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THE LIGHT IN THE NEIGHBORHOOD OF A CAUSTIC [1]

by J. J. DUISTERMAAT

Consider high-frequency waves coming from a point source and traveling through a medium with variable speed of propagation (lenses), boundary conditions (mirrors), etc. At any point $x \in R^n$ the wave pattern will be given locally, and asymptotically as the frequency τ tends to $+\infty$, as a finite sum of integrals of the form

$$(1) \qquad I(x, \tau) = \int e^{i\tau\varphi(x,\alpha)}\, a(x, \alpha, \tau)\, d\alpha ,$$

where $\alpha \in R^k$ is a set of auxiliary integration variables, the amplitude a is a complex-valued C^∞ function on $X \times A \times R_+$ with X, resp. A open in R^n, resp. R^k, and the phase function φ is a real-valued C^∞ function on $X \times A$. It is assumed that $a(x, \alpha, \tau) = 0$ for α outside a fixed compact subset K of A, making I into a C^∞ function of x and τ. Furthermore it is assumed that a has a locally uniform asymptotic expansion of the form

$$(2) \qquad a(x, \alpha, \tau) \sim \sum_{r=0}^{\infty} a_r(x, \alpha)\, \tau^{\mu - r} \qquad \text{as } \tau \to \infty ,$$

and similarly for all derivatives with respect to x and α.

Before the validity of the above representation can be made plausible, some facts have to be collected about the asymptotic behaviour of integrals like (1). For simplification let us consider first

$$(3) \qquad I(\tau) = \int e^{i\tau\varphi(\alpha)}\, a(\alpha)\, d\alpha$$

with $a \in C^\infty(R^k)$, $\operatorname{supp} a \subset K$, $\varphi \in C^\infty(R^k)$ and real-valued.

The first observation is that if there exist $b_j \in C_0^\infty(R^k)$ such that

$$(4) \qquad a(\alpha) = \sum_{j=1}^{k} b_j(\alpha) \cdot \frac{\partial\varphi}{\partial\alpha_j}(\alpha)$$

then, using that

$$(5) \qquad e^{i\tau\varphi}\, \frac{\partial\varphi}{\partial\alpha_j} = \frac{1}{i\tau} \cdot \frac{\partial}{\partial\alpha_j}\, e^{i\tau\varphi}$$

and performing a partial integration in (3) we see that

[1] This lecture is based almost entirely on [2].

(6) $$I(\tau) = -\frac{1}{i\tau}\int e^{i\tau\varphi(\alpha)}\, b(\alpha)\, d\alpha\ , \quad \text{with} \quad b = \sum_j \partial b_j/\partial\alpha_j$$

In other words, if a is in the C_0^∞-ideal $\mathfrak{J} = (\partial\varphi/\partial\alpha_1, \ldots, \partial\varphi/\partial\alpha_k)$ spanned by the first order derivatives of φ with respect to the integration variables, then the oscillatory integral is equal to $\frac{1}{\tau}$ times another oscillatory integral, with the same phase function but another amplitude.

For instance, if $d\varphi(\alpha) \neq 0$ for all $\alpha \in K$ then using a partition of unity it is easy to see that there is a neighborhood A of K such that each $a \in C_0^\infty(A)$ is in $\mathfrak{J}$, and repeating the above observation it follows that $I(\tau)$ is rapidly decreasing (that is of order τ^{-r} for all r) as $\tau \to \infty$. Again using partitions of unity it follows that in general the oscillatory integral modulo rapidly decreasing functions of τ only depends on the behaviour of a in arbitrarily small neighborhoods of the set

(7) $$S_\varphi = \{\alpha \in K\ ;\ d\varphi(\alpha) = 0\}$$

of stationary points of φ. This statement is usually referred to as the <u>principle of stationary phase</u>.

