SÉMINAIRE N. BOURBAKI

A. VAN DE VEN

On the Enriques classification of algebraic surfaces

Séminaire N. Bourbaki, 1978, exp. nº 506, p. 237-251

<http://www.numdam.org/item?id=SB_1976-1977__19__237_0>

Séminaire BOURBAKI 506-01
29e année, 1976/77, n° 506 Juin 1977

ON THE ENRIQUES CLASSIFICATION OF ALGEBRAIC SURFACES

by A. VAN DE VEN

1. Introduction

Among (for the moment : complex) algebraic curves R the rational ones are characterised by the fact that their genus $g(R)$ vanishes. The genus can be interpreted as the dimension of the space of holomorphic 1-forms on R. Is it possible to characterise rational surfaces - i.e. surfaces birationally equivalent to the projective plane P_2 - in a similar way ? This question was raised at the end of the last century, after Noether, Picard, Castelnuovo, Enriques and others had already developed a considerable theory of algebraic surfaces.

If we denote the dimension of the space of holomorphic i-forms on the (smooth, connected) surface X by $g_i(X)$, $i = 1, 2$, then a necessary condition for a surface to be rational is that $g_1(X) = g_2(X) = 0$. It was discovered by Enriques that this condition is not sufficient. But Castelnuovo showed that it is still possible to give a simple criterion for the rationality of a surface X . In fact, let $\mathcal{K}_X$ be the canonical line bundle on X , i.e. the determinant bundle of the covariant tangent bundle of X . Then the n-th plurigenus $P_n(X)$ of X is defined by :

$$P_n(X) = \dim \Gamma(X, \mathcal{K}_X^{\otimes n}) \qquad (n \geq 1) .$$

In particular we have $P_1(X) = g_2(X)$. The $P_n(X)$ are birational invariants of X . It is immediate that for a rational surface X all $P_n(X)$ vanish. Now Castelnuovo proved : ([4])

THEOREM 1.1 (Castelnuovo criterion).- An algebraic surface X is rational if and only if $g_1(X) = P_2(X) = 0$.

A little bit later Castelnuovo and Enriques found other theorems of this type. For example, if we define a ruled surface as a surface birationnally equivalent to the product of P_1 and a curve, then Castelnuovo and Enriques stated

THEOREM 1.2.- An algebraic surface X is ruled if and only if $P_{12}(X) = 0$.

This result implies that the algebraic surfaces X with $P_n(X) = 0$ for all n are exactly the ruled surfaces.

Now the idea of the Enriques classification is, that -more generally - you can say quite a lot about a surface X , once you know the behaviour of $P_n(X)$ for $n \to \sim$. In fact, the Enriques classification divides algebraic surfaces X into four classes, according to the behaviour of $P_n(X)$. Thus the first class is the class of surfaces X for which all $P_n(X)$ vanish, the second class contains all surfaces with $P_n(X)$ either 0 or 1 , but at least one $P_n(X) \neq 0$, and the surfaces in the third and fourth class are characterised by $P_n(X) \sim O(n)$ and $P_n(X) \sim O(n^2)$ respectively. (It has of course to be proved that every surface fits into one of these classes. Observe in any case that if $P_m(X) \neq 0$, $P_n(X) \neq 0$, then $P_{n+m}(X) \geq P_n(X) + P_m(X) - 1$.) We have already seen that the surfaces in the first class can be described rather explicitly. It will be explained later that the same holds for the surfaces in the second and third class ; about the surfaces in the last class much less is known.

To the best of my knowledge, the Enriques classification appears for the first time in an article of Enriques : Sulla classificazione delle superficie algebriche ... I. Rend. Lincei XXIII, 206-214 (1914). You also find it in the survey by Castelnuovo and Enriques : Die algebraischen Flächen vom Gesichtspunkte der birationalen Transformationen aus. Enzykl. Math. Wiss. II, 1, Teubner, Leipzig (1915), and in Enriques book [6].

In the years around 1960 several algebraic geometers treated the Enriques classification in a more modern way, in particular Zariski, Šafarevič and their pupils ([13]). There were important contributions to the case of characteristic p ; for example, Zariski ([19]) proved the Castelnuovo criterion for that case, and M. Artin studied the so called Enriques surfaces for all characteristics $\neq 2$. About the same time, Kodaira ([7], [8]) - using topologicals and analytical tools like sheaf theory, vanishing theorems, deformation theory - not only made things precise, but enriched our knowledge about the surfaces in the second and third class enormously. A major contribution was of course also his treatment of the non-algebraic surfaces.

