(na) = n^2 q_S(a)$ for all $n \in \mathbb{Z}$, $a \in D_S$ and $b_S(a,a') = \frac{1}{2}(q_S(a+a')-q_S(a)-q_S(a'))$ for all $a,a' \in D_S$, where $\frac{1}{2}: \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. We call the pair (D_S,b_S) (resp. (D_S,q_S)) the discriminant bilinear (resp.quadratic) form of S .

More generally, one can define a finite symmetric bilinear (resp. a finite quadratic) form as a pair (A,b) (resp. (A,q)) consisting of a finite abelian group A and a symmetric bilinear form $b: A \times A \to \mathbb{Q}/\mathbb{Z}$ (resp. a quadratic form $q: A \to \mathbb{Q}/2\mathbb{Z}$) . One extends to this case most of the notion introduced for lattices. In particular, we may speak about the orthogonal group O(A) of a finite symmetric bilinear form (every finite quadratic form (A,q) is considered as a finite symmetric bilinear form (A,b), where b is associated to q). We define the homomorphism of groups $\alpha: O(S) \to O(D_S)$ by putting $\alpha(\phi) = \bar{\phi} := \phi^{*-1} \bmod S$, where ϕ^* denotes the transpose of ϕ . We also define the orthogonal sum of finite symmetric bilinear forms and check that

$$D_{S \oplus S'} \simeq D_S \oplus D_{S'} \quad .$$

1.3. Some earlier results.

The following results about unimodular lattices are well-known (see [43]):

Proposition(1.3.1)(J.Milnor [23]). *1) A non-even indefinite unimodular lattice exists for any signature (t_+,t_-) and is isomorphic to the lattice $(<1>)^{t_+} \oplus (<-1>)^{t_-}$.*
2) An even unimodular lattice of signature (t_+,t_-) exists if and only if $t_+ - t_- \equiv 0 \pmod 8$; if indefinite, it is determined by its signature uniquely up to an isomorphism.

It follows immediately from these results that an embedding (necessary primitive) of an (even) indefinite unimodular lattice of signature (t_+,t_-) into a lattice of the same type with signature (t'_+,t'_-) exists and is uniquely determined up to an isomorphism provided $t'_+ > t_+$ and $t'_- > t_-$.

We also mention the following result about primitive embeddings of an even non-unimodular lattice into a unimodular lattice:

Proposition (1.3.2)(D.James[19]). *A primitive embedding of an even non-degenerate lattice of signature (t_+,t_-) into an even unimodular lattice of signature (l_+,l_-) exists if $t_+ + t_- \leq \min\{l_+,l_-\}$. Moreover, if this inequality is strict, then the embedding is unique up to an isomorphism.*

To generalize these results Nikulin uses the technique of discriminant bilinear and quadratic forms. We recall the following results which can be found (or deduced from) in [12]:

Proposition (1.3.3). *1) Two non-degenerate (resp. even non-degenerate) lattices S_1 and S_2 have isomorphic bilinear discriminant forms (resp. quadratic discriminant forms) if and only if $S_1 \oplus L_1 \simeq S_2 \oplus L_2$ for some unimodular (resp. even unimodular) lattices L_1 and L_2 .*
2) For every finite symmetric bilinear form (A,b) (resp. quadratic form (A,q)) there exists a lattice (resp. an even lattice) S such that $(D_S,b_S) \simeq (A,b)$ (resp. $(D_S,q_S) \simeq (A,q)$).
3) The signature and the discriminat quadratic form determine uniquely the genus of an even lattice .

Recall that two lattices S_1 and S_2 belong to the same genus if $\mathrm{sign}(S_1) = \mathrm{sign}(S_2)$ and their p-adic completions $(S_1)_p = S_1 \otimes \mathbb{Z}_p$ and

$(S_2)_p = S_2 \otimes \mathbb{Z}_p$ are isomorphic (as $\mathbb{Z}_p$-bilinear forms) for all primes p. It is known that the genus determines the isomorphism class of $S_{\mathbb{Q}} = S \otimes \mathbb{Q}$ and contains only finitely many isomorphism classes of lattices.

Most of the proofs of the results from the next section are based on the method of passing to the p-adic completions and use the classification of p-adic lattices [15], finite quadratic forms [12,30] and the relation $(D_S)_p \simeq D_{S_p}$ (with the obvious generalization of the definitions to give it sense). It should be said that these methods allowed the experts (e.g. M.Kneser and C.T.C.Wall) to obtain the results below in many special situations. We owe to Nikulin the nice and simple way for expressing this technique in the general case.

1.4. Nikulin's results.

To simplify the exposition we restrict ourselves only to the case of even non-degenerate lattices and state the results not always in their full generality, as proven in [30].

We begin with the classification problem: Given two non-negative integers t_+ and t_- and a finite quadratic form A, we want to find out whether there exists an even non-degenerate lattice S with $\mathrm{sign}(S) = (t_+,t_-)$ and $D_S \simeq A$. Using Proposition (1.3.3) we can define the index $\mathrm{ind}(A)$ of A as the element of $\mathbb{Z}/8\mathbb{Z}$ equal to $t'_+ - t'_- \bmod 8$, where S' is an even lattice with the signature (t'_+,t'_-) and $D_{S'} \simeq A$. Thus, we have a necessary condition: $t_+ - t_- \equiv \mathrm{ind}(A) \pmod 8$. Another obvious necessary condition is $\mathrm{rk}(S) = t_+ + t_- \geq l(A)$.

THEOREM (1.4.1)(Existence). *An even lattice S with $\mathrm{sign}(S) = (t_+,t_-)$ and $D_S \simeq A$ exists providing the following conditions are satisfied:*
(i) $t_+ - t_- \equiv \mathrm{ind}(A) \pmod 8$;
(ii) $t_+ + t_- > l(A)$.

A stronger result ([30],Thm. 1.10.1) replaces (ii) by the non-strict inequality but adds two more local conditions.

To prove the theorem Nikulin applies Proposition (1.3.3) to find a lattice S' with $D_{S'} \simeq A$ and $t'_+ - t'_- \equiv t_+ - t_- \pmod 8$, $\mathrm{rk}(S') \geq \mathrm{rk}(S)$ ($\mathrm{sign}(S') = (t'_+,t'_-)$). Using the classification of finite quadratic forms he finds the p-adic completions S'_p and then possible p-adic completions of S. Finally, the lattice S is reconstructed by the theory of Minkowski-Hasse (see[43]).

THEOREM (1.4.2)(Uniqueness). *An even indefinite lattice S is determined by its signature and its discriminant quadratic form uniquely up to an isomorphism provided* $rk(S) \geq l(S) + 2$.

If $l(S) = 0$, i.e. S is unimodular, this follows from Prposition (1.3.1) . If $l(S) \neq 0$, then $rk(S) \geq 3$ and the result follows from a result of M.Kneser[20]interpreted in the language of discriminant quadratic forms.

COROLLARY (1.4.3) . *Let S be an even indefinite lattice with* $rk(S) \geq l(S) + 3$. *Then* $S \simeq S' \oplus U$ *for some lattice S'. Moreover, if the above inequality is strict, then the lattice S' is determined uniquely up to an isomorphism.*

Before stating Nikulin's results about primitive embeddings of even lattices,we have to make the following simple remarks .

Let S be an even non-degenerate sublattice of an even non-degenerate lattice S' . Suppose that $rk(S) = rk(S')$. The inclusions $S \subset S' \subset S'^{*} \subset S^{*}$ show that the group $H_{S,S'} = S'/S$ is a subgroup of of D_S and the restriction of the quadratic form q_S to $H_{S,S'}$ is identical zero . Conversely, if H is an isotropic subgroup of D_S (i.e. $q_S | H \equiv 0$), then it can be lifted to a subgroup S' of S^{*} such that the $\mathbb{Q}$-valued symmetric bilinear form on S^{*} induces a structure of an even non-degenerate lattice on S' containing S as a sublattice . In this way we obtain

Lemma (1.4.4) . *The correspondence* $S' \to H_{S,S'}$ *defines a bijection between the set of non-degenerate lattices containing S as a sublattice of finite index and the set of isotropic subgroups of* D_S. *Moreover, under this correspondence* $D_{S'}$ *is identified with* $(H_{S,S'})^{\perp}_{D_S}/H_{S,S'}$ *and* $q_{S'} = q_S | D_{S'}$.