The simplest case with stationary points, and the one of the greatest practical importance, is that we have only one stationary point $\alpha^{(0)}$ which is <u>non-degenerate</u>, that is the symmetric bilinear form

(8) $$Q = \frac{\partial^2\varphi}{\partial\alpha^2}\,(\alpha^{(0)})$$

is non-degenerate. This implies that all functions a which vanish at $\alpha^{(0)}$ are in the ideal $\mathfrak{J}$, so modulo oscillatory integrals with the same phase function but a factor $\frac{1}{\tau}$ in front, $I(\tau)$ depends only on the value $a(\alpha^{(0)})$ of the amplitude a at $\alpha^{(0)}$.

In order to actually compute the asymptotic behaviour one can use the Morse lemma to replace φ with a substitution of variables by $\psi(\beta) = \varphi(\alpha^{(0)}) + \langle Q\beta\, , \beta\rangle/2$ and amplitude b such that $b(0) = a(\alpha^{(0)})$. Then

(9) $$\int e^{i\tau\langle Q\beta\, , \beta\rangle/2}\, b(\beta)\, d\beta = \tau^{-k/2}\int e^{i\langle Q\gamma\, , \gamma\rangle/2}\, b(\tau^{-\frac{1}{2}}\gamma)\, d\gamma$$

and it is shown that

(10) $$\lim_{\tau\to\infty} \int e^{i\langle Q\gamma\, , \gamma\rangle/2}\, b(\tau^{-\frac{1}{2}}\gamma)\, d\gamma = c(Q).b(0)\ ,$$

where $c(Q)$ is an analytic function of Q in the domain of complex symmetric bilinear forms Q with Im Q > 0 , and continuous for Im Q ≥ 0 . Since it is elementary that

(11) $$\int_{R^k} e^{-\langle B\gamma , \gamma\rangle/2} \, d\gamma = (2\pi)^{k/2} \, (\det B)^{-\frac{1}{2}} \qquad \text{if } B > 0 ,$$

it follows that

(12) $$c(Q) = (2\pi)^{k/2} \, |\det Q|^{-\frac{1}{2}} \, e^{\frac{1}{4}\pi i \, . \, \operatorname{sgn} Q} \; .$$

In this way one ends up with the asymptotic expansion

(13) $$I(\tau) \sim e^{i\tau\varphi(\alpha^{(0)})} . \left(\frac{2\pi}{\tau}\right)^{k/2} . \sum_{r=0}^{\infty} a^{(r)} \, \tau^{-r} \quad , \quad \tau \to \infty \; ,$$

here $a^{(0)} = |\det Q|^{-\frac{1}{2}} . e^{\frac{1}{4}\pi i \, . \, \operatorname{sgn} Q} . a(\alpha^{(0)})$ and $a^{(r)}$ is a linear form in the derivatives of a at $\alpha^{(0)}$ of order $\leq 2r$. In particular, as $\tau \to \infty$, the function

(14) $$e^{i\tau[\varphi - \varphi(\alpha^{(0)})]} \, . \, \tau^{k/2} / c(Q)$$

tends in the distribution sense to the Dirac measure at $\alpha^{(0)}$. This is a variant of the classical statement that the Gaussian density

(15) $$\gamma \mapsto \left(\frac{\tau}{2\pi}\right)^{k/2} . (\det B)^{\frac{1}{2}} . e^{-\langle B\gamma , \gamma\rangle/2} \qquad \text{with } B > 0$$

tends to the Dirac measure at the origin as $\tau \to \infty$. Such a variant is useful if there is a non-degenerate symmetric bilinear form Q available which however happens to be non-definite. For example, this summer at a conference in Durham (England), Victor Guillemin showed me how to use this in order to prove formula's of the form

(16) $$c.f(e) = (\varpi \, F_f)(e)$$

for Harish-Chandra's transformation, mapping functions f on a semi-simple Lie group G to functions F_f on a Cartan subgroup H . In this case the Killing form is the natural bilinear form. It is non-degenerate, but only definite in the case that G is compact. In fact I believe that the "oscillatory integral" point of view is illuminating in large parts of Harish-Chandra's work. Formula (13) is usually referred to as the method of stationary phase, as I indicated above it should really be regarded as a tool in computations.