Finally, Mumford - partly in collaboration with Bombieri - gave a beautiful unified treatment for all characteristics ([10], [3]).

It should be emphasised that the extension to characteristic p (including $p = 2$ and 3 !) is far from easy, and that this extension could only be completed with the help of an intensive use of the language of schemes and the étale homology.

2. <u>The complex case</u>. <u>Preliminaries</u>

In this and the next section we shall study smooth, complete complex algebraic surfaces. Such a surface is <u>minimal</u>, if it contains no exceptional curves, i.e. irreducible curves which can be blown down to smooth points. Exceptional curves E are characterised by the fact that they are non-singular rational with $E^2 = -1$. Every surface can be obtained from a minimal one by successive blowing up a finite number of times. This minimal model is uniquely determined if the surface is not ruled. In the sequel we shall practically always consider minimal surfaces only.

In this talk the word "fibre space" will be used only for smooth surfaces fibred over smooth curves. Thus a <u>fibre space</u> is a triple $\{X , f , Y\}$, where X is a smooth surface, Y a smooth curve and $f : X \to Y$ a surjective morphism. The fibre space is called (relatively) minimal if no fibre contains an exceptional curve. It is called an <u>elliptic fibre space</u> if almost all fibres are elliptic curves. Sometimes a surface admitting at least one such elliptic fibration is called an elliptic surface.

I shall use the following notations :

$\mathcal{O}_X$: structure sheaf of X

Ω_X^i : sheaf of regular i-forms on X (with $\Omega_X^o = \mathcal{O}_X$)

$\mathcal{V}$, $\mathcal{L}$,..: vector bundles on X

$\mathcal{V}^*$: the dual bundle of $\mathcal{V}$

$\mathcal{K}_X$: canonical line bundle on X (with sheaf of sections Ω_X^2)

$h^i(\mathcal{V})$: $\dim H^i(X ,\mathcal{V})$; $h^{p,q}(X) = h^q(\Omega_X^p)$

$\chi(X ,\mathcal{V})$: Euler characteristic $h^o(\mathcal{V}) - h^1(\mathcal{V}) + h^2(\mathcal{V})$

$q(X)$: $h^1(X , \mathcal{O}_X)$, the <u>irregularity</u> of X

$p_g(X)$: $h^2(X , \mathcal{O}_X)$, the <u>geometric genus</u> of X

$P_n(X)$: $h^o(X ,\mathcal{K}_X^{\otimes n})$, the <u>n-th plurigenus</u> of X $(n \geq 1)$

$e(X)$: topological Euler characteristic of X

$D , E ,\ldots$: divisors on X <u>or</u> divisor classes

DE : intersection number of D and E

$\mathcal{O}_X(D)$: line bundle corresponding to D

nK_X : n-canonical divisors or -divisor classes (hence $\mathcal{K}_X^{\otimes n} = \mathcal{O}_X(nK))$.

Furthermore, we shall make use of the following tools (see for example [9]) :

(i) (special case of <u>Serre duality</u>) $h^i(\mathcal{V}) = h^{2-i}(\mathcal{V}^* \otimes \mathcal{K}_X)$. In particular we have $p_g(X) = P_1(X)$.

(ii) (<u>in the complex case</u>) $h^{p,q}(X) = h^{q,p}(X)$, in particular $q(X) = h^0(\Omega_X^1)$. Furthermore, $\sum_{p+q=r} h^{p,q}(X) = b_r(X)$, the r-th Betti number of X .

(iii) (<u>Riemann-Roch formula for line bundles</u>) :

$$\chi(X, \mathcal{O}_X(D)) = \tfrac{1}{2}D(D-K) + \chi(X, \mathcal{O}_X) .$$

(iv) (<u>Todd formula</u>) :

$$\chi(X, \mathcal{O}_X) = \frac{K_X^2 + e(X)}{12} .$$

(v) (special case of <u>Stein factorisation</u>) If $f : X \to Y$ is a fibre space, then there exists a smooth curve Y' , and regular maps $g : X \to Y'$, $h : Y' \to Y$, with g connected and h finite, such that $f = h \circ g$.

We shall need furthermore

THEOREM 2.1 (<u>Algebraic index theorem</u>).- <u>Let A be the a-dimensional linear subspace of $H_2(X, R)$, generated by the algebraic curves. Then the restriction of the intersection form to A is of signature $(1, a-1)$</u> .