Now, let $i:S \to L$ be a primitive embedding of an even non-degenerate lattice S into an even unimodular lattice L . We identify the lattice S with its image i(S) . Let $K = (S)^{\perp}_{L}$. Then L contains $S \oplus K$ as a sublattice of finite index. The composition of the bijection $i_L:L \to L^{*}$ and the restriction map $L^{*} \to K^{*}$ defines an isomorphism $H_{S \oplus K,L} \to D_K$. Similarly, we construct an isomorphism $H_{S \oplus K,L} \simeq D_S$ and obtain in this way an isomorphism $\gamma(i):D_S \to D_K$. By Lemma (1.4.4), the graph of this isomorphism in $D_S \oplus D_K \simeq D_{S \oplus K}$ is an isotropic subgroup. This shows that $q_K \circ \gamma(i) = -q_S$. Thus, we get

Lemma (1.4.5). *A primitive embedding of an even non-degenerate lattice S into an even unimodular lattice L with $(S)_L^{\perp}$ isomorphic to a fixed lattice K is given by an isomorphism $\gamma: D_S \to D_K$ such that $q_K \circ \gamma = -q_S$. Two such isomorphisms γ and γ' define isomorphic embeddings if and only if there exists $\phi \in O(K)$ such that $\gamma \circ \bar{\phi} = \gamma'$.*

It follows immediately from this lemma that a primitive embedding of an even non-degenerate lattice S of signature (t_+,t_-) into an even unimodular lattice L of signature (l_+,l_-) exists if and only if there exists an even non-degenerate lattice K of signature (l_+-t_+,l_--t_-) with $(D_K,q_K) \simeq (D_S,-q_S)$. Applying Theorem (1.4.1) we obtain

THEOREM (1.4.6) (Primitive embeddings). *A primitive embedding of an even non-degenerate lattice S of signature (t_+,t_-) into an even unimodular lattice L of signature (l_+,l_-) exists provided $l_+ \geq t_+$, $l_- \geq t_-$, $rk(L) - rk(S) > l(S)$.*

The second part of Lemma (1.4.5) implies that a primitive embedding of S into L is uniquely determined up to an isomorphism if the signature (l_+-t_+,l_--t_-) and the quadratic form $q_K \simeq -q_S$ determines uniquely the isomorphism class of a lattice K and the canonical homomorphism α: $O(K) \to O(D_K)$ is surjective. The next result gives a class of lattices for which the latter is true.

Proposition (1.4.7). *Let S be an even indefinite lattice. Suppose that $rk(S) \geq l(S) + 2$. Then the homomorphism $O(S) \to O(D_S)$ is surjective.*

Applying this result, we obtain

THEOREM (1.4.8) (Uniqueness of a primitive embedding). *Let $i: S \to L$ be a primitive embedding of an even non-degenerate lattice S of signature (t_+,t_-) into an even non-degenerate lattice L of signature (l_+,l_-). It is unique up to an automorphism of L provided the following conditions are satisfied:*
(i) $l_+ \geq t_+, l_- \geq t_-$;
(ii) $rk(L)-rk(S) \geq l(S) + 2$.

We leave aside the generalizations of the above results to the case of a primitive embedding into a non-unimodular lattice (see[30], 15^{o}) .

1.5. 2-elementary lattices.

Recall that a lattice S is called 2-elementary if its discriminant group $D_S \simeq (\mathbb{Z}/2\mathbb{Z})^a$ (clearly $a = l(S)$) . We will consider only even 2-elementary lattices . We introduce the invarint δ_S of S by putting $\delta_S = 0$ if q_S takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$ and $\delta_S = 1$ otherwise.

The importance of 2-elementary lattices will be clear later (see also [35]). Partly it is based on the notion of an involution of a lattice. Let L be an even unimodular lattice and $\iota: L \to L$ be its involution, i.e. an isometry of order 2 in O(L) . We define the sublattices of L:

$$L^\iota = \{x \in L: \iota(x) = x\} \quad , \quad L_\iota = \{x \in L: \iota(x) = -x\} \quad .$$

It is clear that these sublattices are primitive sublattices of L and that each of them is the orthogonal complement to another.

Proposition(1.5.1). *The sublattices L^ι and L_ι are even 2-elementary lattices. Conversely, for every primitive 2-elementary sublattice S of an even unimodular lattice L there exists a unique involution ι of L such that $S = L^\iota$ and $(S)^\perp_L = L_\iota$.*

This is a simple corollary of Lemma (1.4.5).

THEOREM (1.5.2)(Classification of 2-elementary lattices). *The genus of an even 2-elementary lattice is determined by the invariants δ_S, $l(S)$ and $sign(S)$. If S is indefinite, then the genus consists of one isomorphism class. An even 2-elementary lattice S with the invariants $\delta_S = \delta$, $l(S) = l$ and $sign(S) = (t_+, t_-)$ exists if and only if the following conditions are satisfied (we assume that $\delta = 0$ or 1 and l, t_+, t_- are nonnegative integers):*

1) $t_+ + t_- \geq l$;
2) $t_+ + t_- + l \equiv 0 \pmod 2$;
3) $t_+ - t_- \equiv 0 \pmod 4$ if $\delta = 0$;
4) $\delta = 0$, $t_+ - t_- \equiv 0 \pmod 8$ if $l = 0$;
5) $t_+ - t_- \equiv 1 \pmod 8$ if $l = 1$;
6) $\delta = 0$ if $a = 2$, $t_+ - t_- \equiv 4 \pmod 8$;
7) $t_+ - t_- \equiv 0 \pmod 8$ if $\delta = 0$ and $l = t_+ + t_-$.

This theorem is deduced from the classification of finite quadratic forms on 2-elementary abelian groups and Proposition (1.3.5).

2. Crystallographic groups in Lobachevsky spaces.

2.1. Discrete reflection groups and convex polyhedra.

Let $E^{1,n}$ be a real (n+1)-dimensional vector space equipped with an inner product $x \cdot x'$ of signature (1,n) . The subset $V = \{ x \in E^{1,n} : x^2 > 0 \}$ is a disjoint union of two open convex cones. We choose one of them and denote it by V^+ . Let $\mathbb{L}^n$ be the subset $V^+/\mathbb{R}_+$ of the real projective space $\mathbb{P}^n(\mathbb{R})$. If $n \geq 2$, then $\mathbb{L}^n$ admits a natural structure of a Riemann manifold of constant negative curvature and is called the n-dimensional Lobachevsky space (or the n-dimensional hyperbolic space). Its group of motions $\mathrm{Aut}(\mathbb{L}^n)$ is identified with the subgroup $O(E^{1,n})^+$ of the orthogonal group $O(E^{1,n})$ consisting of all isometries of $E^{1,n}$ which leave V^+ invariant. Obviously, $O(E^{1,n}) = O(E^{1,n}) \times \{ \pm \mathrm{id} \}$. For every $e \in E^{1,n}$ with $e^2 < 0$ the set

$$H_e = \{ x \in V^+ : x \cdot e = 0 \} / \mathbb{R}_+ \}$$

is a non-empty subset of $\mathbb{L}^n$. The Riemannian structure of $\mathbb{L}^n$ induces a structure of a (n-1)-dimensional Lobachevsky space on H_e (if $n \leq 2$ we still call $\mathbb{L}^n$ the Lobachevsky space) . We call H_e a hyperplane in $\mathbb{L}^n$. We will always assume that e is normalized in such a way that $e^2 = -2$. If $H_e \cap H_{e'}$ is a hyperplane in both H_e and $H_{e'}$, then the dihedral angle $\alpha(H_e, H_{e'})$ is defined by putting $\cos(\alpha(H_e, H_{e'})) = \frac{1}{2} e \cdot e'$. A convex polyhedron in $\mathbb{L}^n$ is a subset of $\mathbb{L}^n$ of the form

$$P = \{ x \in V^+ : x \cdot e_i \geq 0 \ , \ i \in I \} / \mathbb{R}_+$$

for some set $\{e_i\}_{i \in I}$ of vectors e_i with $e_i^2 = -2$. We will always assume that the set $\{e_i\}_{i \in I}$ defining P cannot be decreased. The hyperplanes H_{e_i} are called the facets of P . Let $\bar{H}_{e_i}$ be the natural extension of H_{e_i} to a hyperplane in $\mathbb{P}^n(\mathbb{R})$ and $\bar{P}$ be the convex polyhedron in $\mathbb{P}^n(\mathbb{R})$ bounded by the hyperplanes $\bar{H}_{e_i}$. Then P is of finite volume with respect to the Riemannian metric in $\mathbb{L}^n$ if and only if $\bar{P}$ is contained in the closure $\bar{\mathbb{L}}^n$ of $\mathbb{L}^n$ in $\mathbb{P}^n(\mathbb{R})$. Moreover, P is compact if and only if $\bar{P} \subset \mathbb{L}^n$. If P is of finite volume, then the set of its facets H_{e_i} is finite and we can define the Gram matrix of P by putting

$$G(P) = (a_{ij}) \ , \text{ where } \ a_{ij} = e_i \cdot e_j \ , \ i,j \in I \ .$$

A reflection with respect to a hyperplane H_e (or a vector $e \in E^{1,n}$ with $e^2 = -2$) is a motion of $\mathbb{L}^n$ given by the formula:

$$s_e(x) = x + (x \cdot e)e \ , \quad x \in E^{1,n} \ .$$

Every polyhedron P defines a subgroup $\Gamma(P)$ of $\mathrm{Aut}(\mathbb{L}^n)$ generated by the reflections with respect to the facets of P . The group $\Gamma(P)$ is a discrete subgroup of the Lie group $O(E^{1,n})^+$ if and only if all the dihedral angles of P are of the form π/m , where m is an integer ≥ 2 . Conversely, for any discrete subgroup Γ of $\mathrm{Aut}(\mathbb{L}^n)$ generated by reflections, the union of the hyperplanes defining all reflections from Γ divide $\mathbb{L}^n$ into a union of convex polyhedra. Each of them, say $P(\Gamma)$, is a fundamental domain for Γ in $\mathbb{L}^n$ and $\Gamma(P(\Gamma)) = \Gamma$. If $\{H_{e_i}\}_{i\in I}$ is the set of facets of $P(\Gamma)$, then the pair $(\Gamma,\{s_{e_i}\}_{i\in I})$ is a Coxeter system [7,46] .