The statements for (3) have versions for (1) which depend smoothly, resp. locally uniformly on x in a rather obvious way. Write

(17) $$S_\varphi = \{(x , \alpha) \in X \times A \; ; \; \frac{\partial\varphi}{\partial\alpha}(x , \alpha) = 0\} \; .$$

If $\operatorname{supp} a_r \cap S_\varphi = \emptyset$ for all r then $I(x,\tau)$ is rapidly decreasing as $\tau \to \infty$, locally uniformly in x , this is called a shadow.

Secondly, if for each $x \in X$ there is exactly one $\alpha = \alpha(x) \in A$ such that $(x,\alpha) \in S_\varphi$, and if $Q(x) = \frac{\partial^2\varphi}{\partial\alpha^2}(x,\alpha(x))$ is non-degenerate for each $x \in X$, then $x \mapsto \alpha(x)$ is C^∞ , and

$$(18) \qquad I(x,\tau) \sim e^{i\tau\varphi(x,\alpha(x))} \cdot \left(\frac{2\pi}{\tau}\right)^{k/2} \cdot \sum_{r=0}^{\infty} a^{(r)}(x) \cdot \tau^{\mu-r}$$

as $\tau \to \infty$, here $a^{(r)} \in C^\infty$ and the asymptotics is locally uniformly in x . In particular

$$(19) \qquad a^{(0)}(x) = |\det Q(x)|^{-\frac{1}{2}} e^{\frac{1}{4}\pi i \cdot \operatorname{sgn} Q(x)} \cdot a_0(x,\alpha(x)) .$$

So in this case we have an asymptotic sum of simple progressing waves with phase $\tilde{\varphi}(x) = \varphi(x,\alpha(x))$. Note that the order of the amplitude changes when integrating away the α-variables, for this reason the number $\nu = \mu - \frac{k}{2}$ will be called the order of the oscillatory integral (1), even in the general case when we make no assumptions on the derivatives of φ .

In order to understand more general oscillatory integrals one can test them against oscillatory functions of the form $e^{-i\tau\psi(x)} \cdot b(x)$, with $\psi \in C^\infty$ and real-valued, $b \in C_0^\infty$ and $b(x^{(0)}) = 1$. This resembles the investigation of the singularities of a distribution u near $x^{(0)}$ by cutting it off near $x^{(0)}$ with b and then looking at the asymptotic behaviour of the Fourier transform of bu at infinity. So we consider the asymptotic behaviour of

$$(20) \qquad \langle I(\,.\,,\tau), e^{-i\tau\psi} b\rangle = \iint e^{i\tau[\varphi(x,\alpha)-\psi(x)]} a(x,\alpha,\tau)\, b(x)\, d\alpha\, dx$$

as $\tau \to \infty$. This can be computed with the method of stationary phase if there is only one stationary point $(x^{(0)},\alpha^{(0)})$ for $\varphi - \psi$, that is

$$(21) \qquad \frac{\partial\varphi}{\partial\alpha}(x^{(0)},\alpha^{(0)}) = 0 \quad \text{and} \quad \frac{\partial\varphi}{\partial x}(x^{(0)},\alpha^{(0)}) = d\psi(x^{(0)}) ,$$

which moreover is non-degenerate. Now the latter can only happen if

$$(22) \qquad \operatorname{rank} \frac{\partial}{\partial(x,\alpha)} \frac{\partial}{\partial\alpha} \varphi(x^{(0)},\alpha^{(0)}) = k ,$$

in which case φ is called a non-degenerate phase function. It can be proved that the property (22) is generic, and in particular it will be satisfied when dealing with wave patterns originating at a point source, as we will see. (22) implies that S_φ is a C^∞ manifold through $(x^{(0)},\alpha^{(0)})$ of codimension $= k$, and also that