Using some linear algebra, one can derive from this ([2], p. 345)

PROPOSITION 2.2.- <u>Let D be a divisor on X with $DE \geq 0$ for all non-negative divisors E . Then $D^2 \geq 0$</u> .

Of importance will also be the <u>Albanese torus</u> $Alb(X)$; however we shall only use that there exists an abelian variety $Alb(X)$ of dimension $q(X)$ and a regular map $f : X \to Alb(X)$, such that $f(X)$ is not contained in any (translated) proper subtorus of $Alb(X)$.

Then I recall that for every irreducible curve C on X the <u>adjunction formula</u>

$$KC + C^2 = 2\tilde{\pi}(C) - 2$$

holds, where $\tilde{\pi}(C)$ - the virtual genus of C - is at least equal to the actual genus of C . The virtual genus $\tilde{\pi}(C) = 0$ if and only if C is non-singular rational, and $\tilde{\pi}(C) = 1$ if and only if C is non-singular elliptic, or rational with one ordinary double point, or rational with one cusp.

Let for a moment X be any connected compact complex manifold, and let $\mathcal{K}_X$ again be the canonical line bundle of X. If we take for $\mathcal{K}_X^{\otimes 0}$ the trivial line bundle on X, then $\sum_{n \geq 0} \Gamma(X, \mathcal{K}_X^{\otimes n})$ is in a natural way a graded ring, the canonical ring of X. The Kodaira dimension $\mathrm{Kod}(X)$ of X is defined as follows :

$$\mathrm{Kod}(X) = \begin{cases} -\infty & \text{if all } P_n(X) \text{ vanish} \\ \mathrm{tr}_{\mathbb{C}} \sum_{n \geq 0} \Gamma(X, \mathcal{K}_X^{\otimes n}) - 1 & \text{otherwise .} \end{cases}$$

Slightly more down to earth $\mathrm{Kod}(X)$ can be defined by means of the rational maps, associated to $\mathcal{K}_X^{\otimes n}$, $n \geq 1$ (see for example [15], Ch. II, § 6). We always have that $-\infty \leq \mathrm{Kod}(X) \leq \dim X$. $\mathrm{Kod}(X)$ is a birational invariant of X.

Iitaka and others (see [15]) have developed a classification theory for compact, complex manifolds, based on the Kodaira dimension. From this theory emerges the central position of the following conjecture ([15], p. 132).

CONJECTURE $C_{n,m}$ (Iitaka).- Let X and Y be smooth connected compact algebraic varieties, of dimension n and m respectively, and let $f : X \to Y$ be a regular, connected surjective map. Then

$$\mathrm{Kod}(X) \geq \mathrm{Kod}(Y) + \mathrm{Kod}(F) ,$$

where F is a general fibre of f.

Up to now only a few special cases of $C_{n,m}$ have been proved, in particular the case $C_{n,n-1}$ (Viehweg [18]). The case $C_{2,1}$ follows from the Enriques classification. Conversely, this classification becomes much easier once an independent proof of $C_{2,1}$ is available. Three such proofs have been given : one by Ueno ([16]), and two by Viehweg ([18]), one of the last two being of course a specialisation of Viehweg's argument for the case $C_{n,n-1}$. Since I shall base my approach to the Enriques classification on $C_{2,1}$, I shall give a short sketch of the simplest of the three forementioned proofs, i.e. Viehweg's second proof.

Let $f : X \to Y$ be a fibre space, such that the general fibre is connected and has genus $g \geq 1$. In this situation we can consider :

(i) the relative canonical bundle $\mathcal{K}_{X/Y} = \mathcal{K}_X \otimes f^*(\mathcal{K}_Y^{-1})$;

(ii) the Weierstrass divisor $W_{X/Y} = W$, inducing on the general fibre the Weierstrass divisor, and containing no component of any fibre of f (For $g = 1$ the divisor $W = 0$).

Let S be the union of the singular fibres of f , i.e. those fibres in which there lies at least one point where f is not of maximal rank. Then we have

PROPOSITION 2.3.- <u>There exists a line bundle</u> $\mathcal{D}$ <u>on</u> Y , <u>a non-negative divisor</u> E <u>on</u> X <u>with</u> $\operatorname{supp}(E) \subset S$, <u>and natural numbers</u> N , M <u>such that</u>

$$\mathcal{K}_{X/Y}^{\otimes N} = f^*(\mathcal{D}) \otimes \mathcal{O}_X(W)^{\otimes M} \otimes \mathcal{O}_X(E) .$$

<u>Moreover</u>, $\deg(\mathcal{D}) \geq 0$ <u>and if</u> $\deg(\mathcal{D}) = 0$, <u>then</u> $\mathcal{D}$ <u>is trivial modulo torsion</u>.

