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NEW RESULTS ON THE CLASSICAL PROBLEM OF PLATEAU ON THE EXISTENCE OF MANY SOLUTIONS

by Reinhold BÖHME

§ 1 THE PROBLEM OF PLATEAU

The notion of a classical minimal surface is not exactly defined. Generally one understands a classical minimal surface to be a two-dimensional surface of mean curvature-zero in Euclidean N-space. These "classical" surfaces need not to be embedded or immersed. However there is only one type of singularities admitted, the so called "branch points". This notion excludes certain singularities, where different pieces of minimal surfaces build up a system of surfaces intersecting "minimally" at angles of 120^0, their edges possibly meeting at angles of 109^0 ,(as discussed and classified in [47]).

One reason for the choice of this class of surfaces is its link to the theory of analytic functions of one complex variable. Namely, it is easy to show that a minimal surface $F \subset \mathbb{R}^N$ as above allows a conformal parametrization, i.e. for such F there exists a Riemann surface R (or possibly a subset $\phi \subset R$) with a fixed conformal structure and a conformal parametrization $f : \phi \to F \subset \mathbb{R}^N$.
The equation "mean curvature $\equiv 0$ in all regular points of F" implies that "$f : \phi \to \mathbb{R}^N$ is harmonic".
If $N = 2$ and f is harmonic and conformal, then f is complex analytic. Therefore, the existence theorems for minimal surfaces can be understood as a generalization of the Riemann mapping theorem. Many conjectures about minimal surfaces (on boundary behavior, on singularities, on their Jacobi fields) have arisen from the examples in the case $N = 2$. The recent work of A. Fisher and A.J. Tromba on conformal structures indicates that the methods of minimal surfaces theory will shed a new light on the classical Teichmüller theory.

The second reason for the above choice of the definition of a minimal surface are the existing existence theorems. They have their origin in the fact that the equation "mean curvature $\equiv 0$" is the Euler-Lagrange equation for the area function on the space of 2-surfaces with fixed boundary. Therefore, one can construct minimal surfaces with a minimizing procedure. Even if today there exist more general existence theorems (due to Reiffenberg, de Giorgi, Federer, Fleming, Almgren) the subsequent approach is the one where the topological type can be prescribed in advance. We refer to [3], too, for the limitations of this approach.

Theorem 1.1 (J. Douglas [12]): If $\Gamma \subset \mathbb{R}^N$ is a Jordan curve, then Γ bounds a classical minimal surface of the type of the disc, i.e. there exists a continuous parametrization $g : S^1 \to \mathbb{R}^N$ such that $g(S^1) = \Gamma$ and that the harmonic extension $x : \bar{D} \to \mathbb{R}^n$ from g to the unit disc D in $\mathbb{R}^2$ is harmonic and conformal on D and continuous on $\bar{D}$, i.e. $x(\bar{D}) \subset \mathbb{R}^N$ is a classical minimal surface.

There exists a more general theorem (J. Douglas [13]) proving the existence of minimal surfaces of higher connectivity k ($k \geq 1$) and of higher genus g, when the boundary set Γ consists of $(k+1)$ Jordan curves and the parameter domain is of genus $g \geq 1$. Such an existence theorem makes assumptions about Γ so that the infimum of the area on all surfaces bounded by Γ and of genus g is smaller then the infimum on all surfaces of a genus bounded by $g_1 < g$. (See [10], [34], [45]).

Theorem 1.2: A major achievement was the proof that - exactly as in the case of linear elliptic systems - the solutions of the Plateau problem are regular up to the boundary, i.e. the surface is smooth up to the boundary, if the boundary is smooth (H^k, $C^{k+\alpha}$, C^∞, C^ω) (H. Lewy, St. Hildebrandt, J.C.C. Nitsche, R. Hardt and L. Simon). [25, 22, 41, 19].