$$(23) \qquad i_\varphi : S_\varphi \ni (x,\alpha) \mapsto \left(x, \frac{\partial\varphi}{\partial x}(x,\alpha)\right) \in T^*X$$

is an immersion on a neighborhood of $(x^{(0)}, \alpha^{(0)})$ in S_φ. So the image Λ_φ is a smooth n-dimensional submanifold of T^*X which turns out to be a Lagrange manifold, that is the restriction of the canonical 2-form $\sum_j d\xi_j \wedge dx_j$ to Λ_φ vanishes identically. Conversely every Lagrange manifold Λ in the (x, ξ)-space T^*X is locally of the form Λ_φ for some non-degenerate phase function φ. Writing $\xi^{(0)} = \frac{\partial\varphi}{\partial x}(x^{(0)}, \alpha^{(0)}) = d\psi(x^{(0)})$, we note that (21) means that $d\psi$ (or rather its graph in T^*X) intersects Λ_φ at $(x^{(0)}, \xi^{(0)})$, and the non-degeneracy of the stationary point just means that the intersection is transversal.

Since Λ_φ can be found from looking for which ψ (and b) the integral (20) is not rapidly decreasing, it is an invariant for I in the sense that $\Lambda_\varphi = \Lambda_{\tilde\varphi}$ if the classes of oscillatory integrals I defined by φ, resp. $\tilde\varphi$ are the same. Here it is even allowed to have different integration dimensions k , resp. $\tilde k$. As Hörmander showed, there is also a strong converse here : if $\Lambda_\varphi = \Lambda_{\tilde\varphi}$ and say $k \leq \tilde k$, then one can regroupe the $\tilde\alpha$-variables such that with the notation $\tilde\alpha' = (\tilde\alpha'_1, \ldots, \tilde\alpha'_k)$, $\tilde\alpha'' = (\tilde\alpha''_{k+1}, \ldots, \tilde\alpha''_{\tilde k})$ the function $\tilde\alpha'' \mapsto \tilde\varphi(x, \tilde\alpha', \tilde\alpha'')$ has a non-degenerate stationary point at $\tilde\alpha'' = \tilde\alpha''(x, \tilde\alpha')$. Writing $\tilde\varphi'(x, \tilde\alpha') = \tilde\varphi(x, \tilde\alpha', \tilde\alpha''(x, \tilde\alpha'))$ one then has a regular substitution of variables $\alpha = \alpha(x, \tilde\alpha')$ such that $\tilde\varphi'(x, \tilde\alpha') = \varphi(x, \alpha(x, \tilde\alpha')) + \text{const.}$. So integrating away the $\tilde\alpha''$-variables with the method of stationary phase and then making a substitution of variables one obtains that modulo factors of the form $e^{i\tau \cdot \text{const.}}$ (phase shifts) the class of oscillatory integrals defined by φ is the same as that defined by $\tilde\varphi$.

The coefficient of the top order term in the asymptotic expansion of (20) regarding to ψ only depends on $\frac{\partial^2\psi}{\partial x^2}(x^{(0)})$, which can be identified with the tangent space of $d\psi$ at $(x^{(0)}, \xi^{(0)})$. This can be any Lagrange space in $T_{(x^{(0)}, \xi^{(0)})}(T^*X)$ transversal to the fiber (= ξ-space) and to $T_{(x^{(0)}, \xi^{(0)})}\Lambda$. Going from one ψ to another leads to multiplication by a factor depending only on these Lagrange spaces. The functions of these Lagrange spaces with these transition properties form a 1-dimensional complex vector space $L_{(x^{(0)}, \xi^{(0)})}$ and it is rather obvious that their union over all $(x^{(0)}, \xi^{(0)}) \in \Lambda$ is a smooth complex line bundle L over Λ. In this way the coefficients in the top order terms in (20) are regarded as a section of L, which is called the principal symbol σ of the oscillatory integral I of order ν. If Λ is some (global) Lagrange manifold in T^*X then an oscillatory function u defined by Λ of order ν is defined as a locally (in x) finite sum of integrals like in (1) with Λ_φ an open piece in Λ. Its symbol is defined as the sum of the symbols of each of the integrals.

