

SÉMINAIRE N. BOURBAKI

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Séminaire N. Bourbaki, 1981, exp. n° 559, p. 273-292

<http://www.numdam.org/item?id=SB_1979-1980__22__273_0>

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RIGIDITY AND COCYCLES FOR ERGODIC ACTIONS
OF SEMI-SIMPLE LIE GROUPS
[after G.A. Margulis and R. Zimmer]

by Harry FURSTENBERG

Introduction

Margulis' theory of rigidity of lattices of semi-simple Lie groups of $\mathbb{R}$ -rank ≥ 2 ([10], [14], [15]) has been recast by Zimmer [16] in a form that becomes meaningful for arbitrary finite-measure-preserving ergodic actions of these groups. If Γ is a lattice in a locally compact group, then G/Γ carries a finite G -invariant measure, and being homogeneous, it consists of a single G -orbit. If one replaces transitivity by ergodicity, then an ergodic action of G on a space X carrying a finite G -invariant measure may be regarded as the action on the quotient of G by a "virtual lattice". Certain theorems about lattices become meaningful in this broader context provided all the concepts involved can be translated to a form in which there is no explicit reference to the lattice itself. For example, as we shall see, homomorphisms of the subgroup Γ into a group H can be replaced by cocycles of the action of G on G/Γ with values in H . From this point of view we can summarize Zimmer's work as reinterpreting Margulis' rigidity results for homomorphisms of lattices as theorems about the cohomology of cocycles of ergodic measure-preserving actions of $\mathbb{R}$ -rank ≥ 2 groups with values in a semi-simple group. Inasmuch as such cocycles occur naturally in several contexts, Margulis' rigidity theorems have implications that are quite unrelated to lattices.

Suppose that Γ is a subgroup of an arbitrary group G , and let $\theta : G/\Gamma \rightarrow G$ be a cross-section for the natural map $G \rightarrow G/\Gamma$, so that $\theta(x)\Gamma = x$ for $x \in G/\Gamma$. For $g \in G$, $x \in G/\Gamma$, we have $\theta(gx)\Gamma = g\theta(x)\Gamma$ so that

$$(0.1) \quad \gamma(g, x) = \theta(gx)^{-1} g \theta(x)$$

takes values in Γ . The function $\gamma(g, x)$ satisfies the cocycle equation

$$(0.2) \quad \gamma(g_1 g_2, x) = \gamma(g_1, g_2 x) \gamma(g_2, x)$$

by (0.1). If we now have any homomorphism of Γ into a third group H , $\rho : \Gamma \rightarrow H$, then $\sigma(g, x) = \rho(\gamma(g, x))$ is a map from $G \times G/\Gamma \rightarrow H$ satisfying the cocycle equation. Let us say that two cocycles σ, σ' with values in H are equivalent (cohomologous)

if there is a map $\psi : G/\Gamma \rightarrow H$ with

$$(O.3) \quad \sigma'(g, x) = \psi(gx)^{-1} \sigma(g, x) \psi(x) .$$

We see that different choices of the cross-section θ lead to equivalent cocycles. Conversely, suppose that $\sigma : G \times G/\Gamma \rightarrow H$ is a cocycle with values in the group H . Restricting σ to $\Gamma \times \{x_0\}$, $x_0 = (\Gamma)$, we obtain a homomorphism $\rho(\gamma) = \sigma(\gamma, x_0)$ of Γ to H . Moreover, equivalent cocycles lead to equivalent homomorphisms, where we say that two homomorphisms ρ, ρ' are equivalent if $\rho'(\gamma) = h_0^{-1} \rho(\gamma) h_0$, $h_0 \in H$. We thus find a bijection

$$(O.4) \quad \{\text{cocycles } \sigma : G \times G/\Gamma \rightarrow H\} / \text{cohomology} \longleftrightarrow \{\text{homomorphisms } \rho : \Gamma \rightarrow H\} / \text{equivalence} .$$

In this bijection it may be seen that equivalence classes containing cocycles $\sigma(g, x)$ which are independent of x , so that $\sigma(g, x) = \sigma(g)$, $\sigma(g_1 g_2) = \sigma(g_1) \sigma(g_2)$, correspond to homomorphisms of Γ which are extendable to homomorphisms of G .

Taking this into account we find that a natural framework for rigidity theorems is the cohomology theory of cocycles $\sigma : G \times X \rightarrow H$, where G is a semi-simple Lie group and X is a measure space on which G acts ergodically, preserving a finite measure, and H is another topological group. Under certain conditions we shall be able to show that a cocycle σ is cohomologous to one which is independent of x , that is to say, one which corresponds to a homomorphism of $G \rightarrow H$. This may then be regarded as a "rigidity" theorem, implying in special cases that a homomorphism of the lattice Γ extends to G .

In addition to cocycles that occur when X is a homogeneous space, one also encounters cocycles for other measurable actions. For example suppose we have measurable actions of two groups G and H on two measure spaces X and Y , respectively. Suppose that H acts freely so that $h \neq \text{identity} \Rightarrow hy \neq y$ for every $y \in Y$. Assume there is an orbit preserving map of X to Y , i.e., $\pi : X \rightarrow Y$, with $\pi(Gx) = H\pi(x)$ for $x \in X$. Then $\pi(gx) = \sigma(g, x)\pi(x)$ defines a unique $\sigma(g, x)$ and it is easily checked that σ satisfies the cocycle equation. The same cohomology phenomenon that leads to Margulis' rigidity theorems now yields the following result of Zimmer.

THEOREM O.1.- Let G and H be a semi-simple Lie group with trivial centers and assume $\text{IR-rank } G \geq 2$. Assume that G acts freely on a measure space X preserving a finite measure and that every simple component of G acts ergodically on X . Let H act freely on a measure space Y preserving a finite measure and assume that there exists a 1-1 measurable orbit preserving map of X into Y . There is then a homomorphism of $G \rightarrow H$ such that the resulting action of G on Y is conjugate to the action of G on X .

Actions of G on two spaces X and Y are conjugate if there is a 1-1 map $\pi : X \rightarrow Y$ with $\pi(gx) = g\pi(x)$. Zimmer's result is particularly striking if one

bears in mind that in the category of amenable group actions matters are quite different. By [2] and [4] any two ergodic, measure-preserving free actions of any two infinite discrete amenable groups are orbit equivalent. On the other hand even for the same group, two such actions need not be conjugate. For example, two $\mathbb{Z}$ -actions of different entropy or different spectral characteristics cannot be conjugate.

Another example of a cocycle for a measurable action arises in considering smooth actions of a group G on a manifold X . If we partition X measurably into coordinate patches then we can identify the tangent space $T(X)$ with $X \times \mathbb{R}^m$, $m = \dim X$. For each $g \in G$, the differential dg defines a map of $T(X)_x \rightarrow T(X)_{gx}$ which is a linear transformation $\sigma(g, x) \in GL_m(\mathbb{R})$. Again it is easily verified that σ satisfies the cocycle equation, and we shall refer to it as the tangent cocycle. Under suitable hypotheses we shall be able to obtain information about the cohomology class of this cocycle. In § 8 we formulate an "entropy rigidity" theorem for smooth actions which appears as a consequence of these considerations.

This report is based on Zimmer [16]. There are some modifications which enable one to obtain slightly sharper results. Thus there are some implications for cocycles of semi-simple groups having $\mathbb{R}$ -rank one. Also we study general matrix valued cocycles without assuming the range is Zariski dense in a semi-simple Lie group. A more detailed exposition is currently in preparation.