A convex polyhedron P in $\mathbb{L}^n$ is called a crystallographic polyhedron if it is of finite volume and $\Gamma(P)$ is a discrete subgroup of $\mathrm{Aut}(\mathbb{L}^n)$. A discrete subgroup of $\mathrm{Aut}(\mathbb{L}^n)$ generated by reflections is called a crystallographic group if $P(\Gamma)$ is of finite volume.

PROBLEM. *Find all crystallographic polyhedra in $\mathbb{L}^n$, or, equivalently, all crystallographic subgroups of $O(E^{1,n})^+$.*

It is known that for any discrete arithmetic subgroup of $O^+(E^{1,n})$ its fundamental domain in $\mathbb{L}^n$ is of finite volume [6] . Thus, any discrete arithmetic subgroup of $\mathrm{Aut}(\mathbb{L}^n)$ generated by reflections is a crystallographic group . We notice the following striking result due to E.Vinberg [47]:

THEOREM (2.1.1). *Let Γ be a crystallographic group in $\mathbb{L}^n$. Suppose that $n > 30$. Then Γ is not arithmetic and $P(\Gamma)$ is not compact.*

Many examples of crystallographic groups were constructed by various geometers (F.Lanner,V.Makarov,E.Vinberg,E.Andreev,H.-C. im Hof are some of them). No examples of such groups are known for $n > 19$.

2.2. Arithmetic crystallographic groups.

Let P be a crystallographic polyhedron in $\mathbb{L}^n$ and G(P) be its Gram matrix. Assume that $\Gamma(P)$ is an arithmetic subgroup of $O(E^{1,n})^+$. Then all the cyclic products $b_{i_1\dots i_m} = a_{i_1 i_2}\cdot\dots\cdot a_{i_m i_1}$ are real algebraic integers [45]. In particular, all the entries a_{ij} of G(P) are real algebraic integers . Let $K = \mathbb{Q}(\{b_{i_1\dots i_m}\})$ be the extension of $\mathbb{Q}$ generated by all the products $b_{i_1\dots i_m}$ and $\tilde{K} = \mathbb{Q}(\{a_{ij}\})$ be the extension of $\mathbb{Q}$ generated by all the entries of G(P) . The extension $\tilde{K}/K$ is a Galois extension, its Galois group G is a 2-elementary abelian group. Let $\tilde{O}$ be the ring of integers of $\tilde{K}$ and S be the

$\tilde{O}$-submodule of $E^{1,n}$ spanned by the vectors e_i defining the facets of P (we assume for simplicity that they span $E^{1,n}$). The inner product on $E^{1,n}$ induces a symmetric bilinear form on S . The Galois group G acts on S by acting on $\tilde{O}$ and sending the e_i's to $\pm e_i$. The group $\Gamma(P)$ is a subgroup of finite index of the group O(S,G) of G-isometries of S . Moreover, $\Gamma(P)$ is generated by the G-reflections s_{e_i} . The triple $(K,\tilde{K},S)$ is called the lattice of P (or $\Gamma(P)$). Two arithmetic crystallographic groups are called equivalent if they have the same fields K and $\tilde{K}$, and their lattices are G-isomorphic.

THEOREM(2.2.1) (V.Nikulin [33,34]). *For a fixed $N = [K:\mathbb{Q}]$ the number of equivalence classes of arithmetic crystallographic groups in $\mathbb{L}^n$ $(n \geq 2)$ is finite . Moreover, there exists a number N_0 such that for $n > 9$ there are no arithmetic crystallographic groups with $[K:\mathbb{Q}] > N_0$.*

Notice that P is always compact if $K \neq \mathbb{Q}$. If $\tilde{K} = K = \mathbb{Q}$, then S is a non-degenerate even hyperbolic lattice and $\Gamma(P)$ is a subgroup of finite index of O(S) generated by reflections s_{e_i} , $e_i \in S$.

Let F^r denote the set of isomorphism classes of non-degenerate even hyperbolic lattices of rank r such that the subgroup W(S) of O(S) generated by all reflections s_e , $e \in S$ (the reflection group of S),is of finite index in O(S) . We put $F^{\geq m} = \bigcup_{r \geq m} F^r$ and $F = \bigcup_{r=2} F^r$.

Since O(S) is an arithmetic subgroup of the group $O(S_{\mathbb{R}})$, the group W(S) and any its subgroup of finite index generated by reflections is an arithmetic crystallographic group if and only if $S \in F$. The following result of V.Nikulin lists the set $F^{\geq 5}$:

THEOREM (2.2.2). *The set $F^{\geq 20}$ is empty . The set $F^{\geq 5}$ is finite and consists of all 2-elementary lattices S with $rk(S)+\ell(S) \leq 18$ except the lattice $S = U \oplus E_8(2)$, the 2-elementary lattice $S = U \oplus A_1 \oplus E_8^2$ and the following non-2-elementary lattices:*

$U(4)\oplus A_1^3$, $\langle 6\rangle \oplus A_2^2$, $U\oplus A_1\oplus A_2^2$, $\langle 2^k\rangle \oplus D_4$ $(k = 2,3,4)$, $U\oplus A_3$ (of rank 5);

$U(4)\oplus D_4$, $U(3)\oplus A_2^2$, $U\oplus A_2\oplus A_1^2$, $U\oplus A_2^2$, $U\oplus A_1\oplus A_3$, $U\oplus A_4$ (of rank 6) ;

$U\oplus A_1\oplus A_2^2$, $U\oplus A_3\oplus A_1^2$, $U\oplus A_2\oplus A_3$, $U\oplus A_1\oplus A_4$, $U\oplus A_5$, $U\oplus D_5$ (of rank 7);

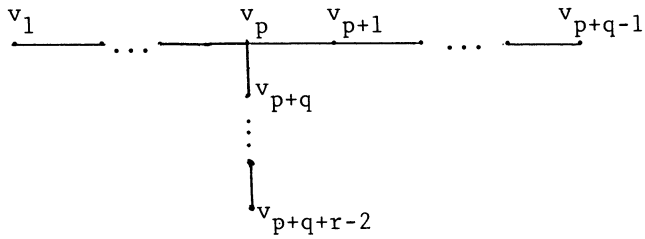
$U\oplus A_2^3$, $U\oplus A_3^2$, $U\oplus A_2\oplus A_4$, $U\oplus A_6$, $U\oplus A_1\oplus A_5$, $U\oplus A_2\oplus D_4$, $U\oplus A_1\oplus D_5$ (of rank 8);

$U\oplus A_1\oplus E_6$, $U\oplus A_2\oplus E_6$, $U\oplus A_2\oplus E_8$, $U\oplus A_3\oplus E_8$ (of rank 9,10,12 and 13).

Notice that F^2 consists of lattices representing the lattice $\langle 0\rangle$ or the lattice $\langle -2\rangle$. The set F^3 is finite by Theorem (2.2.1) , however, it is not known yet . The set F^4 was found by E.Vinberg (unpublished). It consists of 14 lattices. We refer to [48] for the problem of

classifying odd unimodular lattices whose orthogonal group contains a crystallographic subgroup.