It is not surprising in view of the factor $c(Q)$ (see (12)) in the method of stationary phase, that $L = \Omega_{\frac{1}{2}} \otimes M$, where $\Omega_{\frac{1}{2}}$ denotes the line bundle of densities of order $\frac{1}{2}$ on Λ and M, the Maslov bundle, is a bundle over Λ with constant local transition functions which are powers of i. In particular, if σ is the principal symbol of a compactly supported oscillatory function u defined by Λ and of order ν, then $|\sigma|^2 = \sigma . \bar{\sigma}$ is a density of order 1 on Λ because $M.\bar{M} = 1$, and

$$(24) \qquad \int_X |u|^2 = \tau^{2\nu}\left(\int_\Lambda |\sigma|^2 + O(\tau^{-1})\right) \qquad \text{as } \tau \to \infty .$$

This shows that, since σ is smooth, the energy of u in a domain U in x-space is asymptotically proportional to the n-dimensional volume of $\pi^{-1}(U)$ in Λ, where π denotes the projection $\Lambda \ni (x, \xi) \longmapsto x$ from Λ onto the base (x-) space. For instance, if U shrinks as a ball with decreasing radius to the point $x^{(0)}$ then the ratio between the energy in U and the volume in U (asymptotically as $\tau \to \infty$, let's say for $\nu = 0$) tends to $+\infty$ if and only if $x^{(0)}$ is a singular value of π. Such points will be called caustic points for Λ because there the light "burns". Because it is easily verified that the kernel of $Q(x^{(0)})$ and the kernel of the tangent mapping of π at $(x^{(0)}, \xi^{(0)})$ have the same dimension, these are exactly the points where we do not have simple progressing waves as in (18).

Up till now nothing has been said about the wave mechanics, that is the equations which have to be satisfied by u. It will be assumed that these are of the form " Pu is asymptotically small as $\tau \to \infty$ ". Here P is linear partial differential operator with coefficients depending smoothly on x and polynomially on τ. On has

$$(25) \qquad e^{-i\tau\varphi(x)}\, P(e^{i\tau\varphi})(x) = \tau^m . p(x, d\varphi(x)) + O(\tau^{m-1})$$

for an invariantly defined C^∞ function p on T^*X, which is a polynomial in ξ, called the principal symbol of P. It follows directly from the definition that if u is an oscillatory function defined by Λ of order ν with principal symbol σ, then Pu is an oscillatory function defined by Λ of order $\nu + m$ with principal symbol $p . \sigma$. So if we want to satisfy $Pu = 0$ asymptotically we need $p . \sigma = 0$, that is p = 0 on Λ if we take $\sigma \neq 0$.

Now suppose that p is real and $dp \neq 0$ on Λ . Because Λ is a Lagrange manifold, p = 0 on Λ implies that Λ is invariant under the solution curves of

(26) $$\frac{dx}{dt} = \frac{\partial p}{\partial \xi}(x, \xi) \quad , \qquad \frac{d\xi}{dt} = -\frac{\partial p}{\partial x}(x, \xi) \ ,$$

that is the Hamilton system defined by the function p . Its solution curves in $p^{-1}(\{0\})$ are called the bicharacteristic strips, and their projections in x-spaces the bicharacteristic curves of the operator P .