§ 1. Cocycles of measurable actions

Let $(X, \mathcal{B}, \mu)$ denote a probability space; i.e., X is an abstract space, $\mathcal{B}$ is a σ -algebra of subsets of X , and μ is a σ -additive non-negative measure on $\mathcal{B}$ with $\mu(X) = 1$. When we speak of the measure space X we have in mind the probability triple. If G is a topological group, the Borel sets of G constitute the measurable sets of G . We say G acts on X if there is a map $G \times X \rightarrow X$, $(g, x) \rightarrow gx$, which is jointly measurable and which satisfies $(g_1 g_2)x = g_1(g_2 x)$, $ex = x$. We suppose furthermore that for $A, B \in \mathcal{B}$, $\mu(A \cap g^{-1}B)$ is a continuous function of $g \in G$. We also say that X is a measurable G -space. The action is measure-preserving if $\mu(g^{-1}A) = \mu(A)$ for $g \in G$ and $A \in \mathcal{B}$, and it is ergodic if $gA = A$ for all $g \in G$ (or, equivalently, for a dense set of g) implies $\mu(A) = 0$ or 1 . In the sequel X will always denote a measurable G -space.

Throughout our discussion H will denote an algebraically closed real linear group, for example, a connected Lie group with trivial center. Let $\mathcal{L}(X, H)$ denote the space of measurable functions from X to $H \subset GL_m(\mathbb{R}) \subset \mathbb{R}^{m^2}$. On $\mathcal{L}(X, H)$ one can introduce the topology of convergence in measure:

$$U_\epsilon(f) = \{f' \in \mathcal{L}(X, H) : \mu\{x : \|f(x) - f'(x)\| > \epsilon\} < \epsilon\}.$$

A measurable function $\sigma : G \times X \rightarrow H$ will be called a cocycle if the map of

$G \rightarrow \mathcal{Z}(X, H)$ given by $g \rightarrow \sigma(g, \cdot)$ is continuous and

$$(1.1) \quad \sigma(g_1 g_2, x) = \sigma(g_1, g_2 x) \sigma(g_2, x)$$

for every $g_1, g_2 \in G$ and a.e. x . In practice G will have a countable dense subgroup and on account of the continuity hypothesis it doesn't matter whether the exceptional set of x depends on g_1, g_2 or not.

Two cocycles σ, σ' are equivalent (or cohomologous) if there exists a measurable map $\psi : X \rightarrow H$ with

$$(1.2) \quad \sigma'(g, x) = \psi(gx)^{-1} \sigma(g, x) \psi(x).$$

Since H will usually not be commutative, the cohomology classes do not form a group.

Let M be a topological H -space (i.e. we have a continuous map $H \times M \rightarrow M$ with $(h_1 h_2)u = h_1(h_2 u)$, $eu = u$) or a measurable H -space, and suppose that σ is a cocycle on $G \times X$ with values in H . If we define

$$(1.3) \quad g(x, u) = (gx, \sigma(g, x)u), \quad g \in G, \quad x \in X, \quad u \in M,$$

then

$$(1.4) \quad (g_1 g_2)(x, u) = (g_1 g_2 x, \sigma(g_1, g_2 x) \sigma(g_2, x) u) = g_1(g_2(x, u))$$

by the cocycle equation. Hence we obtain an action of G on $X \times M$. This action is sometimes called the skew product action of G on $X \times M$, and we denote the latter space by $X \times_G M$ to specify the action of G on it. $X \times_G M$ is an extension of X in the sense that there is a morphism $\pi : X \times_G M \rightarrow X$ (or a G -equivariant map) commuting with the action of G . Note that equivalent cocycles define essentially isomorphic extensions in the sense of a commuting diagram of G -equivariant maps :

$$\begin{array}{ccc} X \times_G M & \xleftrightarrow{\quad} & X \times_{\sigma'} M \\ & \searrow & \swarrow \\ & X & \end{array}$$

§ 2. Cocycles for groups with property T

Let G be a group with property T ([3], [8]). This means that there exists a compact subset $Q \subset G$ and an $\varepsilon > 0$, such that if $R : G \rightarrow \text{Aut}(\mathcal{H})$ is any strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}$ for which some vector $u \in \mathcal{H}$, $\|u\| = 1$, satisfies $\|R(g)u - u\| < \varepsilon$ for all $g \in Q$, then there exists $v \in \mathcal{H}$, $v \neq 0$, with $R(g)v = v$ for all $g \in G$. A semi-simple group all of whose simple factors have $\mathbb{R}$ -rank ≥ 2 has property T. The real and complex hyperbolic groups do not have property T and an amenable group has property T iff it is compact.

We can illustrate the program of studying the cohomology of cocycles for measure preserving actions with the following simple result. It generalizes the fact that if Γ is a lattice in a group with property T then $\Gamma/[\Gamma, \Gamma]$ is finite.

PROPOSITION 2.1.- Assume that G has property T and that the action of G on X is measure preserving and ergodic. Let σ be a cocycle on $G \times X$ with values in the positive reals. Then $\sigma \sim 1$.

Proof.- Let $Q \subset G$ and $\varepsilon > 0$ be given as above. Define a representation of G on $L^2(X \times \mathbb{R})$ by

$$(2.1) \quad R(g^{-1})f(x,t) = \sigma(g,x)^{it} f(gx,t) .$$

If we choose $u_\delta \in L^2(X \times \mathbb{R})$ to be constant for $|t| \leq \delta$ and $u_\delta = 0$ otherwise, it can be seen that $\|R(g)u_\delta - u_\delta\| < \varepsilon$ for $g \in Q$ as soon as δ is sufficiently small. This implies the existence of some $v(x,t)$, measurable on $X \times \mathbb{R}$, with

$$(2.2) \quad \sigma(g,x)^{it} v(gx,t) = v(x,t) .$$

For each t , by ergodicity of the G -action $|v(x,t)|$ is a constant and since $v \not\equiv 0$ we can take $|v(x,t)| = 1$ for a set Δ of t with $m(\Delta) > 0$. Also by ergodicity, for any fixed t , two solutions of (2.2) are proportional, and so the function $v(x,t)\overline{v(y,t)}$ on $X \times X$ is uniquely determined by (2.2). Next we see that we can solve (2.2) for $t \in \Delta - \Delta$ and hence for all t by taking quotients and products of $v(x,t)$, $t \in \Delta$. By measurability we have a measurable function $V(x,t)$ defined for all t , $|V(x,t)| = 1$ with (2.2) satisfied for every t and for a.e. x . By uniqueness,

$$(2.3) \quad V(x,t)\overline{V(y,t)} \cdot V(x,s)\overline{V(y,s)} = V(x,t+s)\overline{V(y,t+s)}$$

which now holds for a.e. x,y and for a.e. t,s . But this implies that

$$V(x,t)V(y,t) = e^{ip(x,y)t}$$

with $p(x,y)$ uniquely determined. From $p(x,y) + p(y,z) = p(x,z)$ we deduce that $p(x,y) = k(x) - k(y) + c$ which implies that

$$V(x,t) = \omega(t)e^{-ik(x)t} ,$$

and finally that

$$\sigma(g,x) = e^{ik(gx)} / e^{ik(x)} .$$

The following corollary will be used in § 8.

COROLLARY.- Let G have property T and suppose that the action of G on X is measure preserving and ergodic. Let σ be a cocycle on $G \times X$ with values in $GL_m(\mathbb{R})$ then σ is equivalent to a cocycle the values of which are matrices of determinant ± 1 .