$C_{2,1}$ follows immediately from this result.

As to the proof of the relation above, after suitable blowing up, this is first brought back to the case where all fibres are reduced by applying (a simple case of) the stable reduction theorem. This case on its turn is brought back to the case where all fibres are reduced, where f is minimal (no fibre of f contains an exceptional curve) and where $W = \sum k_i K_i$, the K_i 's being sections of f . In this last case a more precise formula - holding more generally for every case where all fibres are reduced - is used, namely :

$$\mathcal{K}_{X/Y}^{\otimes \frac{1}{2}g(g+1)} = f^*\left(\bigwedge^{g} (f_*(\mathcal{K}_{X/Y}))\right) \otimes \mathcal{O}_X(W) \otimes \mathcal{O}_X(E)$$

($f_*\mathcal{K}_{X/Y}$ is known to be locally free here), where E is again a non-negative divisor with support in the singular fibres. This formula is proved by establishing explicitly an isomorphism between the bundle on the left hand side and the one on the right hand side outside of the finitely many points where f is not of maximal rank. It turns out that the statement about the degree of $\mathcal{D}$ in Proposition 2.3 will be proved as soon as it is proved for the special case under consideration, where $\mathcal{D} = \bigwedge^{g} f_*(\mathcal{K}_{X/Y})$. For this purpose an explicit formula for $\deg(\mathcal{D})$ is given, from which it follows that $\deg(\mathcal{D}) \geq 0$ and $\deg(\mathcal{D}) = 0$ if and only if all fibres of f are smooth. For $g \geq 2$ the proof is completed by applying a theorem of Parshin, concerning exactly $\deg(\mathcal{D})$ in this smooth situation ([11], Prop. 5). For $g = 1$ a special argument is needed.

3. <u>The complex case</u>

It will be clear by now that the division of algebraic surfaces X into four classes, mentioned in section 1, is the division according to the value of $\operatorname{Kod}(X)$, which can be $-\infty$, 0 , 1 or 2 (we don't need the interpretation given for the last two classes in section 1). Taking into account that curves Y have

$\mathrm{Kod}(Y) = -\infty$, 0 or 1 if they have genus 0 , 1 or ≥ 2 respectively, we obtain from $C_{2,1}$ and the Castelnuovo criterion already some important information.

Let X be a minimal algebraic surface with $\mathrm{Kod}(X) \leq 0$. Then we have the following table :

Kod(X)	
$-\infty$	$q = 0$: rational surfaces
	$q > 0$: ruled surfaces (of genus $q \geq 1$)
0	$q = 0$ a) $K \neq 0$, $2K = 0$ $\overset{\text{def}}{\Longrightarrow}$ Enriques surfaces b) $K = 0$ $\overset{\text{def}}{\Longrightarrow}$ K_3-surfaces b') there is a positive divisor $2K$
	$q > 0$ c) image $f(X) \subset \mathrm{Alb}(X)$ has dimension 2 d) $q(X) = 1$ and X is an elliptic fibre space over an elliptic curve

Next we want to study cases b') and c) of the case $\mathrm{Kod}(X) = 0$. But first a more general remark.

PROPOSITION 3.1.- <u>If on the minimal surface</u> X <u>there is a non-negative divisor</u> D <u>with</u> $KD < 0$, <u>then</u> $\mathrm{Kod}(X) = -\infty$.

<u>Proof</u>. We may assume that D is an irreducible curve. Suppose that $\mathrm{Kod}(X) \geq 0$. Then there exists for some n a positive n-canonical divisor $\sum c_i C_i$. Since $(\sum c_i C_i) D < 0$, we have that D is contained in $\sum c_i C_i$, hence for some $a > 0$ we have that $(\sum c_i C_i - aD) D \geq 0$, i.e. $D^2 < 0$. From the adjunction formula we find $KD = D^2 = -1$, $\tilde{\pi}(D) = 0$, i.e. D is an exceptional curve, contradicting the fact that X is minimal.

From Proposition 3.1 and Proposition 2.2 we derive

PROPOSITION 3.2.- <u>If</u> $\mathrm{Kod}(X) \geq 0$, <u>then</u> $K_X^2 \geq 0$.

By Riemann-Roch the case $K_X^2 > 0$ leads to $\mathrm{Kod}(X) = 2$, hence for

$\mathrm{Kod}(X) = 0$ or 1 we find $K_X^2 = 0$.