If p = 0 on Λ then Pu is actually an oscillatory function defined by Λ of order $\nu + m - 1$, and its symbol of that order is equal to

(27) $$\frac{1}{i} \mathcal{L}_{H_p} \sigma + q.\sigma \ .$$

Here $\mathcal{L}$ denotes Lie-derivative (of densities of order $\frac{1}{2}$), H_p is the Hamilton vector field on the right hand sides in (36), and $q = q(x, \xi)$ is another invariantly defined function on T^*X called the subprincipal symbol of P . (The fact that everything is invariantly defined means that the above is all valid in the same way if X is a smooth n-dimensional manifold.) Demanding that (27) be = 0 means solving an ordinary linear differential equation for σ along the bicharacteristic strips. In particular σ is determined along the whole strip if it is given at one of its points. One says that σ propagates along the bicharacteristic strips. Regarding Λ together with its bicharacteristic strips as the geometrical optics of the oscillatory solution u , then the statement is that the geometrical optics describes the propagation of the high-frequency asymptotics of waves. The light rays are identified with the bicharacteristic curves.

For global oscillatory solutions we have to take the full flow-out of a local Lagrange manifold along the bicharacteristic strips. Following Λ along such a bicharacteristic strip it may "turn over", that is its tangent space may get an intersection with the fiber of positive dimension, the projection in X will then be a caustic point. This will also correspond to a singular concentration of light rays there, which is the geometrical optics definition of a caustic point.

By induction on lower order terms one can construct global oscillatory functions defined by Λ such that Pu is not only of order $< \nu + m - 1$ but even rapidly decreasing as $\tau \to \infty$, and there are also results about the uniqueness of such solutions.

Waves coming from a point $x^{(0)}$ are obtained by defining Λ as the flow-out of

$$(28) \qquad \Lambda^{(0)} = \{x^{(0)}, \xi) \; ; \; p(x^{(0)}, \xi) = 0\}$$

by the bicharacteristic strips. If $\frac{\partial p}{\partial \xi}(x^{(0)}, \xi) \neq 0$ when $p(x^{(0)}, \xi) = 0$, $\dim \Lambda^{(0)} = n-1$ and the bicharacteristic strips are transversal to it, so $\dim \Lambda = n$. Because $\Lambda^{(0)}$ is isotropic for the canonical 2-form and $p = 0$ on $\Lambda^{(0)}$ it follows that Λ is a Lagrange manifold contained in $p^{-1}(\{0\})$, and the proof of the statement that the wave pattern at an arbitrary point is given by locally finite sums of integrals as in (1) now is in sight. (Except for the mirrors. If the light rays hit them transversally it is not hard to prove that one gets oscillatory solutions of the same kind defined by a "reflected" Lagrange manifold, see for instance Chazarain [1]. But in the case of tangential light rays the solutions are of a much more complicated nature, see for instance Melrose [4] and Taylor [5].)

We now turn to the problem of determining the actual asymptotic behaviour of $I(x,\tau)$ near a caustic point. The most general result is that if the function φ in (3) is real analytic then, using Hironaka's theorem on resolution of singularities, one has (assuming that $\varphi = 0$ on S_φ):

$$(29) \qquad I(\tau) \sim \sum_{t,s} \sum_{r=0}^{\infty} c_{r,s,t}(a)\, \tau^{\alpha_s - r} (\log \tau)^t \;, \quad \tau \to \infty \;,$$

where t, s range over finitely many natural numbers, $\alpha_s \in Q$. For analytic functions with an isolated singularity in the complex domain there is a relation between the α_s, t and the monodromy of the singularity. See Malgrange [3]. However, in general it is hard to compute the α_s, t or $c_{r,s,t}(a)$, and also these approaches do not seem to give information whether the estimates are locally uniform in the presence of parameters. (See erratum, p. 490-11.)