§ 3. Minimal and proximal cocycles

If M is a compact metric H -space we can consider various "dynamical" properties of the action. We say the action is minimal if all H -orbits on M are dense. This is equivalent to requiring that no proper closed subset of M may be H -invariant - hence, the term "minimal". Another property is proximality; we say H acts proximally on M if for every $u, v \in M$ there exists a sequence $h_n \in H$ with the distance $d(h_n u, h_n v) \rightarrow 0$. Equivalently, H acts proximally if every H -invariant subset of $M \times M$ meets the diagonal $\Delta_M \subset M \times M$. We shall now extend these notions to cocycles of a G -action on X with values in H . We now have an action of G on the (measurable $\times$ topological) space $X \times_G M$ and minimality and proximality correspond to behavior of this action relative to the base X . We assume throughout that the action of G on X is ergodic.

In order to define minimality and proximality we need the analogue of a closed invariant set for a skew product action. We shall denote by 2^M the compact metric space of closed subsets of M endowed with the Hausdorff metric. We shall be concerned with subsets of $X \times_G M$ whose fibre over each point $x \in X$ corresponds to a closed subset of M .

DEFINITION.- A measurable function $\Phi : X \rightarrow 2^M$ is said to be σ -invariant if $\sigma(g, x)\Phi(x) = \Phi(gx)$, $g \in G$, and a.e. $x \in X$.

If Φ is a σ -invariant map then the set

$$X_\Phi = \bigcup_{x \in X} x \times \Phi(x)$$

is G -invariant in $X \times_G M$. We write $\Phi_1 \leq \Phi_2$ if $\Phi_1(x) \subset \Phi_2(x)$ a.e.

DEFINITION.- A σ -invariant map $\Phi : X \rightarrow 2^M$ is minimal if $\Phi_1 \leq \Phi$ and Φ_1 σ -invariant $\Rightarrow \Phi_1 = \Phi$.

The following lemma will be useful.

Lemma 3.1.- If Φ is any σ -invariant map $\Phi : X \rightarrow 2^M$, there exists a minimal Φ_1 which is σ -invariant and with $\Phi_1 \leq \Phi$.

Proof.- One uses Zorn's lemma and the separability of the metric space $\mathcal{L}(X, 2^M)$.

Lemma 3.2.- The map $\Phi(x) \equiv M$ is minimal $\Leftrightarrow$ for every $A \subset X$ with $\mu(A) > 0$ and measurable function $u : X \rightarrow 2^M$ and open set $V \subset M$ there exists $g \in G$, $x \in A$ with $gx \in A$ and $\sigma(g, x)u(x) \in V$.

Proof.- Suppose Φ is not minimal and $\Phi_1 \leq \Phi$ is σ -invariant, and that for a set A of positive measure $\Phi_1(x) \neq M$. Restricting A we can find an open set $V \subset M$ with $\Phi_1(x) \cap V = \emptyset$ for $x \in A$. Choose $u(x) \in \Phi_1(x)$, then $\sigma(g, x)u(x) \in \Phi_1(gx)$ so we cannot have $\sigma(g, x)u(x) \in V$ for $gx \in A$. Conversely, suppose Φ is minimal and A , $u(x)$, V are given, and suppose $\sigma(g, x)u(x) \notin V$ if $x, gx \in A$. Set

$$(3.1) \quad \Phi_1(x) = M \setminus \bigcup_{gx \in A} \sigma(g, x)^{-1}V$$

this being measurable since by our hypotheses on cocycles we can replace the union in (3.1) by a countable union. We check that Φ_1 is σ -invariant and $\Phi_1(x) \neq \emptyset$ for $x \in A$. By ergodicity of the action of G on X , $\Phi_1(x) \neq \emptyset$ a.e. so that Φ_1 is well defined. This contradicts the minimality of Φ .

When the map $\Phi(x) \equiv M$ is minimal we say that σ is minimal on M .

DEFINITION.- A σ -invariant map Φ is proximal if $\Phi_1 \leq \Phi \times \Phi$ where $\Phi_1 : X \rightarrow 2^{M \times M}$ and Φ_1 σ -invariant implies $\Phi_1(x) \cap \Delta_M \neq \emptyset$ for a.e. x .

The proof of the following is analogous to that of lemma 3.2.

Lemma 3.3.- The map $\Phi(x) \equiv M$ is proximal $\iff$ for every $A \subset X$ with $\mu(A) > 0$ and measurable functions $u, v : X \rightarrow 2^M$, and $\varepsilon > 0$, there exists $g \in G$, $x \in A$ with $gx \in A$ and $d(\sigma(g, x)u(x), \sigma(g, x)v(x)) < \varepsilon$.

Again, we say in this case that σ is proximal on M . With either characterization it is easy to see that if σ is minimal or proximal on M the same is true of any equivalent cocycle.

§ 4. Boundaries of Lie groups

Let G be a topological group and let B be a compact G -space. We say B is a projective G -space if there is a representation of G into $\text{PGL}_m(\mathbb{R})$ and an injection of B into $P^{m-1}(\mathbb{R})$ so that the action of G on B comes from the action of $\text{PGL}_m(\mathbb{R})$ on $P^{m-1}(\mathbb{R})$.

DEFINITION.- If B is a compact G -space we say the action is strongly proximal if for any probability measure π on M there exists a sequence $g_n \in G$ with $g_n \pi \rightarrow$ point measure.

The following is not difficult to prove.

Lemma 4.1.- For projective G -spaces, proximality and strong proximality are equivalent.

Recall that a group P is amenable if whenever P acts by affine transformations on a compact convex set, there exists a fixed point. Equivalently, when P acts by homeomorphisms on a compact space, there exists an invariant measure.

PROPOSITION 4.2.- If G is a connected Lie group, there exists a closed subgroup, unique up to conjugacy, satisfying

- (i) P is amenable,
- (ii) G/P is compact,
- (iii) the action of G on G/P is strongly proximal.

This is proved in [6]. We make some remarks regarding the proposition. Suppose P satisfies (i), (ii), and (iii) and suppose S is any normal amenable subgroup of G .

S has a fixed measure on G/P and since S is normal all the translates by G of this measure are S -invariant. By (iii) it follows that S has a fixed point and so all the points of G/P are S -invariant. It follows that the space G/P will be the same for G as for $G/\text{radical } G$. In the semi-simple case we write $G = KAN$, the Iwasawa decomposition of G . Then if $M = \text{normalizer of } AN \text{ in } K$, $P = MAN$ satisfies the conditions of the proposition ([11]). We can readily see that P is unique up to conjugacy. For let P' also satisfy (i), (ii), and (iii). P' has an invariant measure λ on G/P and since G/P' is compact, $G\lambda$ is compact. By (iii) for G/P , there is a point measure in the closure of $G\lambda$; hence λ is a point measure. This proves that $P' \subset \text{conjugate of } P$. The same argument shows that λ is unique, so the conjugate is unique, and reversing the argument we see that $P' = \text{conjugate of } P$.

We denote this uniquely determined space by $B(G)$. P is a minimal parabolic subgroup of G and it is known that G/P is a projective G -space corresponding to an irreducible representation of G ([11]). $B(G)$ reduces to a point iff G is amenable. $B(G)$ can be characterized either as the universal strongly proximal, minimal G -space or as the universal projective, proximal, minimal G -space.

PROPOSITION 4.3.- If B' is either a strongly proximal, minimal compact G -space or a projective, proximal, minimal compact G -space, then B' is an equivariant image of $B(G)$.

The proof is the same as the argument regarding the uniqueness of P . Minimality is used to conclude that the map $B(G) \rightarrow B'$ is onto.

Now suppose that X is a measurable G -space and let $\sigma_0 : G \times X \rightarrow G$ be the cocycle $\sigma_0(g, x) \equiv g$. $B(G)$ is clearly a proximal, minimal G -space. We claim that as a cocycle on $G \times X$, σ_0 is proximal and minimal on $B(G)$, provided the action of G on X is measure preserving.