§ 2 BRANCH POINTS OF CLASSICAL SURFACES

The notion of a classical minimal surface ist not completely satisfactory from the point of view of differential geometry. So, a lot of work went into understanding where branch points are possible for solutions of Plateau's problem under various circumstances. A major success was a theorem of Osserman [42], with later improvements due to H.W. Alt, R. Gulliver,and Gulliver and L.D. Leslie [1, 16, 17]. We summarize

Theorem 2.1: Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve and $F = x(D)$ be one of the solutions of the classical Plateau problem, i.e. x minimizes Dirichlet's integral $\int_D (x_u^2 + x_v^2)\, du\, dv$ among all mappings in $\{H^1(D, \mathbb{R}^3) \cap C^0(\bar{D}, \mathbb{R}^3);\ x_{|\partial D}$ a parametrization of $\Gamma\}$. Then x has no interior branch points.
If Γ is a real analytic curve, then x has no branch points on the boundary either, i.e. $F = x(D)$ is a real analstic immersion of the closed disc.

The idea of the proof is easy to understand. In soap film experiments one never can observe branch points. When looking at a branched surface F with boundary Γ and with one branch point P of order $m \geq 1$ on F, then in the neighborhood of P the surface F looks locally "like" a(m+1)-fold cover of the tangent plane to F through P (which does exist). If looking for an absolutely area minimizing surface with boundary Γ, then this (m+1)-fold cover obviously is not an economic way of using the area, and with some "cutting and pasting" one can decrease the area of the surface. The question is only whether one gets again a surface of the type of the disc. These problems got resolved in the proof of theorem 2.1.

Surprisingly the theorem 2.1 depends heavily on the fact that the surface F is situated in $\mathbb{R}^3$, i.e. has codimension 1; it is wrong in $\mathbb{R}^4$. Namely, H. Federer [14] ovserved that a piece of a complex curve L in $\mathbb{C}^2$(or in $\mathbb{C}^n$) is absolutely area minimizing when the boundary ∂L is fixed, even if the surface L has branch points. Namely:

An integral current (of even dimension) in $\mathbb{C}^n$ (or in a Kähler manifold), which has a complex tangent space almost everywhere is a minimal current.

Osserman's theorem together with 1.2 gives a solution of the disc type for any Plateau problem in $\mathbb{R}^3$ (one boundary curve) which is immersed. If the boundary curve is knotted, there is no hope for the disc type solution to be embedded. But when giving up the condition of disc type there should be a better answer. It was given by R. Hardt and L. Simon [19].

Theorem 2.2: There exists an a priori bound $b(\Gamma)$ for the genus g depending only on the geometry of a C^2-curve Γ in $\mathbb{R}^3$, such that any such Γ bounds an embedded minimal surface (i.e. a minimal submanifold) of genus $g \leq b(\Gamma)$ which is absolutely area minimizing.

Theorem 2.2 is part of a much broader approach to the boundary regularity of minimal surfaces of codimension 1 in $\mathbb{R}^N$ where classical methods of minimal surface theory and the methods of geometric measure theory meet. That such a bound on the genus is not at all trivial follows from an example of W. Fleming [15], which describes a Jordan curve Γ in $\mathbb{R}^3$-rectifiable but not smooth - such that the problem of least area has no solution with a finite topological type. The estimate of Hardt and Simon is not helpful for deciding whether a specific curve Γ bounds an embedded (absolutely area minimizing) disc. But we now know a large class of curves which bound a minimally embedded disc.

Definition: A smooth Jordan curve Γ in $\mathbb{R}^3$ is called extreme, if Γ is situated on the boundary of a convex body (or more generally in a surface with everywhere non negative mean curvature).

Theorem 2.3: (Meeks-Yau, Almgren-Simon, Tromba-Tomi [2, 28, 51]):
Any extreme curve bounds at least one minimally embedded disc (which is absolutely area minimizing).