Let $\varphi \in C^\infty(X \times A)$, real-valued. Considering $\varphi(x, \alpha)$ as a family of functions of α, or an <u>unfolding</u> of a function of α in Thom's terminology, one may call φ <u>equivalent</u> as such to $\tilde{\varphi} \in C^\infty(X \times A)$, if there exists a diffeomorphism of the form $H : (x, \alpha) \mapsto (\tilde{x}(x), \tilde{\alpha}(x, \alpha))$ and a function $\psi \in C^\infty(X)$ such that $\tilde{\varphi} \circ H = \varphi + \psi$. If $\tilde{I}(\tilde{x}, \tau)$ is an oscillatory integral with phase function $\tilde{\varphi}$ and amplitude $\tilde{a}$, then $\varphi \sim \tilde{\varphi}$ implies that

$$(30) \qquad I(\tilde{x}(x), \tau) = e^{i\tau\psi(x)}\, I(x, \tau) \;,$$

where I is an oscillatory integral with phase function φ and corresponding amplitude a. So if we disregard x- dependent phase shifts $\psi(x)$ then the oscillatory integrals defined by equivalent phase functions can be transformed into one another just by a diffeomorphism of x-space. In particular also their asymptotic growth at corresponding points is the same and for instance their caustic sets must be diffeomorphic. φ will be called a stable unfolding if there is a neighborhood U of φ in $C^\infty(X \times A)$ in the Whitney topology such that $\tilde{\varphi} \sim \varphi$ for every $\tilde{\varphi} \in U$. φ will be called stable at $(x^{(0)}, \alpha^{(0)}) \in X \times A$ if there exist open neighborhoods $X^{(0)}$, resp. $A^{(0)}$ of $x^{(0)}$, resp. $\alpha^{(0)}$ such that the restriction of φ to $X^{(0)} \times A^{(0)}$ is stable. If $n \leq 5$ the generically φ is stable at every point. For arbitrary n generically φ can at least be embedded in a family with more parameters which is stable at any point. Finally Mather theory shows that φ is stable if and only if φ is infinitesimally stable, that is :

$$(31) \qquad C^\infty(X \times A) = \sum_j C^\infty(X \times A) \frac{\partial \varphi}{\partial \alpha_j} + \sum_\ell C^\infty(X) \frac{\partial \varphi}{\partial x_\ell} + C^\infty(X) \ .$$

Here we recognize the ideal spanned by the derivatives of φ with respect to the α_j again. Writing

$$(32) \qquad a_r(x, \alpha) = \sum_j a_{rj}(x, \alpha) \frac{\partial \varphi}{\partial \alpha_j}(x, \alpha) + \sum_\ell b_{r\ell}(x) \frac{\partial \varphi}{\partial x_\ell}(x, \alpha) + c_r(x) \ ,$$

cutting off in the α-variables, performing a partial integration with the $\partial\varphi/\partial\alpha_j$ - terms and repeating the procedure with the amplitude of the newly obtained oscillatory integral, one obtains for stable φ an asymptotic expansion of the form

$$(33) \quad I(x, \tau) \sim \sum_{r=0}^{\infty} \tau^{\mu-r} \left[\nu_r(x)\, F(x, \tau) + \frac{1}{i\tau} \sum_{\ell=1}^{n} \nu_{r\ell}(x) \frac{\partial F}{\partial x_\ell}(x, \tau)\right] .$$

Here ν_r, $\nu_{r\ell} \in C^\infty(X)$, the expansion is locally uniformly in x and

$$(34) \qquad F(x, \tau) = \int e^{i\tau\varphi(x, \alpha)} \chi(\alpha)\, d\alpha$$

for some $\chi \in C_0^\infty(A)$ such that $\chi = 1$ in a neighborhood of the set of $\alpha \in A$ for which there is an $x \in X$ with $(x,\alpha) \in S_\varphi$. If we restrict to suitable neighborhoods X, resp. A of $x^{(0)}$, resp. $\alpha^{(0)}$, this can be arranged. (Everything under the assumption that φ is stable at $(x^{(0)}, \alpha^{(0)})$.) This reduces the problem to the asymptotic expansion of the oscillatory integral (34) with the phase function φ and amplitude essentially equal to 1 , and its x-derivatives which is an oscillatory integral with phase function φ and amplitude essentially equal to $i\tau \frac{\partial \varphi}{\partial x_\ell}(x,\alpha)$.