PROPOSITION 4.4.- Assume the action of G on X is measure preserving. Then the cocycle σ_0 is proximal and minimal on $B(G)$.

Proof.- We prove minimality, the proof of proximality being similar. Let $\Phi : X \rightarrow 2^{B(G)}$ be a σ_0 -invariant map. Consider all probability measure supported on the set X_Φ and which project onto μ under the map $(x, y) \mapsto x$. This is a non-empty convex set and it can also be seen to be compact, passing to a compact model of X for which Φ is continuous. Since μ is G -invariant, the action of G on X_Φ takes this set of measures to itself. The amenable subgroup P has a fixed measure in this set, say λ . Projecting λ to $B(G)$ gives a P -invariant measure on $B(G)$. But we have already seen that there is a unique P -invariant measure on $B(G)$, namely δ_{u_0} where $u_0 = (P)$. Hence $\Phi(x) \ni u_0$ for a.e. x and so $\Phi(gx) \ni gu_0$ for a.e. x . We conclude that $\Phi(x) = B(G)$ and so $B(G)$ is minimal.

Taking $X = G/\Gamma$ for a lattice Γ in G we can conclude from proposition 4.4 that Γ acts minimally and proximally on $B(G)$.

DEFINITION.- If X is a measurable G -space, M a topological H -space and $\sigma : G \times X \rightarrow H$ a cocycle, we say that a measurable map $\psi : X \rightarrow M$ is σ -invariant if $\psi(gx) = \sigma(g, x)\psi(x)$.

The following theorem is a basic tool for studying cocycles with values in non-amenable Lie groups.

THEOREM 4.5.- Let G be a Lie group and assume that the action of G on X is measure preserving and ergodic. Let σ be a cocycle on $G \times X$ with values in H and let M be a projective H -space. Let $\Phi : X \rightarrow 2^M$ be a proximal σ -invariant map. Then there exists a unique measurable σ -invariant map $\psi : X \times B(G) \rightarrow M$ with $\psi(x, u) \in \Phi(x)$, $x \in X$, $u \in B(G)$.

For the special case $X = G/\Gamma$, the foregoing theorem has the following corollary.

COROLLARY.- Let Γ be a lattice in the Lie group G and let $\tau : \Gamma \rightarrow \text{PGL}_m(\mathbb{R})$ be a representation of Γ for which the range $\tau(\Gamma)$ acts proximally on a compact $M \subset \mathbb{P}^{m-1}$. Then there exists a measurable equivariant map $\alpha : B(G) \rightarrow M$; namely α satisfies $\alpha(\gamma u) = \tau(\gamma)\alpha(u)$ for $\gamma \in \Gamma$, $u \in B(G)$.

Proof of corollary.- Let $\theta : G/\Gamma \rightarrow G$ be a Borel cross-section so that $\gamma(g, x) = \theta(gx)^{-1}g\theta(x) \in \Gamma$. Set $\sigma(g, x) = \tau(\gamma(g, x))$ so that σ is a cocycle on $G \times G/\Gamma$ to H . Apply the theorem to σ with $\Phi(x) \equiv M$ to obtain a measurable σ -invariant map $\psi : G/\Gamma \times B(G) \rightarrow M$. For a.e. $g_0 \in G$, $\psi(g_0\Gamma, u)$ is defined for a.e. $u \in B(G)$ and

$$\psi(g_0\Gamma, gu) = \sigma(g, g_0\Gamma)\psi(g_0\Gamma, u)$$

holds for all $g \in g_0\Gamma g_0^{-1}$. It is readily seen that for appropriate γ_0 the map $\alpha(u) = \tau(\gamma_0)\psi(g_0\Gamma, g_0u)$ satisfies the conclusion of the corollary.

For the proof of the theorem 4.5 we need two lemmas. We denote by $\mathcal{P}(F)$ the compact metrizable space of probability measure on a compact metric space F .

Lemma 4.6.- Let G be a Lie group and assume the action of G on X is measure preserving. Let $\sigma : G \times X \rightarrow H$ be a cocycle and let M be any compact metric H -space, and $\Phi : X \rightarrow 2^M$ a σ -invariant map. Then there exists a σ -invariant measurable map $\psi : X \times B(G) \rightarrow \mathcal{P}(M)$ with $\psi(x, u) \in \mathcal{P}(\Phi(x))$ for a.e. $x \in X$, $u \in B(G)$.

Proof.- Consider the set $\mathcal{S}$ of all probability measures on X_Φ projecting to μ on X . As in the proof of prop. 4.4, let $\lambda \in \mathcal{S}$ be P -invariant. If we disintegrate $\lambda = \int \delta_x \times \lambda_x d\mu(x)$ so that $x \mapsto \lambda_x$ is defined a.e. from $X \rightarrow \mathcal{P}(M)$, we will have $\lambda_x \in \mathcal{P}(\Phi(x))$. $p\lambda = \lambda$ for $p \in P$ implies

$$(4.1) \quad \sigma(p, x)\lambda_x = \lambda_{px}.$$

Let $\psi(x, g_1p) = \sigma(g_1, g_1^{-1}x)\lambda_{g_1^{-1}x}$. This is well defined on account of (4.1), and

$$\psi(gx, gg_1P) = \sigma(gg_1, g_1^{-1}x) \lambda_{g_1^{-1}x} = \sigma(g, x) \sigma(g_1, g_1^{-1}x) \lambda_{g_1^{-1}x} = \sigma(g, x) \psi(x, g_1P) .$$

this completes the proof.

In the next lemma M is assumed to be a projective H -space and we furthermore assume the action on X is ergodic.

Lemma 4.7.- If $\psi : X \times B(G) \rightarrow \mathcal{P}(M)$ is σ -invariant and $\psi(x, u) \in \mathcal{P}(\Phi(x))$ where Φ is a proximal σ -invariant map $\Phi : X \rightarrow 2^M$, then for a.e. $(x, u) \in X \times B(G)$, $\psi(x, u)$ is a point measure.

Clearly these two lemmas imply theorem 4.5. A sketch of the proof of lemma 4.7 is given in an appendix. The idea of the argument, which is based on probabilistic considerations, is that "in general" $\sigma(g, x)$ tends to contract $\psi(x, u)$ to a point measure, so a solution to

$$\psi(gx, gu) = \sigma(g, x) \psi(x, u)$$

for all g, x, u implies $\psi(x, u)$ is a point measure.

§ 5. Examples of proximal cocycles

1. Orbit equivalence

Let X be an ergodic measurable G -space and let H be a Lie group acting on Y and preserving measure on Y . Suppose that the action of H on Y is free and that the two actions are orbit equivalent so that there exists an invertible measurable map $\pi : X \rightarrow Y$ with $\pi(Gx) = H\pi(x)$. Define the cocycle $\sigma : G \times X \rightarrow H$ by $\pi(gx) = \sigma(g, x)\pi(x)$.

PROPOSITION 5.1.- σ is minimal and proximal on $B(H)$.

Proof.- We prove proximality, the proof of minimality being similar. Suppose $A \subset X$, $\mu(A) > 0$, and $u(x), v(x)$ are measurable functions $u, v : X \rightarrow B(H)$. For $\varepsilon > 0$ we want to find $x \in A$, $g \in G$ with $gx \in A$ and $d(\sigma(g, x)u(x), \sigma(g, y)v(y)) < \varepsilon$. By prop. 4.4, the identity cocycle $H \times Y \rightarrow H$ is proximal on $B(H)$. So $\exists h \in H$, $y \in \pi(A)$ with $hy \in \pi(A)$ and $d(hu\pi^{-1}(y), hv\pi^{-1}(y)) < \varepsilon$. Let $x = \pi^{-1}y$ and let g satisfy $\pi(gx) = hy$ and we obtain the desired inequality.