Example (2.2.3). Let $S = T_{p,q,r}$, the lattice spanned by vectors e_i, $i = 1,\ldots,p+q+r-2$, with $e_i^2 = -2$ and $e_i \cdot e_j = 1$ or 0 depending on whether the vertices v_i in the following graph are joint by an edge or not:



It is easy to check that $T_{p,q,r}$ is a hyperbolic lattice if and only if $p^{-1}+q^{-1}+r^{-1} < 1$. The order of its discriminant group is equal to $|pqr-pq-pr-qr|$ and $l(T_{p,q,r}) \leq 3$. We leave it as an exercise to verify that $T_{p,q,r} \in F$ if and only if (p,q,r) is one of the following 9 triples: $(2,3,7)$, $(2,3,8)$, $(2,3,9)$, $(2,3,10)$, $(2,4,5)$, $(2,4,6)$, $(3,3,4)$, $(3,3,5)$, $(3,3,6)$. Notice that the group generated by the reflections s_{e_i} is crystallographic only for the three triples $(2,3,7)$, $(2,4,5)$ and $(3,3,4)$ (see [7], Chap.V , § 4 , Exer.13).

The proof of Theorem (2.2.2) requires a lot of hard work and is very ingeneous. Very roughly, it proceeds as follows. First, one separates the class F' of lattices S such that $S \simeq U \oplus K$, where K is an even negative definite non-2-elementary lattice. It is proved that such S belongs to F if and only if for every representation as above the sublattice of K spanned by vectors v with $v^2 = -2$ is of finite index in K. Using the results of § 1, one classifies $F \cap F'$. Next, one consisders 2-elementary lattices S. It is proved that if such S belongs to F, then $rk(S)+l(S) \leq 20$. In particular, such S can be realized as the Picard lattice on a K3-surface (see below, § 4) . Then , applying Proposition (1.5.1) and the Global Torelli Theorem for K3-surfaces one studies the fixed-point set of the involution on a K3-surface defined by the lattice S. The analysis of this set allows to classify all 2-elementary lattices from F. Finally, one considers the lattices S which are neither 2-elementary nor belonging to F'. The corresponding list of such lattices was found earlier by

E.Vinberg.

3. Applications to surface singularities.

3.1. The Milnor lattices.

Let (X,x) be an isolated n-dimensional complex singularity, i.e. a germ of a n-dimensional complex space X at its isolated singular point x . We choose the representative X in the class of contractible Stein spaces. A smoothing of (X,x) is a flat holomorphic map $f:Y \to \Delta$ of a contractible Stein space Y of dimension n+1 onto the unit disk Δ such that $X = f^{-1}(0)$ and f is of maximal rank outside the point x . A singularity (X,x) is called smoothable if there exists a smoothing of it . For example, every complete intersection singularity (i.e. X can be chosen as a closed subvariety of $\mathbb{C}^{n+k}$ given by k equations) is obviously smoothable.

Let $f:Y \to \Delta$ be a smoothing of (X,x), $Y_t = f^{-1}(t)$, $t \in \Delta$. If $t \neq 0$, then Y_t is a complex Stein manifold of dimension n . The intersection pairing defines an integral bilinear form on $M = H_n(Y_t)$ (if not stated otherwise, all the homology and the cohomology are considered with the coefficients in $\mathbb{Z}$). This form is symmetric (resp. alternate) if n is even (resp.odd) . By the universal coefficient formula $\mathrm{Tors}(M) = \mathrm{Tors}(H^{n+1}(Y_t,\mathbb{Z})) = 0$ (because Y_t is a Stein manifold).Thus, if n is even, M has a structure of a lattice. We will call this lattice the Milnor lattice of the smoothing f of (X,x). Obviously, its isomorphism class is independent of a choice of $t \neq 0$.

From now on we assume that (X,x) is a surface singularity, i.e. $n = 2$. Also we assume that (X,x) is a normal singularity, i.e. X is a normal analytic surface. Recall that a resolution of (X,x) is a proper bimeromorphic map $p:\bar{X} \to X$ of a complex manifold $\bar{X}$ which maps $\bar{X}-p^{-1}(x)$ isomorphically onto $X-\{x\}$. We can always assume that $E = p^{-1}(x)$ is a union of nonsingular curves on $\bar{X}$ intersecting each other transversally. We define the genus $p_g(X,x)$ of (X,x) by putting $p_g(X,x) = \dim_{\mathbb{C}} H^1(\bar{X},\mathcal{O}_{\bar{X}})$, where $\mathcal{O}_{\bar{X}}$ is the structure sheaf of $\bar{X}$. Let $c_1(\bar{X})$ be the first Chern class of $\bar{X}$ considered as an element of $H^2(\bar{X})$ and let $b_i(E)$ be the Betti numbers of the 2-complex E . Finally, recall that (X,x) is called a Gorenstein singularity if there exists a nowhere vanishing holomorphic 2-form on $X-\{x\}$. In this case $c_1(\bar{X})$ can be represented as a linear integral combination

$\Sigma n_i[E_i]$ of the classes of irreducible components E_i of E in $H^2(\bar{X})$. The coefficients n_i can be easily found by using the adjunction formula $c_1(\bar{X})\cdot[E_i] = (E_i^2) + 2 - 2g(E_i)$ and the non-degeneracy of the matrix $A = ((E_i \cdot E_j))$. Here E_i considered as 2-cycles on $\bar{X}$ and $g(E_i)$ denotes the genus of E_i.

The next result allows to compute the signature of the Milnor lattice of a Gorenstein surface singularity:

THEOREM (3.1.1)(J.Steenbrink [44]). *Let (X,x) be a smoothable surface singularity and (t_+,t_-,t_0) be the signature of its Milnor lattice. Then*

$$t_0 = b_1(E) \ , \ t_+ = 2p_g(X,x) - b_1(E) \ .$$

Moreover, if (X,x) is a Gorenstein singularity, then

$$t_- = 10p_g(X,x) + c_1(\bar{X})^2 - b_1(E) + b_2(E) \ .$$

Notice that these formulas are applicable to any complete intersection singularity, since they are known to be Gorenstein. In this latter case the formulas are due to A.Durfee [13]. In particular, we can apply these formulas to the Milnor lattice of an isolated critical point of an analytic function in three variables as defined in [24].

Proposition (3.1.2). *The Milnor lattice of a smoothing of a Gorenstein singularity is an even lattice.*

The proof is simple. One extends a non-vanishing holomorphic 2-form on $X - \{x\}$ to a similar form on Y_t. Then the assertion easily follows from Wu's formula (cf. [1], Chap.IX, §3).

By Proposition (1.3.3) , the genus of the Milnor lattice M of a smoothing of a Gorenstein singularity is determined by its signature and the discriminant quadratic form $q_{\bar{M}}$, where $\bar{M} = M/\mathrm{Ker}(i_M)$.

Due to E.Looijenga and J.Wahl (unpublished) the latter can be computed via a resolution of (X,x). First of all, by embedding Y into $\mathbb{C}^N$ and replacing it by a sufficiently small ball centered at x , we may asume that Y_t has the same (up to a diffeomorphism) boundary δY_t for all $t \in \Delta$. In particular, we may identify δY_t with δX. Let

$$H_2(Y_t) \to H_2(Y_t,\delta Y_t) \to H_1(\delta Y_t) \to H_1(Y_t) \to H_1(Y_t,\delta Y_t) \qquad (*)$$

be the exact sequence of the pair $(Y_t,\delta Y_t)$. By duality, $H_2(Y_t,\delta Y_t) \simeq H^2(Y_t)$ and $H_1(Y_t,\delta Y_t) \simeq H^3(Y_t) = 0$ (because Y_t is a Stein space). Let $(\ )_{tors}$ denote the torsion part of an abelian group. The group

$H^2(Y_t)/H^2(Y_t)_{tors}$ is identified with the dual group $H_2(Y_t)^* = M^*$ and the first map in (*) induces the canonical map $i_M: M \to M^*$ (we assume that $t \neq 0$). It is known that $H_1(Y_t) = H_1(Y_t)_{tors}$ (see [44]). Passing to the torsion parts in (*), we obtain a complex:

$$H^2(Y_t)_{tors} \xrightarrow{\alpha} H_1(\delta Y_t)_{tors} \xrightarrow{\beta} H_1(Y_t) \quad ,$$

where the first map is injective. Now, the finite group $H_1(\delta Y_t)_{tors}$ has a natural symmetric bilinear form , the <u>linking form</u> of δY_t .

<u>Proposition</u> (3.1.3). *1) The image $Im(\alpha)$ is an isotropic subgroup ;*
2) $Ker(\beta) = (Im(\alpha))^{\perp}_{H_1(\delta Y_t)_{tors}}$;
3) The second map in (*) induces an isomorphism of finite symmetric bilinear forms $\mathcal{D}_{\bar{M}} \simeq Ker(\beta)/Im(\alpha)$.

Notice that in the case of complete intersection singularities we always have $H_1(Y_t) = 0$. Hence, $H^2(Y_t)_{tors} = 0$ and we obtain an isomorphism of finite symmetric bilinear forms $D_{\bar{M}}$ and $H_1(\delta X)_{tors}$.