The next step which one can make is that φ is locally equivalent to an unfolding of the form

$$(35) \qquad \varphi(x, \alpha) = f(\alpha) + \sum_{\ell=1}^{n} x_\ell\, f_\ell(\alpha) \ ,$$

where f is a polynomial which can be taken as the Taylor expansion of $\alpha \mapsto \varphi(x^{(0)}, \alpha)$ at $\alpha^{(0)}$ up to some order $N+1$, and the f_ℓ are arbitrary monomials such that each polynomial of degree $\le N$ can be written as

$$(36) \qquad g(\alpha) = \sum_j c_j(\alpha) \frac{\partial f}{\partial \alpha_j}(\alpha) + \sum_\ell d_\ell f_\ell(\alpha)$$

for some polynomials c_j , constants d_ℓ , and the equality holding only modulo terms of degree $\le N$. This step is the analogue of the use of the Morse lemma with parameters in the method of stationary phase. The polynomials f occuring here are exactly those with an isolated singularity in the complex domain to which Malgrange's observations about the relation with the monodromy apply.

Despite their apparent special character, it is still a very hard problem to determine the asymptotic behaviour of the oscillatory integrals to which we have reduced now. In fact I only know somewhat what happens if one can choose f to be weighted homogeneous, that is

$$(37) \qquad f(t^{r_1}\alpha_1, \ldots, t^{r_k}\alpha_k) = t f(\alpha_1, \ldots, \alpha_k), \qquad t > 0,$$

for some real numbers $r_1, \ldots, r_k$. By an analytic change of α-variables it can be achieved that (37) holds with $0 < r_j \le \frac{1}{2}$ and then the r_j are uniquely determined and rational. Writing

$$(38) \qquad f_\ell(\alpha) = \prod_{j=1}^k \alpha_j^{s_{\ell j}}, \qquad s_\ell = \sum_{j=1}^k s_{\ell j} r_j$$

it is not surprising to find that

$$(39) \qquad \begin{cases} \int e^{i\tau f(\alpha)} \chi(\alpha)\, d\alpha \sim \tau^{-\Sigma r_j} \int e^{if(\alpha)}\, d\alpha \\ \int e^{i\tau f(\alpha)} f_\ell(\alpha) \chi(\alpha)\, d\alpha \sim \tau^{-\Sigma r_j - s_\ell} \int e^{if(\alpha)} f_\ell(\alpha)\, d\alpha \end{cases}$$

as $\tau \to \infty$. Here for any polynomial P we define

$\int e^{if} . P = \int e^{if} . P . \chi + \int e^{if} . P . (1 - \chi)$. The last integral is rewritten using partial integrations, which become possible because f has no stationary points in $\mathrm{supp}(1 - \chi)$, untill the integrand is absolutely integrable. Locally uniform x-dependent versions hold if $s_\ell \le 1$ for all ℓ , if we replace $f(\alpha)$ in the left hand sides of (39) by $f(\alpha) + \sum_{\ell=1}^n x_\ell f_\ell(\alpha)$ and in the right hand sides by $f(\alpha) + \sum_{\ell=1}^n \tau^{1-s_\ell} x_\ell f_\ell(\alpha)$. (Saito has proved that the f with $s_\ell \le 1$ for all ℓ are just those for which any local deformation is equivalent to a weighted homogeneous polynomial again.) If $s_\ell < 1$ for all ℓ then the functions of x appearing in the right hand sides of (39) are entire analytic, and non-zero at $x = 0$. These are exactly the "simple singularities" of Arnol'd, they contain all the "elementary catastrophies", the pictures of which are just the caustic sets.

LITERATURE

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ERRATUM

p. 490.08, line 22 : However, see A.N. Varchenko, Newton polygons and estimates of oscillatory integrals. Funct. Anal. and its Appl., vol. 10, N° 3, 1976, 13-38.