2. Linear cocycles

We now consider cocycles with values in $PGL_m(\mathbb{R})$. We shall denote by $\mathcal{G}_{m,r}$ the Grassman variety of r -dimensional subspaces of $\mathbb{R}^m$.

DEFINITION.- $\sigma : G \times X \rightarrow PGL_m(\mathbb{R})$ is reducible if for some r , $0 < r < m$, there exists a measurable σ -invariant map $\Phi : X \rightarrow \mathcal{G}_{m,r}$.

A finite union of linear subvarieties of P^{m-1} will be called a quasi-linear variety. The following is the analogue for cocycles of reducibility of a representation

on a subgroup of finite index.

DEFINITION.- $\sigma : G \times X \rightarrow \text{PGL}_m(\mathbb{R})$ is FI-reducible if there exists a measurable σ -invariant map $\Phi : X \rightarrow 2^{\mathbb{P}^{m-1}}$ with $\Phi(x) = \text{quasi-linear variety} \neq \mathbb{P}^{m-1}$.

It is easy to show that if σ is FI-reducible there exists a finite extension $\tilde{X} \rightarrow X$ and a compatible G -action on $\tilde{X}$ so that the lifted cocycle $\sigma(g, \tilde{x})$ is reducible. If σ is not FI-reducible we say it is FI-irreducible.

DEFINITION.- $\sigma : G \times X \rightarrow H$ is compact if it is equivalent to a cocycle with values in a compact subgroup of H .

It can be shown that if H is semi-simple and $\sigma(g, x)$ is uniformly bounded then σ is compact.

Lemma 5.2.- Let L be any subgroup of $\text{PGL}_m(\mathbb{R})$. Either L is contained in a compact subgroup, or L has a subgroup of finite index which is reducible, or $\exists r$, $0 < r < m$ and L -invariant closed subset $M \subset \mathcal{G}_{m,r}$ so that L acts proximally on M .

Proof.- One can define quasi-projective transformations of $\mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-1}$ as pointwise limits of projective transformations. If τ is a quasi-projective transformation (q.p.t.) it can be described as follows. One has a sequence of subspaces $\mathbb{R}^m = W_0 \supsetneq W_1 \supsetneq \dots \supsetneq W_{k-1} \supsetneq \{0\}$ and linear transformations $A_i : W_i \rightarrow \mathbb{R}^m$ with $W_{i+1} = \text{Ker } A_i$, $i = 0, 1, \dots, k-1$. If $\bar{W}_i \subset \mathbb{P}^{m-1}$ is the corresponding linear subvariety, then $\tau|_{\bar{W}_i \setminus \bar{W}_{i+1}}$ is the transformation induced by A_i . The set of all q.p.t. form a semigroup which is compact in the topology of pointwise convergence. Form the closure $\bar{L}$ in this semigroup. $\bar{L}$ is again a semigroup (the enveloping semigroup of L). If no non-singular elements occur in $\bar{L}$, then $\bar{L}$ is a compact subgroup of $\text{PGL}_m(\mathbb{R})$. Otherwise consider the elements $\tau \in \bar{L}$ for which $\dim W_0 - \dim W_1 = \dim A_0(W_0)$ is minimal. Let r be this dimension, and let M be the set of r -dimensional subspaces of the form $A_0(W_0)$ obtained in this way. It is easily seen that this is a closed L -invariant set.

Assume now that no subgroup of finite index of L is reducible. This implies that if U_1, U_2, W_1 are three subspaces of $\mathbb{R}^m$, then for some $g \in L$, neither $gU_1 \subset W_1$ nor $gU_2 \subset W_1$. For if the contrary were true, then in the dual space to $\mathbb{R}^m$ we would have ${}^t_{gW_1} \subset U_1^\perp \cup U_2^\perp$. Now every subset of a finite dimensional vector space has a unique minimal finite union of subspaces containing it (by the ascending chain condition on polynomial ideals). Let Q be this minimal finite union containing $\bigcup_{g \in L} {}^t_{gW_1}$; then $Q \subset U_1^\perp \cup U_2^\perp$. By uniqueness, Q is t_L invariant, so t_L permutes the subspaces of Q . So t_L has reducible subgroup of finite index, and the same would be true of L . Now take $U_1, U_2 \in M$ and $W_1 = W_1(\tau)$ for some τ with $\dim A_0(W_0) = r$. Because of the minimality of r it follows that no $\sigma \in \bar{L}$ can collapse U_1 or U_2 to a subspace of dimension $< r$. On the other hand for

$gU_1 \not\subset W_1$, $gU_2 \not\subset W_1$, $Tg\bar{U}_1 \subset \overline{A_0(W_0)}$ and $Tg\bar{U}_2 \subset \overline{A_0(W_0)}$. It follows that L is proximal on $M \subset \mathcal{G}_{m,r}$.

Lemma 5.2 has the following generalization in terms of cocycles. We omit the proof which involves probabilistic considerations.

PROPOSITION 5.3.- Let σ be a cocycle on $G \times X$ with values in $PGL_m(\mathbb{R})$ where G is measure preserving and ergodic on X . Then either σ is compact, or σ is FI-reducible, or $\exists r$, $0 < r < m$ and a σ -invariant map $\Phi : X \rightarrow 2^{\mathcal{G}_{m,r}}$ which is proximal.

We now apply theorem 4.5 to the proximal spaces of prop. 5.1 and prop. 5.3.

THEOREM 5.4.- Let (X,G) and (Y,H) be measure preserving actions where G acts ergodically and H acts freely. Assume G and H are Lie groups and that there is an invertible measurable map π from X to Y taking G orbits onto H orbits. Then if $\sigma : G \times X \rightarrow H$ is defined by $\sigma_{g,x}\pi(x) = \pi(gx)$, there exists a unique measurable σ -equivariant map $\psi : X \times B(G) \rightarrow B(H)$.

THEOREM 5.5.- Let (X,G) be an ergodic measure preserving action and let σ be a cocycle on $G \times X$ with values in $PGL_m(\mathbb{R})$. If σ is FI-irreducible and non-compact, $\exists r$, $0 < r < m$ and a σ -equivariant map $\psi : X \times B(G) \rightarrow \mathcal{G}_{m,r}$.

§ 6. Smooth actions and rationality

A group action for a locally compact group on a complete metric space is smooth if all orbits are locally closed. This is very different from ergodicity, and, in fact, if an action is both smooth and ergodic with respect to a measure, the measure must be concentrated on a single orbit. Examples of smooth actions are : actions of algebraic groups on algebraic varieties, the action of $GL_m(\mathbb{R})$ on the space of probability measures on $\mathbb{P}^{m-1}$. The following result of Margulis provides further examples [10, p. 42].

Lemma 6.1.- Let Z be a measure space and let $M = H/L$ where H is a real algebraic group and L an algebraic subgroup. Let $\Sigma = \mathcal{L}(Z,M)$ be the space of all measurable maps from Z to M with the topology of convergence in measure. H acts on Σ by $(hf)(z) = h(f(z))$. This actions of H is smooth.

When an action is smooth the orbit space of the action is a standard Borel space with a countably generated algebra of Borel sets [5]. Since functions with values in such a space which are constant along the orbits of an ergodic action must be constant, we have the following.:

Lemma 6.2.- Let X be an ergodic measurable G -space and let σ be a cocycle on $G \times X$ with values in H and suppose that H acts smoothly on M . If $\varphi : X \rightarrow M$ is a σ -invariant map, then there exists $x_0 \in X$ and a map $\eta : X \rightarrow H$ with $\varphi(x) = \eta(x)\varphi(x_0)$ for a.e. $x \in X$.