There are three very different proofs, the proof [51] not showing that the solution actually is an absolute minimum for the area.
The proof of Meeks and Yau is part of a general study of 3-manifolds, depending on Dehn's lemma and the tower construction of topology. The proof of Tomi and Tromba gives a weaker result, but is very easy. They construct the Hilbert manifold of "all" disc type immersions, use a homotopy argument in it and a closedness property of minimal embeddings due to Gulliver and Spruck [18, 52].
Clearly the class of branched surfaces is much too small to cover all singularities which are met with in soap film experiments. We only refer to the important work of J. Taylor [47, 48], and to recent work of F. Morgan [33].

§ 3 UNIQUENESS THEOREMS

Generally it is easier to prove existence (just by constructing a solution) than to show its uniqueness (you would have to look for solutions anywhere in the function space). There are two classical uniqueness theorems.

Theorem 3.1 (T. Radó): If the boundary curve $\Gamma \subset \mathbb{R}^3$ has a convex projection then the solution of Plateau's problem is unique (and a graph).

The proof depends on the maximum principle. (See e.g. [41]).

Theorem 3.2 (J.C.C. Nitsche [39]): If the boundary curve $\Gamma \subset \mathbb{R}^3$ is real analytic and the total curvature K of Γ satisfies $K \leq 4\pi$, then Γ bounds a unique immersed disc. (Probably there are no solutions of higher genus.)

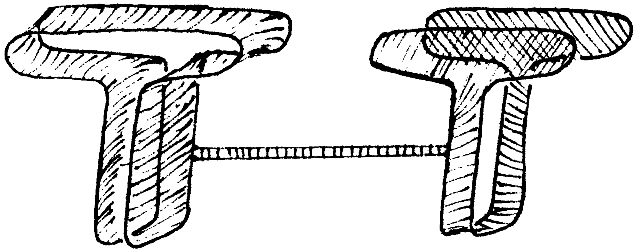
Nitsche's proof involves global methods of Morse theory. He excludes a second minimum by using the fact that then there exists an "unstable" minimal surface, which cannot exist for geometric reasons. If the total curvature is larger than 4π, then the uniqueness theorem is false [5].
In the last years serveral authors have worked on the question of generic uniqueness for the absolute minimum for Plateau's problem, while generic uniqueness in the class of all classical solutions of disc type does not hold.

Theorem 3.3 (F. Morgan - A.J. Tromba [30, 54]): In the space A of all smooth curves in $\mathbb{R}^N$, $N \geq 3$, there exists an open and dense set A_0 such that for any $\Gamma \in A_0$ there exists a unique area minimizing minimal disc. Moreover there exists a subset $A_1 \subset A$, with a complement set $A_2 = A \setminus A_1$, where the "measure of A_2" is zero, and any $\Gamma \in A_1$ bounds a unique absolutely area minimizing minimal surface of some finite genus.

The measure which is involved has to be defined using the measure induced by Brownian motion of the space A. It was introduced by F. Morgan. He works in the class of area minimizing integral currents. A boundary regularity theorem is important. The proof of Tromba makes use of the index theorem which we will discuss below. A result of interest related to 3.3 is due to L.P. de M. Jorge [23]. See also [32].
A non-existence theorem for a solution of genus 1 with a specific boundary consisting of two circles was a announced by Nitsche [40].

§ 4 PLATEAU PROBLEMS WITH MANY SOLUTIONS

Geometric intuition as well as soap film experiments can convince anyone immediately that curves Γ in $\mathbb{R}^3$ exist such that Γ bounds more than one minimal surface (which are even all locally area minimizing). The book of R. Courant contains many beautiful examples [10].



The first rigorous treatment of such an example seems to be due to Nitsche for a boundary curve lying in the Enneper surface [37, 38].
Am important means of constructing minimal surfaces bounding many solutions is the so called bridge theorem.