The linking form on $H_1(\delta Y_t)_{tors} = H_1(\delta X)_{tors}$ can be computed via a resolution of (X,x). First of all, we identify the group $H_2(\bar{X})$ with the free abelian group R spanned by the classes of the irreducible components of E . The canonical map $i_R: R \to R^* = H^2(\bar{X})$ is given by the resolution matrix A . We identify the groups $H_i(\delta X)$ with the groups $H_i(X-\{x\}) \simeq H_i(\bar{X}-E)$. Then the kernel of the canonical map $H_1(\delta X) \to H_1(\bar{X}) \simeq H_1(E)$ is the torsion part $H_1(\delta X)_{tors}$. The composition of the duality map $H^2(\bar{X}) \to H_2(\bar{X}, \delta\bar{X})$ and the boundary map $H_2(\bar{X}, \delta\bar{X}) \to H_1(\delta\bar{X}) \simeq H_1(\delta X)$ defines a map $R^* \to H_1(\delta X)_{tors}$. It is not difficult to verify (cf. [27]) that this map induces an isomorphism

$$\phi: R^*/R \to H_1(\delta X)_{tors} \quad .$$

<u>Proposition</u> (3.1.4). *The isomorphism ϕ induces an isomorphism between the discriminant bilinear form $\mathcal{D}_R = R^*/R$ and the linking form $H_1(\delta X)_{tors}$. Moreover, if (X,x) is a Gorenstein singularity, then the map $q: R \to \mathbb{Z}$, $x \to x^2 - c_1(\bar{X}) \cdot x$, extends to a quadratic function $q: \mathcal{D}_R \to \mathbb{Q}/2\mathbb{Z}$. The associated bilinear form of q is the discriminant bilinear form b_R . The homogeneous part of q induces via the isomorphism ϕ and (3.1.3) the discriminant quadratic form $q_{\bar{M}}$.*

Notice that the lattice R is <u>not</u> an even lattice even for Gorenstein singularities (in general case).

Applying Proposition (3.1.3) and Theorem (3.1.1) , we obtain a necessary condition for smoothability of a Gorenstein singularity:

$$12\,p_g(X,x) + c_1(\bar{X})^2 - 2b_1(E) + b_2(E) \geq 1(R) \quad .$$

This allows to construct many examples of non-smoothable Gorenstein singularities.

Proposition (3.1.5). *Let (X,x) be an isolated even-dimensional hypersurface singularity . Then the following properties are true:*
(i) M is spanned by vectors v with $v^2 = -2$;
(ii) if M is definite, then $M \simeq A_n, D_n$, or E_n ;
(iii) if M is semi-definite, then $M \simeq <0> \oplus E_n$;
(iv) if $\bar{M} = M/Ker(i_M)$ is indefinite, then M contains $<0> \oplus E_6$.

The first three assertions are well-known (see [2]). The forth follows from the fact that every non-simple and non-parabolic critical point is adjacent to the parabolic point with $M = <0> \oplus E_6$ (see [3]).

It follows from the above proposition that for a hypersurface singularitry (X,x) with indefinite $\bar{M}$ we always have $rk(\bar{M}) \geq 1(\bar{M}) + 5$. Applying Corollary (1.4.3) and Theorem (1.4.7) , we obtain:

THEOREM (3.1.6) (V.Nikulin) . *The Milnor lattice M of an isolated hypersurface even-dimensional singularity is determined uniquely by its signature (t_+,t_-,t_0) and the discriminant form $D_{\bar{M}}$. If $t_+ > 1$ and $t_- > 1$, then $M \simeq U \oplus M'$ for a unique lattice M' .*

It follows from (3.1.1), (3.1.3) and (3.1.4) that the Milnor lattice of a hypersurface surface singularity is uniquely determined by the resolution data .

3.2. Smoothing of triangle and cusp singularities.

Recall that a singularity (X,x) is called quasi-homogeneous if one can choose X as a closed subvariety of $\mathbb{C}^N$ given by weighted homogeneous equations . A quasi-homogeneous surface singularity (X,x) is called a triangle singularity $D_{p,q,r}$ if there exists a resolution of (X,x) such that the exceptional curve E is a union of 4 nonsingular rational curves E_i with $E_0^2 = -1$, $E_1^2 = -p$, $E_2^2 = -q$, $E_3^2 = -r$, $E_0 \cdot E_i = 1$ $(i = 1,2,3)$, $E_i \cdot E_j = 0$ for $i \neq j \neq 0$. Here p,q, and r are natural numbers with $1/p + 1/q + 1/r < 1$.

It is known [9] that for any such triple the singularity $D_{p,q,r}$ exists and is uniquely defined . The construction of triangle singularities is reminded in [22] . There are exactly 22 triples (p,q,r) for which $D_{p,q,r}$ is a complete intersection singularity [9,

41,49].For all of them $p+q+r \leq 15$.

Let (X,x) be a $D_{p,q,r}$-singularity . One can find a $\mathbb{C}^*$-equivariant compactification $\bar{X}$ of X such that $\bar{X} - X$ is a union of $p+q+r-2$ nonsingular curves intersecting each other according to the diagram $T_{p,q,r}$ from Example (2.2.3). A smoothing $f:Y \to \Delta$ of (X,x) is called <u>good</u> if it can be extended to a flat proper map $\bar{f}:\bar{Y} \to \Delta$ such that $\bar{f}^{-1}(0) = \bar{X}$ and $\bar{f}$ induces an isomorphism $\bar{Y} - Y \simeq (\bar{X} - X) \times \Delta$ (over Δ). It can be shown that all nonsingular fibres $\bar{Y}_t$ of a good smoothing are algebraic K3-surfaces (see[41]). In particular, the irreducible components of the curve $\bar{X} - X$ span the sublattice of the lattice $L = H_2(\bar{X}) \simeq U^3 \oplus E_8^2$ (see §4) isomorphic to the lattice $T_{p,q,r}$ from (2.2.3) .

THEOREM (3.2.1)(E.Looijenga [22]) . *Let $L = U^3 \oplus E_8^2$. Suppose that there exists a primitive embedding $T_{p,q,r} \to L$. Then there exists a good smoothing of the triangle $D_{p,q,r}$-singularity.*

This is an improvement of an earlier result of H.Pinkham [41] . In fact, a more general result of E.Looijenga assumes something less than the primitivity of an embedding $T_{p,q,r} \to L$. The proof is based on the surjectivity of the period mapping for K3-surfaces and the technique of deformations of quasi-homogeneous singularities (cf. [38]) .

Obviously, the condition $p+q+r \leq 22$ is necessary for the existence of a good smoothing of a $D_{p,q,r}$-singularity (we use that the sublattice of algebraic 2-cycles on a K3-surface is of rank ≤ 20). The necessary condition of smoothability from 3.1 shows that the same is true for any smoothing (cf.[50]).

THEOREM (3.2.2). *The singularity $D_{p,q,r}$ is smoothable if and only if $p+q+r \leq 22$ and $(p,q,r) \neq (2,10,10)$.*

For $(p,q,r) \leq 21$ this immediately follows from the previous theorem and Nikulin's results (cf.[22]). A more careful application of Nikulin's results allows to do the same for the triples (p,q,r) with $p+q+r = 22$ but $(p,q,r) \neq (2,6,14),(2,10,10)$ and $(6,6,10)$ ([40]). Furthermore H.Pinkham shows in [40]that Theorem(3.2.1) (in its general form) allows to prove the smoothability of $D_{2,6,14}$ and $D_{6,6,10}$. By other methods he proves that $D_{2,10,10}$ has no smoothings at all.

Now let us turn to <u>cusp singularities.</u> A surface singularity (X,x) is called a cusp singularity if there exists a resolution of (X,x)

such that the exceptional curve E is a union of nonsingular rational curves E_i , $i = 1,\ldots,r \geq 2$, with $E_1 \cdot E_2 = \ldots = E_{r-1} \cdot E_r = E_r \cdot E_1 = 1$ and $E_i \cdot E_j = 0$ otherwise for $r \geq 2$, and E_1 intersects E_2 transversally at two points for $r = 2$. An example of a cusp singularity is a hypersurface singularity $T_{p,q,r}$: $z_1^p + z_2^q + z_3^r + z_1 z_2 z_3 = 0$, where $1/p + 1/q + 1/r < 1$.