We now return to the situation of theorem 4.5 where we obtained a σ -invariant map from $X \times B(G)$ to the H -space M . We impose two more conditions which will enable us to deduce that the map in question is well behaved. We assume first of all that G is a semi-simple Lie group with $\mathbb{R}$ -rank $G \geq 2$. We also assume that each simple factor of G acts ergodically on X .

Lemma 6.3 (Moore [12]).- If G is a semi-simple Lie group acting on a measure space X and preserving the measure and if every simple component of G acts ergodically on X , then for any $\tau \in G$, either $\tau \in$ compact subgroup or τ acts ergodically on X .

The hypothesis $\mathbb{R}$ -rank $G \geq 2$ appears in the following observation. Consider a σ -invariant map $\psi : X \times B(G) \rightarrow M$, assuming H algebraic and M a homogeneous space of H by an algebraic subgroup. Let P denote a parabolic subgroup of G so that $B(G) = G/P$. If $\mathbb{R}$ -rank $G \geq 2$ there will exist abelian subgroups $T \subset P$ with centralizer $C \not\subset P$. Now if $\tau \in T$, $c \in C$

$$(6.1) \quad \psi(\tau x, cP) = \sigma(\tau, x) \psi(x, \tau^{-1} cP) = \sigma(\tau, x) \psi(x, c \tau^{-1} P) = \sigma(\tau, x) \psi(x, cP).$$

Thus along a T -orbit in X the function $\psi(x, \cdot)$ on $CP \subset B(G)$ remains in the same H -orbit. By lemma 6.3, T will be ergodic on X and applying lemma 6.2 we will be able to write

$$(6.2) \quad \psi(x, cP) = \eta(x) \psi_0(cP)$$

More generally we have

$$\psi(g \tau g^{-1} x, g c P) = \sigma(g \tau g^{-1}, x) \psi(x, g c P)$$

and since $g \tau g^{-1}$ is ergodic on X we can write

$$(6.3) \quad \psi(x, g c P) = \eta(x, g) \psi_0(g, cP)$$

Let

$$\omega(x, g, c') = \eta(x, g c') \eta(x, g)^{-1},$$

so that

$$(6.4) \quad \psi(x, g c' c P) = \omega(x, g, c') \psi(x, g c P), \quad c, c' \in C.$$

From (6.4) we deduce that for $c, c'_1, c'_2 \in C$,

$$\omega(x, g, c'_1 c'_2) \psi(x, g c P) = \omega(x, g, c'_1) \omega(x, g, c'_2) \psi(x, g c P).$$

Thus $\omega(x, g, \cdot)$ is a map of $C \rightarrow H$ which modulo the normalizer of the stability group in H of the function $\psi(x, g c P)$ is multiplicative. This implies that the map of C to the quotient group which was a priori measurable is, in fact, rational. Hence $\psi(x, g c' c P)$ depends rationally on c' , or $\psi(x, g c P)$ depends rationally on $c \in C$. By varying T so as to "exhaust" G/P , and using the fact ([14], Appendix, lemma 17) that a function of two variables rational in each variable separately is rational in the pair, one proves, as in [10] that $\psi(x, g P)$ is rational in g for a.e. x . One thus obtains

THEOREM 6.4.- Let G be semi-simple with $\mathbb{R}$ -rank $G \geq 2$ and let G act in a measure preserving way on X so that each simple factor of G is ergodic on X . Let $\sigma : G \times X \rightarrow H$ be a cocycle with values in an algebraic subgroup $H \subset \text{PGL}_m(\mathbb{R})$ and let M be an H -invariant subvariety of $\mathbb{P}^{m-1}$. If $\psi : X \times B(G) \rightarrow M$ is a σ -invariant measurable map, then for a.e. x , $\psi(x, u)$ is a rational function from $B(G)$ to M .

§ 7. Straightening out the cocycle σ

A cocycle σ defines a skew action of G on $X \times_G M$. If σ is equivalent to a cocycle σ' in which the variable x does not appear, we may say that we have straightened out the action; i.e., there is an action of G on M so that $X \times_G M$ is the product action. A further argument of Margulis now shows that this occurs when theorem 6.4 holds.

THEOREM 7.1.- Let G , X , H , and M be as in theorem 6.4, and assume that the cocycle σ is irreducible and minimal on M . If there exists a σ -invariant, measurable map $\psi : X \times B(G) \rightarrow M$, then σ is equivalent to a cocycle $\sigma'(g, x) = \xi(g)$ where $\xi : G \rightarrow H$ is a homomorphism.

Proof.- By theorem 6.4, $\psi(x, u)$ is rational in u for almost every x . Regard ψ as a map from $X \rightarrow \Sigma$ where Σ consists of all rational functions from $B(G)$ to M . The group H acts on Σ as before and the action is smooth. This time G also acts on Σ by $gf(u) = f(g^{-1}u)$, and the combined action of $G \times H$ on Σ is smooth. (See [16] for details.) The σ -equivariance: $\psi(gx, gu) = \sigma(g, x)\psi(x, u)$ together with lemma 6.2 imply that there is a single element $\psi_o \in \Sigma$ with

$$(7.1) \quad \psi(x, u) = \eta(x)\psi_o(\gamma(x)u)$$

where $\eta : X \rightarrow H$ and $\gamma : X \rightarrow G$ are measurable maps. η and γ must satisfy

$$(7.2) \quad \psi_o(\gamma(gx)gu) = \eta(gx)^{-1}\sigma(g, x)\eta(x)\psi_o(\gamma(x)u),$$

and so we see that the maps $\psi_o(\gamma(gx)gu)$ and $\psi_o(\gamma(x)u)$ are in the same H -orbit in Σ . Let $H \backslash \Sigma$ denote the space of H -orbits of Σ and let $\overline{\psi}_o$ be the image of ψ_o in $H \backslash \Sigma$. For every $g \in G$ and a.e. $x \in X$, $\gamma(gx)g$ and $\gamma(x)$ have the same effect on $\overline{\psi}_o$ as elements of G acting on $H \backslash \Sigma$. Now it can be seen ([16]) that the stabilizer in G of any point of $H \backslash \Sigma$ is an algebraic subgroup of G . On the other hand the map $x \rightarrow \gamma(x)^{-1}\overline{\psi}_o$ is a G -equivariant map, so that if G_o denotes the stabilizer of $\overline{\psi}_o$, then G/G_o carries a finite G -invariant measure. By the Borel density theorem [1], G_o is Zariski dense in G ; so $G_o = G$. This means that $\psi_o(gu) = \xi(g)\psi_o(u)$ for some measurable map $\xi : G \rightarrow H$. Since the cocycle σ is minimal we must have $\psi_o(B(G)) = M$ and since σ is irreducible M cannot be contained in a proper linear subvariety of $\mathbb{P}^{m-1}$. Hence an element of H is uniquely

determined by what it does on M . This shows that ξ is a homomorphism of G to H . Finally we can see from (7.2) that $\sigma \sim \sigma'$ where $\sigma'(g, x) = \xi(g)$:

$$(7.3) \quad \sigma(g, x) = \eta(gx) \xi(\gamma(gx)) \xi(g) \xi(\gamma(x))^{-1} \eta(x)^{-1}.$$

This completes the proof.

Note that in the foregoing argument it was essential that no non-trivial element of H leaves every point of $\psi_0(B(G))$ fixed. Nevertheless, we can sometimes obtain a partial straightening of σ . The next lemma will be useful for the situation where we have a σ -invariant map $\psi: X \times B(G) \rightarrow \mathcal{G}_{m,r}$. It will be more convenient to treat groups of complex matrices.