Theorem 4.1: It states that if Γ_1 and Γ_2 are two disjoint curves in $\mathbb{R}^N$, and if Γ results from Γ_1 and Γ_2 by joining them with a thin "bridge", if F_1 is a local minimum for the area among the surfaces bounded by Γ_1, F_2 a local minimum among the surfaces bounded by Γ_2, then suitably connecting the surfaces F_1 and F_2 by a thin "ribbon" along the bridge produces a surface F bounded by Γ which is close to a minimal surface F_* bounded by Γ. The theorem goes back to R. Courant, was in part proved by F. Kruskal [26]. Am complete proof in the classical formulation appears not to exist.
The theorem is difficult in its classical form because conformal parameters are used. The result looks much more natural in the setting of geometric meausure theory. Here a related theorem was proved by F. Almgren.
Using the bridge theorem for constructing Plateau problems with infinitely many solutions is technically difficult and apparently nobody worked it out (see [38]).

H.C. Wente proved an existence theorem [58] for a Plateau problem where the boundary curve Γ satisfies a finite symmetry group G:

Theorem 4.2: It can happen that the minimal surface F of minimal area with boundary Γ is not G-invariant. Then there can exist up to $|G|$ different, but G-equivalent minimal surfaces, and in addition one of larger area which is G-invariant. Another new existence theorem for many solutions was given by Böhme [6] based on the index theorem [8]:

Theorem 4.3: If Γ is Jordan curve in $\mathbb{R}^3$, if Γ_2 denotes its double cover, then for any $N_0 \in \mathbb{N}$ there exists a real analytic Jordan curve Γ_* close to Γ_2 which bounds at least N_0 minimal surfaces of the type of the disc. Taking $\Gamma = S^1$ as the standard circle one can show that for any $\varepsilon > 0$ there exists a real analytic Jordan curve in $\mathbb{R}^3$ of total curvature $K < 4\pi + \varepsilon$ which bounds N_0 minimal surfaces.

The proof depends on a careful bifurcation analysis where the Hilbert manifold of minimal surfaces with exactly one branch point of order one has to be studied. If the index theorem for minimal surfaces of higher genus (6.5) will be used in a similar manner, one should be able to prove the same result as above for minimal surfaces not of genus zero but of genus one.
Finally there are two striking ecamples of Plateau problems with infinitely many solutions, due to F. Morgan [31].

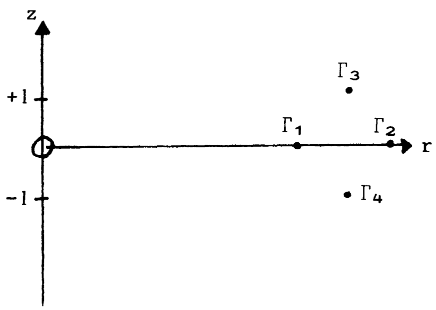
Theorem 4.4: The curve $\Gamma \subset \mathbb{R}^4$, $\Gamma = \gamma(S^1)$, $\gamma : S^1 \to \mathbb{R}^4 = \mathbb{C}^2$ being defined by $\gamma(e^{i\phi}) := (e^{i\phi}, M \cdot e^{4i\phi})$ $(M >> 0)$
bounds a continuum of (unoriented) distinct minimal surfaces.
Namely, the curve has an S^1-symmetry and the minimizing solution has not.
Even more surprising is the following example. Let's introduce cylinder coordinates (γ,ϕ,z) in $\mathbb{R}^3$. Let the boundary Γ consist of 4 circles which can be described by

$\Gamma_1 = \{r=19, z=0\}$, $\Gamma_2 = \{r=21, z=0\}$, $\Gamma_3 = \{r=20, z=1\}$ and $\Gamma_4 = \{r=20, z=-1\}$.