It is known that every cusp singularity is a Gorenstein singularity. Let $\bar{E}$ be the curve obtained from E by blowing down the components of E which are exceptional curves of the first kind on $\bar{X}$. The number $-(\bar{E} \cdot \bar{E})$ is called the degree of (X,x) . The number of irreducible components of $\bar{E}$ is called the length of (X,x). We denote them by d and ℓ respectively . One checks immediately that $c_1(\bar{X}) = E$ and hence:

$$-d + \ell = c_1(\bar{X})^2 + b_2(E) \quad .$$

Also, it is easy to verify that $p_g(X,x) = 1$ (a cusp singularity is a so-called minimal elliptic singularity). Then , as it follows simply from 3.1, we obtain that any smoothable cusp singularity satisfies the inequality $d \leq \ell + 9$ (cf.[51]) . In [21] E.Looijenga conjectures that the converse is true provided that the exceptional curve of a resolution of the dual cusp (see the next section) lies on a rational surface and represents its first Chern class. He also proves that the latter condition is necessary for smoothability. In the case $\ell \leq 3$ this conjecture has been verified by the combined methods of [16] and [17] . In this case there exists a rational surface V with $c_1(V) = E'$, where E' is a curve isomorphic to the exceptional curve E of a resolution of (X,x) itself (with the same intersection matrix of the irreducible components). Let L be the sublattice of $H_2(V)$ spanned by the irreducible components of E . By direct computions one verifies that its orthogonal complement R in $H_2(V)$ is isomorphic to the lattice $T_{p,q,r}$ for some (p,q,r) .

THEOREM (3.2.3)(R.Friedman,H.Pinkham [17]) . *Suppose that R admits an embedding (not necessary primitive) into the lattice $U^2 \oplus E_8^2$ and the image of the embedding is not contained in the image of the lattice R' corresponding to another cusp singularity with $\ell \leq 3$. Then (X,x) is smoothable.*

We will explain in the next section why the condition on the dual cusp from Looijenga's conjecture implies the existence of the embedding of R . By explicit computations based on Nikulin's results in §1 it is verified that all cusp singularities with $\ell \leq 3$, $d \leq \ell + 9$ satisfy the assumptions of (3.2.3) except the cases $(-E_1^2,\ldots,-E_r^2) =$

= (4,11),(7,8),(2,4,12),(2,8,8),(3,3,12),(3,4,11),(3,7,8),(4,4,10), (4,6,8),(4,7,7),(5,5,8),(2,5,11),(4,5,9). Then it is checked which cusps from this list are smoothable. This is done by the methods of [16] or [17]. It turns out that all of them are non-smoothable except the last two cases.

3.3. The strange duality.

Let (p,q,r) be one of the following 14 triples:(2,3,7),(2,3,8), (2,3,9),(2,4,5),(2,4,6),(2,4,7),(2,5,5),(2,5,6),(3,3,4),(3,3,5), (3,3,6),(3,4,4),(3,4,5),(4,4,4). The corresponding triangle $D_{p,q,r}$-singularities are the only hypersurface triangle singularities and can be realized as the zero level sets of the 14 exceptional unimodal critical points of Arnold [3] . The Milnor lattice $M_{p,q,r}$ of these singularities was computed by A.Gabrielov [18]. It turns out that $M_{p,q,r} \simeq U \oplus T_{p',q',r'}$, where (p',q',r') is another one of the 14 triples above. The correspondence $(p,q,r) \to (p',q',r')$ is an involutive map . This obsevation was due to V.Arnold, who called it the strange duality [4]. The following explanation of this duality was found by H.Pinkham [37]and independently by V.Nikulin and myself [11].

Let $f:\bar{Y} \to \Delta$ be a good smoothing of a hypersurface singularity $D_{p,q,r}$ (which always exists). As was explained before the lattice $T_{p,q,r}$ can be embedded into the lattice $L = U^3 \oplus E_8^2$ by identifying the lattter lattice with the lattice $H_2(\bar{Y}_t)$ and the former one with the sublattice spanned by the irreducible components of the curve $C = \bar{Y}_t - Y_t$ $(t \neq 0)$. The Milnor lattice $M_{p,q,r}$ is realized as the lattice $H_2(Y_t)$. Since $H_1(Y_t) = 0$ by Milnor's result [24], the embedding $T_{p,q,r} \to L$ is primitive and $(T_{p,q,r})^{\perp}_L$ is isomorphic to $M_{p,q,r}$. By Theorem (3.1.6) there exists a unique lattice $M'_{p,q,r}$ such that $M_{p,q,r} \simeq U \oplus M'_{p,q,r}$. Computing the signature of $M'_{p,q,r}$ and its discriminant quadratic form (which must be isomorphic to $D_{T_{p,q,r}(-1)}$), we find that they are the same as ones computed for the lattice $T_{p',q',r'}$. Applying Theorem (1.4.2), we find that the lattices $M'_{p,q,r}$ and $T_{p',q',r'}$ are isomorphic. This explains the duality though the reason of the existence of an isomorphism $T_{p',q',r'} \simeq M'_{p,q,r}$ remains mysterious.

Now let us turn to the cusp duality. As was observed by I.Nakamura [28](and independently by E.Looijenga, W.Neumann and J.Wahl) the exceptional curve E of a resolution of any cusp singularity (X,x)

is isomorphic to a curve on a certain Inoue-Hirzebruch surface F (a compact analytic surface with no meromorphic nonconstant functions with the first Betti number equal to 1 and the second one is positive) . At the same time the surface F contains another curve disjoint from the first which is isomorphic to the exceptional curve E' of a resolution of another cusp singularity (X',x') (all the isomorphisms preserve the intersection matrices of the irreducible components of the curves). In [39] H.Pinkham gives a lattice-theoretical interpretation of this duality under the assumption that the dual cusps "sit on a rational surface" . The latter means that the both curves E and E' can be realized as the curves lying on rational surfaces V and V' respectively and represent the first Chern class of these surfaces. Let R (resp. R') be the orthogonal complements of the lattice S (resp. S') spanned by the irreducible components of E (resp. E') in the lattice $H_2(V)$ (resp. $H_2(V')$).

THEOREM (3.3.1). *Suppose that the lattices S and S' are primitive in $H_2(V)$ and $H_2(V')$ respectively. Then there exists a primitive embedding of R into the lattice $U^2 \oplus E_8^2$ such that the orthogonal complement of the image is isomorphic to R'* .

It follows from this theorem that, if S is primitive in $H_2(V)$, then the existence of a primitive embedding of $R = (S)^{\perp}_{H_2(V)}$ into $U^2 \oplus E_8^2$ is a necessary condition for smoothing of (X,x). We notice that the primitivity assumption is satisfied if the length of a cusp singularity is at most 5 .

4. Automorphisms of K3-surfaces.

4.1. The period mapping.

By a K3-surface we will always mean an algebraic K3-surface, i.e. a nonsingular projective surface F with $H^1(F,\mathcal{O}_F) = 0$, $c_1(F) = 0$. The base field is assumed to be the field of complex numbers $\mathbb{C}$. The following facts are standard (see, for example, [1], Chap.IX):

$H_2(F) \simeq L = U^3 \oplus E_8^2$;

$H^2(F,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, where $H^{2,0} = \mathbb{C}\omega$ for some nowhere vanishing holomorphic 2-form ω on F .

Let us fix an isomorphism $\phi: L \to H_2(F)$. The pair (F,ϕ) is called a marked K3-surface. The Hodge structure on F defines the line

$\phi_{\mathbb{C}}^*(H^{2,0})$ in $L_{\mathbb{C}}^* = L^* \otimes \mathbb{C}$, where $\phi_{\mathbb{C}}^*$ denotes the map induced by the transpose map $\phi^*: H_2(F) = H^2(F) \to L^*$ after tensoring with $\mathbb{C}$. We consider this line as a point $P(F,\phi)$ in the projective space $\mathbb{P}(L_{\mathbb{C}}^*) \simeq \simeq \mathbb{P}^{21}(\mathbb{C})$. The Hodge relations $\omega\wedge\bar{\omega} > 0$ and $\omega\wedge\omega = 0$ show that $P(F,\phi)$ belongs to the set

$$D = \{x \in L_{\mathbb{C}}^* : x\cdot x = 0 \ , \ x\cdot\bar{x} > 0\}/\mathbb{C}^* \quad .$$

Here we consider L^* as a lattice isomorphic to the lattice L (by means of the isomorphism i_L) and extend the product to $L_{\mathbb{C}}^*$.

The set D is an open subset of a 20-dimensional complex quadric hypersurface in $\mathbb{P}(L_{\mathbb{C}}^*)$ and has a natural structure of a symmetric homogeneous space isomorphic to the coset space $SO(2)\times SO(1,19)\backslash SO(3,19)$.

Let S_F denote the Picard group of F considered as the sublattice of $H_2(F)$ spanned by algebraic cycles. Let T_F be its orthogonal complement in $H_2(F)$, the transcendental lattice of F . By Hodge Index theorem

$$\text{sign}(S_F) = (1,\text{rk}(S_F)-1) \ , \quad \text{sign}(T_F) = (2,20-\text{rk}(S_F)) \ .$$

The number $\text{rk}(S_F)$ is denoted by $\rho(F)$ (the Picard number of F).

