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SOME RECENT RESULTS ON SURFACES OF GENERAL TYPE

by A. VAN DE VEN

1. Introduction

The problem behind this whole talk is the problem of classifying compact complex surfaces, i.e. compact 2-dimensional complex manifolds.

If we start by looking at the case of (smooth, compact, connected) complex curves X , then the classification is according to their genus $\pi(X)$, which can have all values $0,1,2,\dots$ and then (apart from the simple cases $\pi = 0,1$) there is for each genus π a Teichmüller space, the points of which parametrise the curves X with $\pi(X) = \pi$.

For complex surfaces (here always to be taken compact and connected) we look for something similar : first a coarse classification according to the value of some basic numerical invariants, and then for each coarse class a certain number of parametrising families (not necessarily finitely or even countably many, and not necessarily as good as a Teichmüller space).

Many numerical invariants are available : topological ones (e.g. betti numbers), analytic ones (e.g. dimensions of cohomology groups of coherent sheafs) and mixed ones like Chern numbers. It has become customary to take as a first numerical invariant the Kodaira dimension, which is defined in the following way.

Let X be a connected, compact complex manifold, of dimension n , $\mathcal{L}$ a holomorphic rank-one vector bundle (line bundle, invertible sheaf) on X , $\Gamma(X,\mathcal{L})$ the space of sections of $\mathcal{L}$, $N = \dim \Gamma(X,\mathcal{L})$ and $S \subset X$ the analytic subset where all elements of $\Gamma(X,\mathcal{L})$ vanish. If $N > 0$, there is associated with $\mathcal{L}$ a map $f_{\mathcal{L}} : X - S \to P_{N-1}$. The closure of the image is an algebraic variety of dimension $d(\mathcal{L})$ ($f_{\mathcal{L}}$ is not uniquely determined, but two of these maps differ by a projective transformation of P_{N-1}). $\mathcal{L}$ is the pullback of the hyperplane bundle of P_{N-1} on $X - S$.

Now let $\mathcal{K}_X$ be the canonical bundle on X , i.e. the determinant bundle of the covariant tangent bundle. Then the Kodaira dimension $\mathrm{Kod}(X)$ is defined by

$$\mathrm{Kod}(X) = \begin{cases} -\infty & \text{if } \Gamma(X, \mathcal{K}_X^{\otimes n}) = 0 \text{ for all } n \geq 1, \\ \max_{n \geq 1} d(\mathcal{K}_X^{\otimes n}) & \text{otherwise}. \end{cases}$$

We have always that $\mathrm{Kod}(X) \leq n$.

Examples.- 1) For curves we have : $\mathrm{Kod}(X) = -\infty$ if $\pi(X) = 0$, $\mathrm{Kod}(X) = 0$ if $\pi(X) = 1$ and $\mathrm{Kod}(X) = 1$ if $\pi(X) \geq 2$.

2) $\mathrm{Kod}(P_2) = -\infty$, $\mathrm{Kod}(\text{torus}) = 0$, Kod(product of an elliptic curve with a curve of genus ≥ 2) = 1 , Kod(non-singular surface of degree ≥ 5 in P_3) = 2 .

The surfaces X with $\mathrm{Kod}(X) \leq 1$ are called special surfaces. Their classification has been completed, at least in principle, except for two types (K_3-surfaces and surfaces of class VII). We shall not consider this classification here. The situation is completely different, however, for surfaces X with $\mathrm{Kod}(X) = 2$. These surfaces, the analogues of curves with genus ≥ 2 , are called surfaces of general type. They are all known to be projective algebraic ([1], p. 415).