Lemma 7.2.- Let L be a subgroup of $\text{PGL}_m(\mathbb{C})$ for which no subgroup of finite index is reducible. Let $M \subset \mathcal{G}_{m,r}(\mathbb{C})$ be an L -invariant subset, let $J \subset \text{PGL}_m(\mathbb{C})$ be the subgroup of elements which fix each point of M , and let J' be the centralizer of J in $\text{PGL}_m(\mathbb{C})$. There is a subgroup L_0 of finite index in L , and there are homomorphisms $\theta: L_0 \rightarrow J$, $\theta': L_0 \rightarrow J'$, so that for each $\ell \in L_0$, $\ell = \theta(\ell)\theta'(\ell)$. Moreover if $W \in M$ is some r -dimensional subspace of $\mathbb{C}^m$, each $j \in J$ is determined by its restriction to W .

We sketch the proof. The Lie algebra $\mathcal{J}$ of J consists of the trace 0 endomorphisms of $\mathbb{C}^m$ taking each $W \in M$ into itself. Clearly $\ell \mathcal{J} \ell^{-1} = \mathcal{J}$ for $\ell \in L$. If an abelian subalgebra of $\mathcal{J}$ were invariant under L , it would have an L -invariant eigenspace contradicting the irreducibility of L . From this it is easy to conclude that $\mathcal{J}$ is semi-simple. We can replace L by the normalizer of $\mathcal{J}$ so that L is an algebraically closed subgroup, and we take L_0 to be the connected component of the identity. Since conjugation in $\mathcal{J}$ by elements of L_0 corresponds to $\text{Ad } J$ we conclude that $L_0 = JJ'$. Finally if $j \in J$ and j is the identity on W , then j is the identity on $j'W$ for $j' \in J'$. Hence j is the identity on ℓW for all $\ell \in L_0$, and since L_0 is irreducible, $j = 1$.

Now suppose G and X satisfy the hypotheses of theorem 6.4 and let $\sigma: G \times X \rightarrow \text{PGL}_m(\mathbb{C})$ be a cocycle which is neither compact nor FI-reducible. We apply the argument in the proof of theorem 7.1 to the σ -equivariant map $\psi: X \times B(G) \rightarrow \mathcal{G}_{m,r}(\mathbb{C})$ of theorem 5.5, and we conclude that σ is equivalent to a cocycle σ' whose action on $\psi_0(B(G))$ is independent of x :

$$\psi_0(gu) = \sigma'(g, x) \psi_0(u).$$

We apply the foregoing lemma to the group generated by $\{\sigma'(g, x)\}$ and $M = \psi_0(B(G))$, and we find that for a finite extension of X (corresponding to a subgroup of finite index of L) we can write

$$(7.4) \quad \sigma'(g, x) = \theta(\sigma'(g, x)) \theta'(\sigma'(g, x))$$

where $\theta'(\sigma'(g, x)) = \xi(g)$ is independent of x . Finally, it is not hard to show

that if the minimal value of r was chosen in prop. 5.3 giving a proximal action on $\mathcal{G}_{m,r}(\mathbb{C})$, then the cocycle $\theta(\sigma'(g,x))$ which is determined by its restriction to a subspace $W \in \Psi_0(B(G))$, will be compact. We obtain the following result.

THEOREM 7.3.- Let G be semi-simple with $\mathbb{R}$ -rank $G \geq 2$ and let G act in a measure preserving way on X so that each simple component of G is ergodic on X . Let $\sigma : G \times X \rightarrow \text{PGL}_m(\mathbb{C})$ be an FI-irreducible cocycle. There exists a finite extension $\tilde{X} \rightarrow X$ of the G -action and the lifted cocycle is equivalent to a cocycle $\sigma'(g,\tilde{x})$ where

$$\sigma'(g,\tilde{x}) = \rho(g,\tilde{x})\xi(g)$$

with $\rho(g,\tilde{x})$ taking values in a compact subgroup $U \subset \text{PGL}_m(\mathbb{C})$, and ξ is a homomorphism of G into a subgroup $H \subset \text{PGL}_m(\mathbb{C})$ where H and U commute elementwise.

§ 8. Applications

1. Rigidity of lattices

Let Γ be an irreducible lattice in G . This means that every simple component of G acts ergodically on G/Γ . Let H be a semi-simple Lie group with trivial center and let $\tau : \Gamma \rightarrow H$ be a representation. Extend τ to an H -valued cocycle on $G \times G/\Gamma$ as in the proof of the corollary to theorem 4.5. If $\tau(\Gamma)$ acts proximally and minimally on $B(H)$ we obtain by theorem 4.5 a σ -invariant map $\psi : X \times B(G) \rightarrow B(H)$. Applying theorem 7.1 the cocycle σ can be straightened out to a homomorphism of $G \rightarrow H$. This gives us the following rigidity theorem.

THEOREM 8.1.- Let $\mathbb{R}$ -rank $G \geq 2$ and let Γ be an irreducible lattice in G (the projection of Γ on any direct factor of G is nondiscrete). Let τ be a homomorphism of $\Gamma \rightarrow H$ where H is a semi-simple Lie group with trivial center and such that $\tau(\Gamma)$ acts proximally and minimally on $B(H)$ (e.g., $\tau(\Gamma)$ is a sublattice or is dense in H). Then τ extends to a homomorphism of $G \rightarrow H$.

2. Orbit equivalence

Consider now the situation of theorem 5.4. (X,G) and (Y,H) are orbit equivalent actions where H acts freely on Y and let us now assume $\mathbb{R}$ -rank $G \geq 2$ and that each simple component of G is ergodic on X . $\sigma : G \times X \rightarrow H$ is defined by $\sigma(g,x)\pi(x) = \pi(gx)$, and by theorem 5.4, there exists a σ -equivariant map $\psi : X \times B(G) \rightarrow B(H)$. We can apply theorem 7.1 to conclude that σ is equivalent to a homomorphism $\xi : G \rightarrow H$,

$$\sigma(g,x) = \omega(gx)^{-1}\xi(g)\omega(x)$$

for some measurable $\omega : X \rightarrow H$. Set $\pi'(x) = \omega(x)\pi(x)$; then

$$(8.1) \quad \pi'(gx) = \xi(g)\pi'(x).$$

If we now assume in addition that G acts freely on X then (8.1) implies that the action of G on X , $(g, x) \rightarrow gx$, and the action of G on Y , $(g, y) \rightarrow \xi(g)y$ are conjugate. This gives us Zimmer's theorem, theorem 0.1 of the Introduction.

3. Entropy of smooth actions of Margulis groups

Let X be a compact m -dimensional C^2 -manifold and let $\tau : X \rightarrow X$ be a C^2 -diffeomorphism of X preserving a smooth measure and acting ergodically on X . We denote by $\mathcal{E}(\tau)$ the Kolmogorov-Sinai entropy of the automorphism τ . $\mathcal{E}(\tau)$ is defined in information theoretic terms, but for smooth transformations an alternative characterization has been given by Pesin [13]. Namely, introduce a Riemannian metric in the tangent space $T(X)$. In the space of p -vectors $T^{(p)}(X) = T(X) \wedge \dots \wedge T(X)$ we consider the induced linear transformation

$$\tau_{p,x}^{(n)} : T^{(p)}(X)_x \longrightarrow T^{(p)}(X)_{\tau^n x},$$

and form the norm $\|\tau_{p,x}^{(n)}\|$. It can be proved, using the subadditive ergodic theorem [9], that

$$(8.2) \quad \lambda_p = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\tau_{p,x}^{(n)}\|$$

exists for a.e. x and is independent of x .