Since the value of ω on any algebraic cycle is zero, $P(F,\phi)$ lies in the subspace $\mathbb{P}((L/S_F)_{\mathbb{C}}^*)$ of $\mathbb{P}(L_{\mathbb{C}}^*)$. Conversely, if $P(F,\phi)$ lies in the subspace $\mathbb{P}((L/K)_{\mathbb{C}}^*)$ for certain sublattice K of L , then ω vanishes at $\phi(\gamma)$ for every $\gamma \in K$. By Lefschetz this implies that $K \subset S_F$.

Let M be a primitive sublattice of L of signature $(1,\text{rk}(M)-1)$. A marked K3-surface (F,ϕ) is called a marked K3-surface of type M if $\phi(M) \subset S_F$. If $M = \langle a \rangle$ and $\phi(a)$ is the class of an ample divisor, then a marked K3-surface of type M is called a marked polarized K3-surface of type a and degree a^2 . As was explained before , for any marked K3-surface (F,ϕ) of type M the point $P(F,\phi)$ belongs to the subset $D_M = D \cap \mathbb{P}((L/M)_{\mathbb{C}}^*)$ of D . This subset is a disjoint union of two copies of a bounded symmetric domain isomorphic to the coset space $SO(2)\times SO(1,19-r)^o \backslash SO(2,19-r+1)^o$, where $r = \text{rk}(M)$ and $(\ )^o$ denotes the connected component of the identity. The following fundamental result is due to I.Piatetsky-Shapiro and I.Shafarevich[36]:

THEOREM (4.1.1) (Global Torelli Theorem). *Let (F,ϕ) and (F',ϕ') be two marked polarized K3-surfaces of the same type . Then $P(F,\phi) = P(F',\phi')$ if and only if there exists a unique isomorphism $f: F \to F'$ such that $f_* \circ \phi = \phi'$.*

The next result is also fundamental and is due to various people (see the talk of A.Beauville in the same collection):

THEOREM(4.1.1)(Surjectivity of the period mapping). *Let M be a primitive sublattice of the lattice L with $sign(M) = (1, rk(M)-1)$. Then for any point $x \in \mathcal{D}_M$ there exists a marked K3-surface (F,ϕ) of type M such that $P(F,\phi) = x$.*

Since for every lattice M' strictly containing M of the same type of signature the subset $D_{M'}$ is a proper closed subset of D_M , we see that there exists a dense subset of D_M whose points correspond to marked K3-surfaces of type M with $S_F \simeq M$.

Applying Nikulin's results from §1, we obtain that the following lattices can be realized as the lattices S_F : any lattice of signature $(1,r-1)$ with $r \leq 10$, any lattice from the set F (§2, 2.2), the lattices $T_{p,q,r}$ with $p+q+r \leq 22$, $1/p + 1/r + 1/r < 1$, $(p,q,r) \neq (2,6,14),(6,6,10),(2,10,10)$ (§3, 3.2).

4.2. Automorphisms.

Let F be a K3-surface and $\sigma \in O(H_2(F))$. We say that σ preserves the period of F if $P(F,\phi) = P(F,\sigma\circ\phi)$ for some marked K3-surface (F,ϕ) . It is clear that this condition implies that $\sigma(S_F) \subset S_F$. If we also assume that σ maps the class of an ample divisor on F to the class of an ample divisor, then Theorem (4.1.1) would imply that $\sigma = g_*$ for a unique automorphism g of F . Applying the Riemann-Roch theorem we find that this condition on σ is equivalent to the condition that σ leaves the set of the classes of effective divisors invariant. Thus, we get

THEOREM (4.2.2) . *Let $r: Aut(F) \to O(H_2(F))$ be the natural representation of the automorphism group of F defined by $g \to g_*$. Then ρ is injective and its image consists of the isometries of $H_2(F)$ which preserve the period of F and the set of the classes of effective divisors.*

Now let $\mathbb{L}^{\rho-1}$ be the Lobachevsky space associated to the space $(S_F)_{\mathbb{R}}$ and the half V_F^+ of the cone

$$V_F = \{x \in (S_F)_{\mathbb{R}} : x^2 > 0\}$$

which contains the class of an ample divisor on F . We assume that $\rho > 1$, otherwise everything that follows trivializes. Let

$$R_F = \{x \in S_F : x^2 = -2\}$$

and R_F^+ be the subset of R_F consisting of the classes of nonsingular rational curves on F . Let W_F be the reflection group of S_F , the subgroup of $O(S_F)$ generated by all reflections s_e , $e \in R_F$. Let P_F be one of the fundamental polyhedra of W_F which contains the class of an ample divisor (see §2, 2.1). Let H_e be a facet of P_F . It follows from the Riemann-Roch theorem on F that e is the class of an effective divisor . Considering the irreducible components of e, we easily get

$$P_F = \{x \in V_F^+ : x \cdot e \geq 0 \quad \text{for all } e \in R_F^+\}/\mathbb{R}_+ \quad .$$

In particular, we see that $W_F = W_F^+$, the subgroup of $O(S_F)$ generated by all reflections s_e , where e belongs to R_F^+ .

Let $O(S_F)^+ = O(S_F) \cap O(\mathbb{R}_F)^+$ be the subgroup of $O(S_F)$ of index 2 which preserves the half-cone V_F^+ . Let A(F) be the subgroup of $O(S_F)^+$ of the elements which leave the polyhedron P_F invariant. As was explained above, A(F) can be characterized as the subgroup of $O(S_F)$ of the elements which preserve the set of the classes of effective divisors. It is immediately checked that W_F is a normal subgroup of $O(S_F)^+$ and

$$O(S_F)^+ = A(F) \cdot W_F \quad , \text{ the semi-direct product .}$$

Let $\alpha: O(S_F) \to O(D_{S_F})$ be the homomorphism defined in §1,1.2 . It follows from its definition that $\alpha(W_F) = \{id\}$. Let

$$\bar{\alpha}: A(F) \to O(D_{S_F})$$

be the induced homomorphism. It is clear that the image of the homomorphism $r: Aut(F) \to O(H_2(F))$ preserves the set of the classes of effective divisors . By restriction we define the homomorphism

$$r_S: Aut(F) \to A(F) \quad , \quad g \to g_*|S_F$$

THEOREM (4.2.3)([36]). *1) $Ker(r_\delta)$ is a finite group;*
2) $Im(r_\delta)$ contains the subgroup $Ker(\bar{\alpha})$.

The first assertion follows from the fact that every group of projective automorphisms of F is finite (because F does not have nontrivial holomorphic vector fields, see [1], Chap.IX). To obtain the second one, we notice that every $\sigma \in Ker(\bar{\alpha})$ can be extended to an isometry of $H_2(F)$ which acts as σ on S_F and as the identity map on T_F . In particular, it preserves the period of F and we can apply Theorem (4.2.2).

COROLLARY (4.2.4). *If* $\rho(F) = 1$, *then* $Aut(F)$ *is finite. If* $\rho(F) > 1$, *then the following properties are equivalent:*

(*i*) $Aut(F)$ *is finite;*

(*ii*) W_F *is a subgroup of finite index in* $O(S_F)$;

(*iii*) W_F *is a crystallographic group in* $\mathbb{L}^{\rho-1}$;

(*iv*) P_F *is of finite volume in* $\mathbb{L}^{\rho-1}$;

(*v*) $S_F \in F$.

To get more information on Aut(F) we consider the homomorphism

$$r_t : Aut(F) \to O(T_F) \quad , \quad g \to g_*|T_F$$

induced by the representation r . It follows from Theorem (4.2.3) that the map $r_s \times r_t : Aut(F) \to A(F) \times O(T_F)$ is injective . By integrating along trancendental cycles we get an embedding $T_F \to \mathbb{C}$. Thus, r_t is determined by the character $\chi : Aut(F) \to \mathbb{C}^*$ given by $g^*(\omega) = \chi(g)\omega$. Since $r_s(Ker(r_t))$ contains $Ker(\bar{\alpha})$, χ factors through a finite group . In particular, the image of χ is a finite cyclic group of order n .

THEOREM (4.2.5)(V.Nikulin [31,32]). *1)* $\phi(n) | rk(T_F)$, *where* ϕ *is the Euler function;*

(*ii*) $\#Ker(r_\delta) | n$ *and if a prime p divides* $\#Ker(r_\delta)$ *then* S_F *is a p-elementary lattice;*

(*iii*) *for generic F with* $S_F \simeq M$ (*in the sense of the period domain* $\mathcal{D}_M$) $r_\delta(Aut(F))$ *contains* $Ker(\bar{\alpha})$ *as a subgroup of index* ≤ 2 *and* $n \leq 2$.