To get an idea of how it may be possible to classify surfaces of general type, we first look again at curves of general type, i.e. curves X with $\pi(X) \geq 2$. For these curves we have by Riemann Roch and a vanishing theorem that $\dim \Gamma(X, \mathcal{K}_X^{\otimes n}) = (2\pi - 2)n - \pi + 1$ for $n \geq 2$. Furthermore, the map $f_{\mathcal{K}_X^{\otimes n}} : X \to P_{(2\pi-2)n-\pi}$ is always an isomorphism if $n \geq 3$. Taking $n = 3$ we see that we can classify smooth curves of genus $\pi \geq 2$ in the following way : we take the Chow scheme in dimension 1 and degree $6\pi - 6$ in $P_{5\pi-6}$ (very roughly, the Chow scheme is the algebraic variety that parametrises all positive algebraic cycles $\sum n_i V_i$, with $n_i > 0$ and V_i varieties of a given dimension in P_n , such that $\sum n_i \deg(V_i)$ is fixed), and then the subscheme formed by those smooth curves X of genus π for which the hyperplane bundle is $\mathcal{K}_X^{\otimes 3}$. Now two such curves are isomorphic if and only if they are projectively equivalent, hence we still have to divide by the action of $PGL(5\pi - 6, \mathbf{C})$. This is one approach to the Teichmüller spaces.

Now fixing the genus of a curve means fixing the (first) Chern number ($c_1(X) = 2 - 2\pi(X)$). Therefore it is natural to try to find a rough classification of surfaces of general type X according to their Chern numbers $c_1^2(X)$ and $c_2(X)$. This approach works, but first a few remarks.

First of all, we can restrict ourselves to minimal surfaces of general type. In fact, if X is a 2-dimensional complex manifold, and $x \in X$, then you can blow up $x \in X$, i.e. replace x in a specific way by a curve E, such that the result $\tilde{X}$ is again a complex manifold of dimension 2. The converse is blowing down E in $\tilde{X}$ to a point. These curves E are called exceptional curves (of the first kind) ; they are precisely the non-singular rational curves E with $E^2 = -1$, and no other irreducible curves can be replaced by a point such that the result is smooth. A complex surface is called minimal if there are no exceptional curves on X. Every surface can be obtained from a minimal one by blowing up a finite number of times. In the case of surfaces of géneral type the minimal model is uniquely determined, and blowing up or down in a surface of general type gives again such a surface. So for classification purposes it is reasonable to consider minimal surfaces of general type only.

For these surfaces we cannot have, like in the case of curves, that there is always an n such that $f_{\mathcal{K}_X^{\otimes n}}$ is an isomorphism, for there are many minimal surfaces of general type with (-2)-curves, i.e. rational curves C with $C^2 = -2$, and the restriction $\mathcal{K}_X \mid C$ is trivial.

Nevertheless, there exists a good theory, which I shall sketch now (compare [2], in particular Theorem 1 and Theorem 3). This theory also works in the case of a closed field of characteristic 0.

THEOREM 1.1.- *Let X be a minimal surface of general type. Then for $n \geq 4$ the bundle $\mathcal{K}_X^{\otimes n}$ is spanned by its global sections,* i.e. $f_{\mathcal{K}_X^{\otimes n}}$ *is an everywhere defined map.*

Next, it can be proved that on a surface of general type there is only a finite number of (-2)-curves. Let U be their union (as a subset of X), and let $U = U^{(1)} \cup \ldots \cup U^{(k)}$ be the decomposition of U into connected components (k depends on X).

We consider the line bundle $\mathcal{K}_X^{\otimes n}$, $n \geq 2$. By Riemann Roch and suitable vanishing theorems we have that

$$\dim \Gamma(X, \mathcal{K}_X^{\otimes n}) = \tfrac{1}{2}\, n(n-1)c_1^2(X) + \frac{1}{12}\,(c_1^2(X) + c_2(X)) = N(n, c_1^2(X), c_2(X)) .$$

THEOREM 1.2.- *Let X be a minimal surface of general type, and let $n \geq 5$. Then $f_{\mathcal{K}_X^{\otimes n}} = f$ maps X onto a surface of degree $c_1^2(X)n^2$ in $P_{N(n, c_1^2(X), c_2(X))}$.*

The surface $f(X)$ has k singular points : $x_1, \ldots, x_k$. The restriction $f \mid X - U$

<u>is an isomorphism onto</u> $f(X) - \bigcup_{i=1}^{k} x_i$, <u>and</u> $f(U^{(i)}) = x_i$, $i = 1,\dots,n$.