We return to the case b') of $\mathrm{Kod}(X) = 0$. Applying Riemann-Roch to $\mathcal{K}_X^{\otimes n}$, $n \geq 2$, we find that there is a positive canonical divisor $D^{(n)}$ for all $n \geq 2$, hence exactly one such a divisor. Observing that $3D^{(2)} = 2D^{(3)}$ we see that there is also a positive canonical divisor, hence $p_g = 1$. But applying Riemann-Roch as before to $\mathcal{K}_X^{\otimes n}$ we would find $P_n(X) \geq 2$ for $n \geq 2$, contradicting $\mathrm{Kod}(X) = 0$. Therefore case b') is excluded.

Now the case c) of $\mathrm{Kod}(X) = 0$. Since $K_X^2 = 0$, and $q(X) \geq 2$, we find from the Todd formula

$$12(1 - q(X) + p_g(X)) = e(X)$$

and $e(X) = 2 - 4q + b_2(X)$, that $q(X) = 2$, and $e(X) = 0$. From $q(X) = 2$ it follows that there is a surjective map $f : X \to \mathrm{Alb}(X)$, and $e(X) = 0$ implies that f cannot blow down any curves. Hence : $f : X \to \mathrm{Alb}(X)$ is a (ramified) covering. If f is unramified, then X is an abelian surface (and f an isomorphism). If f is ramified, then let $\sum_{i=1}^{n} r_i R_i$ be the ramification divisor on X ; it is a canonical divisor, hence $K(\sum_{i=1}^{n} r_i R_i) = 0$, and since $KR_i \geq 0$ by Proposition 3.1, we have that $KR_i = 0$, $i = 1,\ldots,n$. If R_i^2 were strictly positive, then we would find by Riemann-Roch that $h^0(\mathcal{O}_X(R_i^{\otimes k})) \geq 2$ for large k , hence $P_k(X) \geq 2$ for large k , contradicting $\mathrm{Kod}(X) = 0$. Therefore $R_i^2 \leq 0$ and by the adjunction formula we find that $\tilde{\pi}(R_i) \leq 1$. Since every map from a rational curve into a torus is constant, it follows that there are elliptic curves on $\mathrm{Alb}(X)$, hence $\mathrm{Alb}(X)$ admits a homomorphism onto an elliptic curve, hence also X admits a map onto an elliptic curve. Then Stein factorisation and $\mathbf{C}_{2,1}$ yield that X is an elliptic fibre space over an elliptic curve, with a positive canonical divisor.

So for $\mathrm{Kod}(X) = 0$ we are left with two cases, both of them elliptic fibre spaces over elliptic curves. To clarify these cases we need some general results on elliptic fibre spaces. A profound study of these structures is one of Kodaira's most important contributions to the theory of surfaces ([7], II, III). A central role is played by the following formula for the canonical bundle of a (minimal) elliptic fibre space.

THEOREM 3.4.- Let $f : X \to Y$ be a (minimal) elliptic fibre space. Then

$$\mathcal{K}_X = f^*(\mathcal{L}) \otimes (\prod_{i=1}^{k} \mathcal{O}_X(F_i)^{\otimes (a_i-1)}) ,$$

where $\mathcal{L}$ is a line bundle on Y of degree $\chi(X, \mathcal{O}_X) - 2\chi(Y, \mathcal{O}_Y)$, and where $F_1, \dots, F_k$ are the multiple fibres of f, of multiplicity $a_1, \dots, a_k$ respectively.

Here a multiple fibre means the following. Every component of a fibre has a certain multiplicity. If the GCD of all components of a fibre is $g \geq 2$, then the fibre is called multiple of multiplicity g.

Theorem 3.4 is already sufficient to show that the case c) of $\mathrm{Kod}(X) = 0$ with ramification over $\mathrm{Alb}(X)$ is impossible. In fact, I have shown before that such a surface is an elliptic fibre space over an elliptic curve, with a positive canonical divisor. Since $\chi(X, \mathcal{O}_X) = 0$, Theorem 3.4 implies that there must be multiple fibres, but this gives immediately $P_n(X) \geq 2$ for n large enough, a contradiction.