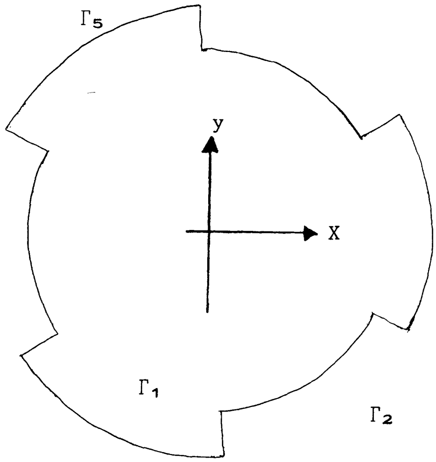
Clearly Γ has an S^1-symmetry. (The integer 20 is convenient, but arbitrary).

Theorem 4.5: There exists an increasing sequence $\{g_n\}_{n\geq 0}$ of natural numbers, such that for any g_n in this sequence, the system Γ of these four curves bounds infinitely many oriented minimal submanifolds of genus g_n.

The proof needs the construction of one such surface, which is not S^1-invariant. Define Γ_5 as indicated below (for some g_n). Solve the Plateau problem for $\Gamma^* = \Gamma_2 \cup \Gamma_5$ and for $\Gamma_* = \Gamma_4 \cup \Gamma_5$. They fit nicely together according a theorem of H. Lewy [26]. This implies 4.5.



{r,z} plane



{z = 0} plane

§ 5 FINITENESS THEOREMS

The major challenge here is the following conjecture which nobody can prove or disprove.

Conjecture: If Γ is a real analytic Jordan curve in $\mathbb{R}^3$ then Γ bounds only finitely many minimal surfaces (of the disc type).

The first result towards proving this conjecture was the following (Böhme and Tomi [9]):

Theorem 5.1: If $\Gamma \subset \mathbb{R}^N$ is a real analytic curve, then there are only finitely many critical values for the area among the surfaces of the type of the disc bounded by Γ.

The idea is to show that the area is constant on any connected component of the solution set using the theory of analytic sets. Recent work of E. Heinz [20,21] on the analycity of generalized minimal surfaces implies:

Theorem 5.2: If $\Gamma \subset \mathbb{R}^N$ is a polygon then there are only finitely many critical values for the area among all surfaces bounded by Γ and of the disc type .

The most important result is due to F. Tomi [49]:

Theorem 5.3: If $\Gamma \subset \mathbb{R}^3$ is a real analytic curve then Γ bounds only finitely many minimal surfaces which are absolutely area minimizing among disc type surfaces.

The idea of the proof is to use the theory of analytic sets and to show first that the connected component of any absolute minimum in the space of minimal surfaces bounded is a one-dimensional or zero-dimensional analytic set. Then studying not only the area of surfaces but also the volume between two such surfaces one can exclude the first possibility. Then 5.1 implies 5.3. There are later improvements made by Tomi and Beeson [50, 4]. One can use [19] in higher genus case with a result analogous to 5.3. But the conjecture above remains open.

§ 6 TOWARDS A GENERAL MORSE THEORY

A fascinating problem seems the generalization of the classical Morse theory which is by now well developped for variational integrals for functions of one

independent variable to those for functions of at least two independent variables. This is not possible today for really nonlinear variational problems, even as simple as the Plateau problem. M. Morse, J. Tompkins and M. Shiffman made an enormous effort in the early fourties in order to prove the Morse inequalities, established for geodesics in [35], for the Plateau problem, too, [36, 59, 60, 61].
Their approach did not give a complete answer due to the fact, that the variational integral of J. Douglas is not smooth, not even continuous, but only lower semicontinuous on the naturally given space $H^1(D,\mathbb{R}^N) \cap C^0(\bar{D},\mathbb{R}^N)$. The only major result of the theory is the so called wall theorem for the Plateau problem, assuring the existence of one non-minimum type surface if two distinct isolated minima are known to exist. A much simpler proof of this theorem was later given by R. Courant [10] using a finite dimensional approximation to the Plateau problem, in the spirit of the paper [35] of Morse on geodesics.
Again a different version of the wall theorem was proved by F. Almgren. It appears that a complete Morse theory modeled e.g. along the lines of Milnor's book is not feasible today. But several important steps were recently achieved.