Assume now that S_F is a 2-elementary lattice. Then Theorem(4.2.2) together with Proposition (1.5.1) imply that there exists a unique involution τ of F such that $\tau_*|S_F = id_{S_F}$ and $\tau_*|T_F = -id_{T_F}$. Moreover, it follows from the uniqueness of τ that τ belongs to the center of Aut(F). Let $\bar{F} = F/(\tau)$ be the quotient surface. It follows from the classification of surfaces that $\bar{F}$ is a rational surface or an Enriques surface (also $\bar{F}$ is nonsingular because $\tau^*(\omega) = -\omega$ and, hence, τ does not have isolated fixed points). In the both cases the projection $F \to \bar{F}$ is a double cover corresponding to the invertible sheaf $\omega_{\bar{F}}^{-1}$ and a section of $\omega_{\bar{F}}^{-2}$. Clearly

$$Aut(\bar{F}) \simeq Aut(F)/(\tau) \quad .$$

Applying Theorem(4.2.5) , we get a description of $Aut(\bar{F})$, where $\bar{F}$ is a "generic" Enriques surface or a "generic" rational surface with non-empty anti-bicanonical system. We refer to [5] for an independent proof of this result in the case of Enriques surfaces.

Finally, we notice that Nikulin finds all finite abelian groups which can be realized as subgroups of $\mathrm{Ker}(r_t)$ for some K3-surface F [31]. His work was extended to the case of non-abelian groups by D.Morrison and S.Mukai (unpiblished).

BIBLIOGRAPHIE

[1] Algebraic surfaces (ed. by I.SHAFAREVICH), Proc.Steklov Math. Inst., vol.75, Moscow. 1965 .

[2] V.ARNOLD - Remarks on the method of stationary phase and Coxeter numbers, Uspehi Mat.Nauk, 28:5 (1973), 17-44 .

[3] V.ARNOLD - Critical points of smooth functions and their normal forms , Uspehi Mat.Nauk, 30:5 (1975) , 3-65

[4] V.ARNOLD - Critical points of smooth functions ,Proc.Int.Congress of Math., Vancouver. 1974, 19-39 .

[5] W.BARTH and C.PETERS - Automorphisms of Enriques surfaces, Invent.Math. (to appear) .

[6] A.BOREL and HARISH-CHANDRA - Arithmetic subgroups of algebraic groups, Ann.Math., 75 (1962), 485-535 .

[7] N.BOURBAKI - Groupes et Algébres de Lie , Chap.IV-VI. Paris.1968.

[8] E.BRIESKORN - Die Milnorgitter der exzeptionellen unimodularen Singularitäten,Bonner Math.Schriften, 150 . Bonn.1983 .

[9] I.DOLGACHEV - Factor-conical singularities of complex surfaces, Funkt.Anal.i ego Priloz., 8:2 (1974), 75-76 .

[10] I.DOLGACHEV - On automorphisms of Enriques surfaces, Invent. Math. (to appear) .

[11] I.DOLGACHEV and V.NIKULIN - Exceptional singularities of V.I.Arnold and K3-surfaces, Proc.Minsk Topology Conf. Minsk. 1977 (abstract) .

[12] A.DURFEE - Bilinear and quadratic forms on torsion modules, Adv.Math., 25 (1977), 133-164 .

[13] A.DURFEE - The signature of smoothings of complex surface singularities, Math.Ann.,232 (1978), 85-98 .

[14] W.EBELING - Quadratische Formen und Monodromiegruppen von Singularittäten,Math.Ann., 255 (1981),463-498 .

[15] M.EICHLER - Quadratische Formen und Orthogonalen Gruppen . Berlin. 1952 .

[16] R.FRIEDMAN and R.MIRANDA - Smoothing cusp singularities of small length , Math.Ann., 263 (1983), 185-212 .

[17] R.FRIEDMAM and H.PINKHAM - Smoothing of cusp singularities via triangle singularities (preprint) .

[18] A.GABRIELOV - Dynkin diagrams for unimodal singularities , Funkt.Anal. i ego Priloz. 8:3 (1974), 1-6 .

[19] D.JAMES - On Witt's theorem for unimodular quadratic forms , Pacif.Journ. Math., 26 (1968), 303-316 .

[20] M.KNESER - Klassenzahlen indefiniter quadratische Formen in drei oder mehr Veränderlichen, Arch.Math.,7 (1956),323-332.

[21] E.LOOIJENGA - Rational surfaces with an anti-canonical cycle , Ann.Math., 114 (1981), 267-322 .

[22] E.LOOIJENGA - The smoothing components of a triangle singularity.I,Proc.Symp.Pure Math., vol.40 (part II),A.M.S. 1983,173--184 .

[23] J.MILNOR - On simply connected 4-manifolds, Int.Symp.Alg.Topology, Mexico City. 1958, 122-128 .

[24] J.MILNOR - Singular points of complex hypersurfaces, Ann. Math. Studies, vol.61, Princeton. 1968 .

[25] D.MORRISON - On K3-surfaces with large Picard number(preprint).

[26] S.MUKAI and Y.NAMIKAWA - On automorphisms of Enriques surfaces which act trivially on cohomology (preprint) .

[27] D.MUMFORD - The topology of normal singularities of an algebraic and a criterion of simplicity, Publ.Math.de l'IHES , 9 (1961)

[28] I.NAKAMURA - Inoue-Hirzebruch surfaces and a duality of hyperbolic unimodal surfaces.I, Math.Ann.,252(1980),221-235.

[29] Y.NAMIKAWA - On the periods of Enriques surfaces (preprint).

[30] V.NIKULIN - Integral symmetric bilinear forms and some of their applications, Izv.Akad.Nauk SSSR, Ser.Math., 43(1979),111-177.

[31] V.NIKULIN - Finite groups of automorphisms of Kahlerian surfaces of type K3 , Proc.Moscow Math.Soc., 38(1979),75-137.

[32] V.NIKULIN - On the quotient groups of automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections. Algebraic-geometrical applications. Modern Problems in Math., vol.18. Moscow. 1981, 1-114 (to be translated in J.Sov.Math.).

[33] V.NIKULIN - On classification of arithmetic groups generated by reflections in Lobachevsky spaces, Izv.Akad.Nauk SSSR, Ser. Math., 45 (1981),113-142 .

[34] V.NIKULIN - On arithmetic groups generated by reflections in Lobachevsky spaces, 44 (1980), 637-668 .

[35] V.NIKULIN - Involutions of integral quadratic forms and their applications to real algebraic geometry, Izv.Akad.Nauk SSSR, Ser.Math., 47 (1983), 109-158 .

[36] I.PIATETSKY-SHAPIRO and I.SHAFAREVICH- A Torelli theorem for algebraic surfaces of type K3,Izv.Akad.Nauk SSSR,35(1971).

[37] H.PINKHAM - Singularites exceptionelles, la dualite etrange d'Arnold et les surfaces K-3 , C.R.Acad.Sci.Paris , 284(1977), 615-618.

[38] H.PINKHAM - Deformations of algebraic varieties with $\mathbb{C}^*$-action, Asterisque, 20 (1974) .

[39] H.PINKHAM - Automorphisms of cusps and Inoue-Hirzebruch surfaces (preprint).

[40] H.PINKHAM - Smoothings of the D_{pqr} singularities ,p+q+r=22 , Proc.Symp.Pure Math., vol.40 (part II), A.M.S. 1983,373-378 .

[41] H.PINKHAM - Groupe de monodromie des singularites unimodulaires exceptionnelles, C.R.Acad.Sci.Paris, 284 (1977), 1515-1518 .

[42] A.RUDAKOV and I.SHAFAREVICH - K3-surfaces over fields of positive characteristics, the same as in [32] , pp. 115-207 .

[43] J.-P.SERRE - A Course in Arithmetic, Springer. 1970 .

[44] J.STEENBRINK - Mixed Hodge structures associated with isolated singularities,Proc.Symp.Pure Math.,40(part 2),A.M.S., 1983 , 513-536 .

[45] E.VINBERG - Discrete groups generated by reflections in Lobachevsky spaces,Mat.Sbornik,72 (1967), 471-488 .

[46] E.VINBERG - Discrete linear groups generated by reflections, Izv.Akad.Nauk SSSR , Ser.Mat., 35 (1971), 1072-1112 .

[47] E.VINBERG - The absence of crystallographic reflection groups in Lobachevsky spaces of large dimension,Funkt.Anal. i ego Priloz., 15:2 (1981), 79-80 .

[48] E.VINBERG - Some arithmetical discrete groups in Lobachevsky spaces,"Discrete subgroups of Lie groups", Oxford,1975,323-348 .

[49] P.WAGREICH - Algebras of automorphic forms with few generators, Trans.A.M.S., 262 (1980) 367-389 .

[50] J.WAHL - Smoothings of normal surface singularities, Topology , 20 (1981), 219-246 .

[51] J.WAHL - Derivations of negative weight and non-smoothability of certain singularities, Math.Ann., 258 (1982) , 383-398 .

I.Dolgachev
Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109
USA