I shall not treat in detail the last remaining case of $\mathrm{Kod}(X) = 0$, that is case d). Here you start by proving that there exists another elliptic fibration $g : X \to P_1$. Using this fact you then prove the following

THEOREM 3.5.- Every surface X in class d) of $\mathrm{Kod}(X) = 0$ is of the form $E_0 \times E_1 / G$, where E_0 and E_1 are elliptic curves, and where G is a finite subgroup of E_0, acting on $E_0 \times E_1$ by

$$g(x,y) = (x + g, \alpha(g)y) ,$$

$\alpha : G \to \mathrm{Aut}(E_1)$ being an injective homomorphism, such that the two elliptic fibrations are the obvious ones : $E_0 \times E_1 / G \to E_0 / G$ and $E_0 \times E_1 / G \to E_1 / \alpha(G)$.

These surfaces are the hyperelliptic surfaces. As early as 1909 Bagnera and de Franchis made a complete list of these surfaces (compare [3], II, p. 37).

We still have to consider the cases $\mathrm{Kod}(X) = 1$ and $\mathrm{Kod}(X) = 2$.

In the case $\mathrm{Kod}(X) = 1$, we know that there exists an index $n_0 \geq 1$, such that $P_{n_0}(X) \geq 2$. This means that there is a 1-dimensional linear system $|F + D_\lambda|$, $\lambda \in P_1$, of n_0-canonical divisors, where F is the fixed component. Let us assume for simplicity that for general λ the divisor D_λ is irreducible. We have $n_0 K_X^2 = K(F + D_\lambda) = 0$ and since $K_X F \geq 0$, $K_X D_\lambda \geq 0$ by Proposition 3.1 we find $K_X D_\lambda = K_X F = 0$. On the other hand, we also have $(F + D_\lambda)^2 = F(F + D_\lambda) + FD_\lambda + D_\lambda^2 = n_0 K_X F + FD_\lambda + D_\lambda^2 = FD_\lambda + D_\lambda^2 = 0$. Now $FD_\lambda \geq 0$, $D_\lambda^2 \geq 0$, hence $D_\lambda^2 = 0$, and the linear system $|D_\lambda|$ has no base points. By Bertini's theorem it follows that in general D_λ is non-singular. Then the adjunction formula yields that almost all D_λ are elliptic.

For the case that the general D_λ is reducible the argument is a little bit more complicated, but the result is the same.

THEOREM 3.6.- The (minimal) surfaces X with $\mathrm{Kod}(X) = 1$ are exactly the (minimal) elliptic surfaces for which the canonical bundle is of positive degree (for every projective embedding of X).

Finally, the surfaces X with $\mathrm{Kod}(X) = 2$ are by definition the surfaces of general type.

Now a word about what is known concerning the surfaces in the different classes.

$\mathrm{Kod}(X) = -\infty$. Every minimal rational surface is either the projective plane or $\mathrm{Proj}(\mathcal{O}_{P_1} \oplus \mathcal{O}_{P_1}(k))$, $k = 0, 1, 2, \ldots$. The classification of non-rational ruled surfaces is equivalent to the classification of rank 2-vector bundles on curves of genus ≥ 1 (compare [13], ch. V, § 7).

$\mathrm{Kod}(X) = 0$. The classification of abelian surfaces is classical, the classification of hyperelliptic surfaces has already been described, for polarised K_3-surfaces there is a well known conjecture about the period map, of which at least one half (i.e. the injectivity) has been proved ([14]). As to Enriques surfaces, M. Artin proved in his thesis the following result ([1], Theorem 3.1.1).

THEOREM 3.7.- Let X be an Enriques surface, defined over an algebraically closed field k of characteristic $\neq 2$, and let $\xi = (\xi_0, \ldots, \xi_3)$ be homogeneous coordinates in P_3. Then there are linear forms $L_1(\xi)$, $L_2(\xi)$, $L_3(\xi)$ and a quadratic form $\varphi(\xi)$, such that X is the minimal smooth model of the surface

$$c_1 M L_2 L_3 \varphi \xi_0 + c_2 M^2 L_2^2 \xi_0^2 + c_3 M^2 L_3^2 \xi_0^2 + c_4 L_2^2 L_3^2 \xi_0^2 + c_5 L_1 L_2^2 L_3^2 \xi_0 + c_6 L_1^2 L_2^2 L_3^2 = 0, \quad \text{where}$$

$M = a\xi_0 + bL_1$, $a, b, c_1, \ldots, c_6 \in k$.

Conversely, such a surface is in general an Enriques surface. The Enriques surfaces, thus obtained, have in general the edges of a tetrahedron as double locus.

Recently, Horikawa has treated the classification of Enriques surfaces from a different point of view. See [20].

Every Enriques surface is elliptic ([1], [13]).