There is an old paper of I. Marx (1955 [27]) (a former student of M. Shiffman) who announced something like a Morse theory for minimal surfaces with a polygonal boundary. His claims depend on analytic properties of a generalized Plateau problem, which recently could be verified by results of E. Heinz [20, 21]. He can embed the solutions of the classical Plateau problem for a fixed polygon Γ in a finite dimensional real analytic manifold M of "generalized minimal surfaces" such that the area is a real analytic function on M. But not all critical points of the area function on M can be accepted as minimal surfaces bounded by Γ, and one cannot see a suitable adjustment for this restriction.

Some time ago A.J. Tromba and the author started a systematic approach to the Morse theory based on the regularity theory for minimal surfaces which allows one to restrict the theory to a space of surfaces with smooth boundaries, [53].

Our main result by now [8] is the following:

Theorem 6.1: Let A denote the space of C^∞-embedding of S^1 into $\mathbb{R}^N$. Then there exists an open and dense subset $A_0 \subset A$ such that for any $\Gamma \in A_0$ the classical Plateau problem has a finite non-degenerate solution set of disc type surfaces, which depends smoothly on Γ.

Locally, i.e. near any Γ, the C^∞-topology can be replaced by some H^k-topology. The method is very different from all older work. We construct a fibre bundle η with base space A, such that for any $\alpha \in A$ we have $\eta(\alpha) = \pi^{-1}(\alpha)$ as the space of smooth disc type harmonic surfaces bounded by α. For any α let $m(\alpha) \in \eta(\alpha)$ denote the set of minimal surfaces bounded by α. Instead of studying the mapping $\alpha \to m(\alpha)$ we study the set $M := \bigcup_{\alpha \in A} m(\alpha)$. We had to make a partition

$$M = \bigcup_{\underline{\lambda},\underline{\nu}} M^{(\lambda_1,\ldots,\lambda_p)}_{(\nu,\ldots,\nu_q)} = \bigcup_{\underline{\lambda},\underline{\nu}} M^{\underline{\lambda}}_{\underline{\nu}} \quad , \text{ where}$$

any $m \in M^{\underline{\lambda}}_{\underline{\nu}}$ is a minimal surface which has exactly p interior branch points with multiplicities $\lambda_1,\ldots,\lambda_p, \lambda_j \geq 1$ in the interior, and exactly q boundary branch points with multiplicities $\nu_1,\ldots,\nu_q$, $\nu_j \geq 2$ and even. If the boundary curve is smooth there cannot exist more than finitely many branch points. We proved the subsequent theorem 6.2, from which 6.1 follows by a Sard theorem.

Theorem 6.2: If the bundle η is endowed with a suitable Hilbert space topology then the set $M^{\underline{\lambda}}_{\underline{\nu}}$ is a Hilbert submanifold of η of infinite dimension and codimension, and the natural projection $\pi : \eta \to A$ induces a projection $\pi^{\underline{\lambda}}_{\underline{\nu}} : M^{\underline{\lambda}}_{\underline{\nu}} \to A$ which is a nonlinear Fredholm operator of

$$\operatorname{index}(\pi^{\underline{\lambda}}_{\underline{\nu}}) = 2(2-n)|\lambda| + (2-n)|\underline{\nu}| + 2p + q + 3.$$

The different (connected) manifolds $M^{\lambda}_{\underline{\nu}}$ fit together like the different strata of an algebraic set.

In fact ideas from algebraic geometry are involved in this approach, since we heavily rely on the fact that the minimal surface equation on the bundle η can formally be written as a purely quadratic equation on η with values in some Hilbert space. This makes it possible to study the zeros and the linearization of this equation by methods from linear functional analysis.