PROPOSITION 8.2 (Pesin).

$$\mathcal{E}(\tau) = \max_{1 \leq p \leq m} \lambda_p.$$

We shall now use theorem 7.3 to obtain information regarding the λ_p where we assume that τ is the restriction to a single element of G of an action of the group G on a compact manifold X . We suppose now that G is simple with $\mathbb{R}$ -rank $G \geq 2$ so that G also has property T , and we assume that G acts smoothly and ergodically on X . Let $\sigma_p(g, x)$ be the tangent cocycle of this action (cf. Introduction) applied to the p -fold exterior product $T^{(p)}(X)$. If this cocycle is FI-reducible, we replace X by a finite extension for which the cocycle becomes reducible. We apply theorem 7.3 to the irreducible components obtained by successively lifting σ_p to appropriate finite extensions. We conclude that after replacing X by a finite extension, $\sigma_p(g, x)$ will be equivalent to a cocycle

$$(8.3) \quad \sigma_p'(g, x) = \begin{pmatrix} \tilde{\sigma}_1(g, x) \xi_1(g) & * & * & \dots \\ 0 & \tilde{\sigma}_2(g, x) \xi_2(g) & * & \dots \\ 0 & 0 & \square & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the $\tilde{\sigma}_i(g, x)$ are unitary matrices and $\xi_i : G \rightarrow \text{SL}_{m_i}(\mathbb{C})$ are homomorphisms, and the matrices $\tilde{\sigma}_i(g, x)$ and $\xi_i(g)$ commute.

Now let $\gamma \in G$ and let τ be the restriction to γ of the action of G on X .

By lemma 6.3, either $\gamma \in$ compact subgroup of G in which case $\mathcal{E}(\tau) = 0$, or τ is ergodic on X . In this case $\mathcal{E}(\tau) = \max_p \lambda_p$, where

$$(8.4) \quad \lambda_p = \lim \frac{1}{n} \log \|\sigma_p^n(\gamma^n, x)\|.$$

Set $\sigma_p(g, x) = \omega(gx)^{-1} \sigma_p'(g, x) \omega(x)$ and let $A \subset X$ be a subset of X with $\mu(A) > 0$ and on which $\omega(x)$ and $\omega(x)^{-1}$ are bounded. Restricting the sequence n in (8.4) so that $\gamma^n x \in A$ (by ergodicity) we find that

$$\lambda_p = \lim \frac{1}{n} \log \|\sigma_p'(\gamma^n, x)\|.$$

Finally it may be seen that for matrices in triangular form, the rate of growth is determined by the diagonal components. Since $\|\tilde{\sigma}_i(g, x)\| = 1$ we obtain

$$(8.5) \quad \mathcal{E}(\tau) = \lim \frac{1}{n} \log \|\xi(\gamma^n)\|$$

for some representation $\xi : G \rightarrow \text{SL}_{m_i}(\mathbb{C})$. This gives the following.

THEOREM 8.3.- Let G be a simple Lie group with $\mathbb{R}$ -rank $G \geq 2$ and let $\tau \in G$. Let G act by C^2 -diffeomorphisms on a compact C^2 -manifold X preserving a smooth measure and ergodically with respect to that measure. Let $\gamma \in G$ and let $\tau : X \rightarrow X$ be the restriction of the G -action to γ . Then either $\mathcal{E}(\tau) = 0$ or

$$\mathcal{E}(\tau) = \log \max |\text{eigenvalue } \xi(\gamma)|$$

where ξ is a representation of G in $\text{SL}_{m_i}(\mathbb{C})$.

This implies that as one ranges over the family of all such actions of G , the set of values of $\mathcal{E}(\tau)$ for any fixed $\gamma \in G$ is discrete.

Appendix. Sketch of proof of lemma 4.7 (see [7])

Let π be a measure on G absolutely continuous with respect to Haar measure and with support equal to all of G . On any Borel G -space Z , if ν is a Borel measure on Z , we can form $\pi * \nu$ on Z by

$$\pi * \nu(A) = \int_G \nu(g^{-1}A) d\pi(g).$$

We say that ν is π -stationary if $\pi * \nu = \nu$. It is shown in [7] that on $B(G)$, there exists a unique π -stationary measure ν . On $X \times B(G)$ we have the π -stationary measure $\mu \times \nu$. Let $Y_1, Y_2, \dots, Y_n, \dots$ be a sequence of G -valued random variables which are independent and have distribution π . Let W_0 be an $X \times B(G)$ -valued r.v. independent of these and with distribution $\mu \times \nu$. Setting $W_n = Y_n Y_{n-1} \dots Y_1 W_0$ we find that $\{Y_n, W_n\}$ forms a stationary sequence of $G \times X \times B(G)$ -valued random variables. This stationary process can be extended to negative values of the index. Now consider the $\mathcal{P}(M)$ -valued r.v.'s $\psi(W_n)$ where ψ is a σ -invariant map. Writing $W_n = (X_n, U_n)$, we have

$$(A.1) \quad \psi(W_{n+1}) = \sigma(Y_{n+1}, X_n) \psi(W_n).$$

Let

$$\lambda_x = E(\psi(W_0) \mid X_0 = x) .$$

This expectation is meaningful since the random variable $\psi(W_0)$ takes its values in a compact convex set. Using the martingale convergence theorem one concludes that

$$(A.2) \quad E(\psi(W_0) \mid X_0 = x, Y_0, Y_{-1}, \dots, Y_{-n})$$

converges as $n \rightarrow \infty$ with probability one. If one shows that the limit

$$E(\psi(Y_0) \mid X_0 = x, Y_0, Y_{-1}, \dots, Y_{-n}, \dots)$$

is a.e. a point measure, it will follow that $\psi(W_0)$ is itself a point measure. By

(A.1) we rewrite (A.2) as

$$\sigma(Y_0 \dots Y_{-n}, (Y_0 \dots Y_{-n})^{-1}x) \lambda_{(Y_0 \dots Y_{-n})^{-1}x}$$

since $Y_0, \dots, Y_{-n}$ are independent of W_{-n-1} . Shifting variables we can say that

$$\sigma(Y_n \dots Y_{-k}, (Y_n \dots Y_{-k})^{-1}x) \lambda_{(Y_n \dots Y_{-k})^{-1}x}$$

and

$$\sigma(Y_n \dots Y_0, (Y_n \dots Y_0)^{-1}x) \lambda_{(Y_n \dots Y_0)^{-1}x}$$

for fixed k , become closer and closer together as $n \rightarrow \infty$. Using the fact that G is measure preserving on X we can assert that

$$(A.3) \quad \sigma(Y_n \dots Y_0, x) \sigma(Y_{-1} \dots Y_{-k}, (Y_{-1} \dots Y_{-k})^{-1}x) \lambda_{(Y_{-1} \dots Y_{-k})^{-1}x}$$

and

$$(A.4) \quad \sigma(Y_n \dots Y_0, x) \lambda_x$$

are close as $n \rightarrow \infty$. By proximality there is positive probability that

$$\sigma(Y_{-1} \dots Y_{-k}, (Y_{-1} \dots Y_{-k})^{-1}x) \lambda_{(Y_{-1} \dots Y_{-k})^{-1}x}$$

is close to a point measure. The variables $Y_0, Y_1, \dots, Y_n$ are independent of $Y_{-1}, \dots, Y_{-k}$ and we consider the set of limits of $\sigma(Y_n \dots Y_0, x)$ as quasi-projective transformations. The limits of (A.3) and (A.4) must be identical and since that of (A.3) will be close to a point measure the limits of (A.4) must be point measures. This in turn implies that the expectations of (A.2) converge to point measures and this completes the proof.

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