$\mathrm{Kod}(X) = 1$. For the classification of elliptic surfaces see Kodaira ([7], in particular III) and Deligne-Rapoport ([5]).

$\mathrm{Kod}(X) = 2$. For a survey of what is known about surfaces of general type I refer to my talk at the preceding Bourbaki seminar ([17]), and to [21].

4. The case of characteristic p

In this section 1 summarise the main results of the papers [10] and [3] by Bombieri and Mumford.

Quite independent of the characteristic there is always the division into four classes, according to whether the Kodaira dimension is $-\infty$, 0 , 1 or 2 . For $\mathrm{Kod}(X) = -\infty$ the result is the same as in the classical case, and the same holds by definition in the last case (but you can say less about the surfaces of general type). As to the case of Kodaira dimension 1, you get, but only in characteristic 2 and 3 , apart from elliptic surfaces also quasi-elliptic surfaces, i.e. fibre spaces with almost all fibres rational with a cusp. The main differences between the classical case and the case of positive characteristic occur for Kodaira dimension 0 , but again for characteristic 2 and 3 only. I now describe this case a little bit more in detail.

Let X be a smooth surface, defined over a closed field of any characteristic. Let again $p_g(X) = h^2(X, \mathcal{O}_X) = h^0(\mathcal{K}_X)$, but let $q(X) = \dim \mathrm{Pic}(X)$ (for the classical case this makes of course no difference). From the Todd formula

$$12(1 - h^{0,1}(X) + p_g(X)) = K_X^2 + e(X)$$

we find

$$12p_g(X) + 10 = 8h^{0,1}(X) + 2(2h^{0,1}(X) - b_1) + b_2 ,$$

where b_i is the i-th Betti number of X .

Observe the term $\Delta = 2h^{0,1}(X) - b_1$, which vanishes always in the classical case, but which need not to vanish in general. Using the theory of the Picard scheme, Bombieri and Mumford show : $b_1 = 2q$ and $0 \leq \Delta \leq 2p_g$, Δ even. Once this is known, it is easy to list all possibilities :

b_2	q	$h^{0,1}$	p_g	Δ
22	0	0	1	0
14	1	1	1	0
10	0	$\begin{cases}0\\1\end{cases}$	$\begin{cases}0\\1\end{cases}$	$\begin{cases}0\\2\end{cases}$
6	2	2	1	0
2	1	$\begin{cases}1\\2\end{cases}$	$\begin{cases}0\\1\end{cases}$	$\begin{cases}0\\2\end{cases}$

The surfaces with $b_2 = 22$ are defined to be the K_3-surfaces (this is in accordance with the usual definition in the complex case). It is shown that no surfaces (always minimal, of Kodaira dimension 0) with $b_2 = 14$ exist, and also that the surfaces with $b_2 = 6$ are exactly the abelian varieties. The surfaces with $b_2 = 10$ are all defined as Enriques surfaces. It turns out that there are three types :

(i) those with $h^{0,1} = 0$. These are called the classical Enriques surfaces ;

(ii) (only present if $\mathrm{char}(k) = 2$) those with $h^{0,1} = 1$, for which the Frobenius operation $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is bijective. These are the singular Enriques surfaces ;

(iii) (also only present if $\mathrm{char}(k) = 2$) those with $h^{0,1} = 1$ and for which $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is zero. These are the supersingular Enriques surfaces.

As to the existence of the three classes for $p = 2$ (for $p \geq 3$ see Theorem 3.7), Bombieri and Mumford give a nice example of a family containing at the same time classical, singular and supersingular Enriques surfaces.

Finally, as to the surfaces with $b_2 = 2$, these are either hyperelliptic in exactly the same sense as before, or quasi-hyperelliptic (which occurs only in characteristics 2 and 3). These last surfaces are the surfaces of the form $E \times C / G$, where E is an elliptic curve, C a rational curve with a cusp, and G a finite subgroup scheme of E , acting by

$$g(e,c) = (e + g , \alpha(g)c)$$

with $\alpha : G \to \mathrm{Aut}(C)$ an injective homomorphism. Such a surface has one elliptic and one quasi-elliptic fibration. Again the authors give a list of all possible hyperelliptic and quasi-hyperelliptic surfaces.

Once the Enriques classification is available, it becomes easy to prove results like Theorem 1.2 for all characteristics. Another example of this type is provided by

Example 4.1. A (minimal) surface X is of Kodaira dimension 0 if and only if $\mathcal{K}_X^{\otimes 12}$ is trivial.