The index theorem is helpful for problems in bifurcation analysis of minimal surfaces and was used for existence theorems for embedded surfaces [52] as well as for branched surfaces [5, 6]. It also makes clear a relation between the branching type $(\underline{\lambda},\underline{\nu})$ of the surface and its stability against perturbation of its boundary.

Theorem 6.3: [4, 8] Let $x \in \eta$ be a minimal surface of branching type (λ,ν). Then x cannot be stable against perturbation of the boundary in the class $M^{\lambda}_{\underline{\nu}}$ unless $N \geq 4$, $p = 0$, $q = 0$, or $N = 3$, $q = 0$, p arbitrary, but $\lambda_1 = \ldots = \lambda_p = 1$, or $N = 2$ and p,q arbitrary.

Any branched minimal surface, even if it is stable against perturbations in the sense above is a degenerate critical point for the variational integral.

This implies that any Morse theory for minimal surfaces in $\mathbb{R}^3$ must cope with degenerate critical points even in a generic situation, due to the possible branch points on the critical surfaces, but in $\mathbb{R}^4$ things get much easier. This seems strange, since absolute minima in $\mathbb{R}^3$ do never have branch points,[42], but they can have some if they are in $\mathbb{R}^4$, [14].

I would like to announce some more recent developments.

Theorem 6.4: (Tromba [56]) If $N \geq 4$ and α is a generic curve as above, such that there exist exactly m minimal surfaces (of disc type)

$u_1, \ldots, u_m$ spanning α having Morse index $\lambda_1, \ldots, \lambda_m$ respectively then

$$\sum_{i=1}^{m} (-1)^{\lambda_i} = 1.$$

The proof is based on the theory of Fredholm structures on Banach manifolds (due to Elworthy and Tromba) and the theorem is essentially a degree theoretic result. The formula above, however, is one of the Morse inequalities.

Theorem 6.5: Let R be an Riemannian surface of genus g, where $k \geq 1$ open dises are removed. So R is a bordered (riemannian) 2-manifold with a conformal structure. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ denote an ordered k-tuple of smooth Jordan curves in $\mathbb{R}^N$, and let A denote the union of all such $\underline{\alpha}$. Let $\eta(\underline{\alpha})$ denote the space of all harmonic surfaces defined on R, having as boundary in $\mathbb{R}^N$ the curves $\underline{\alpha}$, and denote

$$\eta := \bigcup_{\alpha \in A} \eta(\underline{\alpha}) .$$

Let $M_o^{\underline{\lambda}} \subset \eta$ denote the space of all minimal surfaces in η with p interior branch points of multiplicities $\lambda_1, .., \lambda_p$ respectively, and no boundary branch points. Then $M_o^{\underline{\lambda}}$ is a Hilbert submanifold of η and the projection map $\pi : \eta \to A$ induces a Fredholm map

$$\pi_o^{\underline{\lambda}} : M_o^{\underline{\lambda}} \to A \text{ of index}$$

$$\text{index } (\pi_o^{\underline{\lambda}}) = 2(2-n) \, |\underline{\lambda}| - 6g - 3k + 6,$$

reducing to the formula of 6.3 it $g = 0$ and $k = 1$.

The proof depends on the triviality of all line bundles on open surfaces and on the Riemann Roch theorem for the Schottky double of R, [7].
The formula 6.5 again can be used to prove existence theorems for surfaces of higher genus very similar to 4.3.

In order to achieve a complete index theorem for surfaces of higher genus one has to discuss what Douglas and Courant call "the variation of the conformal structure". One has to construct a smooth manifold of conformal equivalence classes of surfaces such that one can differentiate the area integral with respect to the parameters which describe the conformal structure. (see also [46]). Recent work of A. Fisher and A.J. Tromba indicates that this may ultimately be possible. Ultimately one may hope to understand the properties of the solution map for the Plateau problem with its topological properties, its conformal and singularity type, and its bifurcations when varying the boundary.

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