As to the proof, the only thing you have to show is that for a (quasi) hyperelliptic surface $12K = 0$. This can be done by applying a generalisation of Theorem 3.4 to the second fibring (over P_1) of the surface. This extension of Theorem 3.4 (see [3], II, Thm 2) is also in the non-classical case of central importance ; the main new feature is the appearance (for positive characteristic) of wild fibres among the multiple fibres ; these wild fibres F are characterised by the property that $\dim \Gamma(F, \mathcal{O}_F) \geq 2$.

REFERENCES

[1] M. ARTIN - On Enriques surfaces, Thesis, Harvard University (1960).

[2] E. BOMBIERI and D. HUSEMÖLLER - Classification and embeddings of surfaces, Algebraic Geometry, Arcata 1974, Proc. Symp. Pure Math., 29 (1975), 329-420.

[3] E. BOMBIERI and D. MUMFORD - Enriques classification of surfaces : II Complex analysis and algebraic geometry, a collection of papers dedicated to K. Kodaira, Shoten, Tokyo and Cambr. Univ. Press, Cambridge (1977) ; III, Inv. Math., 35 (1976), 197-232.

[4] G. CASTELNUOVO - Sulle superficie di genere zero, Mem. Soc. It. delle Scienze, (3) 10 (1896), 103-126 = Mem. Scelte 307-334, Zanichelli, Bologna (1937).

[5] P. DELIGNE et M. RAPOPORT - Les schémas de modules de courbes elliptiques, Lecture Notes in Math., vol. 349, 1975, 143-316, Springer, Berlin-Heidelberg-New York.

[6] F. ENRIQUES - Le superficie algebriche, Zanichelli, Bologna (1949).

[7] K. KODAIRA - On compact analytic surfaces : I, Ann. of Math., 71 (1960), 111-152 ; II ibid. 77 (1963), 563-626 ; III, ibid. 78 (1963), 1-40.

[8] K. KODAIRA - On the structure of compact complex analytic surfaces:I,Amer.J.Math. 86(1964), 751-198 ; II, ibid., 88 (1966), 682-721 ; III, ibid., 90 (1968), 55-83 ; IV, ibid., 90 (1968), 1048-1066.

[9] D. MUMFORD - Lectures on curves on an algebraic surface, Ann. of Math. Studies, 59 (1966), Princeton Univ. Press, Princeton.

[10] D. MUMFORD - Enriques classification of surfaces in char p : I, Global Analysis Papers in Honour of K. Kodaira, University of Tokyo Press and Princeton Univ. Press (1969).

[11] A.N. PARŠIN - Algebraic curves over function fields I, Irv. Akad. Nauk. U.S.S.R., Ser. Math., 32 (1968), [Translation : Math. U.S.S.R. Irv., 2 (1968), 1145-1170].

[12] H. POPP - On moduli of algebraic varieties, III : Fine moduli spaces, Comp. Math., 31 (1975), 237-286.

[13] I.R. ŠAFAREVIČ et al. - Algebraic surfaces, Proc. Steklov Inst. of Math. Moskwa, (1965),[Translation A.M.S., 1967].

[14] I.R. ŠAFAREVIČ - Le théorème de Torelli pour les surfaces algébriques de type K_3 , Actes du congrès int. des math. Nice I, 413-417, Gauthier-Villars, Paris (1971).

[15] K. UENO - Classification theory of algebraic varieties and compact complex

spaces, Lecture Notes in Math., vol. 439, Springer, Berlin-Heidelber-New York, (1975).

[16] K. UENO - Kodaira dimensions for certain fibre spaces, Complex analysis and algebraic geometry, A collection of papers, dedicated to K. Kodaira, Shoten Tokyo, and Cambr. Univ. Press, Cambridge, (1977).

[17] A. VAN DE VEN - Some recent results on surfaces of general type, Sém. Bourbaki, Exposé 500, fév. 1977, dans ce volume, p. -

[18] E. VIEHWEG - Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension 1, Comp. Math. 35 (1977), 197-223.

[19] O. ZARISKI - On Castelnuovo's criterion of rationality in the theory of algebraic surfaces, III, Ill. J. of Math., 2 (1958), 303-315.

[20] E. HORIKAWA - On the periods of Enriques surfaces I,II. Proc. Japan Acad. 53 (1977), 124-127, and ibid. Ser A 5(1977), 53-55.

[21] D. GIESEKER - Global moduli for surfaces of general type. Invent. Math. 43 (1977), 233-282.