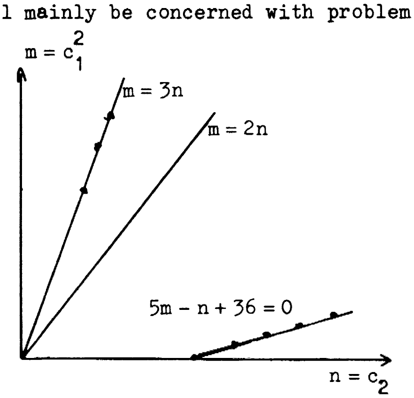
Taking $n = 5$ it follows from this theorem that we can obtain <u>all</u> (isomorphism classes of) <u>minimal surfaces of general type</u> X <u>with given Chern numbers</u> $c_1^2(X)$ <u>and</u> $c_2(X)$ as follows : we take in $P_{N(5\,,\,c_1^2(X)\,,\,c_2(X))}$ the Chow scheme in dimension 2 and degree $25c_1^2(X)$. In this Chow scheme we take the subscheme of surfaces Y , with isolated singularities only, for which on a minimal desingularisation Y the bundle $\mathcal{K}_X^{\otimes 5}$ is the pullback of the hyperplane bundle on P_N . Finally we divide by the action of $PGL(N\,,\mathbf{C})$. It has been proved by Popp ([12], p. 72) that the result is an algebraic $\mathbf{C}$-space of finite type. Roughly we can say : "for a given c_1^2 , c_2 we have a finite number of families of minimal surface of general type".

So we are faced with two problems :

a) For which pairs of integers (m,n) does there exist a minimal surface of general type X with $c_1^2(X) = m$, $c_2(X) = n$? We shall call such a pair (m,n) <u>representable</u>.

b) Find for a given representable pair the structure of the algebraic space described before.

We shall mainly be concerned with problem a).



The following properties of representable pairs are known :

1) $m + n \equiv 0(12)$ (this follows from Riemann Roch, or from topology)

2) $m\,,\,n > 0$ ([1], p. 415 ; [16], p. 285)

3) $n \leq 5m + 36$. There is an infinity of representable pairs on the line $5m - n + 36 = 0$ (see [7], Part. I)

4) There is an infinity of representable points on the line $m = 3n$ (see [11], [16]).

Since nobody ever found a representable pair with $m > 3n$, there was for years the

CONJECTURE.- For every surface of general type X the inequality $c_1^2(X) \leq 3c_2(X)$ holds.

Remark.- If the conjecture is proved for minimal surfaces of general type, then it is true for all surfaces of general type, because blowing up increases c_2 by 1 and it decreases c_1^2 by 1 .

Partial results had been obtained. In 1966 I proved that $c_1^2(X) \leq 8c_2(X)$ ([15], [16]), and a year ago Bogomolov ([14]) obtained the inequality $c_1^2(X) \leq 4c_2(X)$. Finally, last November, Miyaoka ([11]) proved the conjecture. It should be said immediately that, although Bogomolov did not prove the conjecture, his reflections have very much inspired Miyaoka's proof. Bogomolov's considerations remain very interesting, because he relates the problem to the stability of vector bundles, thus obtaining a promising outlook and several nice applications.

I would like to present here a simplified version of Miyaoka's proof. I think it works for a closed field of characteristic 0 , but not in the case of positive characteristic.

2. Miyaoka's proof

To start with, some notations. A divisor on a surface X is a finite sum $D = \sum n_i C_i$, with $n_i \in \mathbb{Z}$, C_i an irreducible curve on X . D is non-negative if all $n_i \geq 0$, and it is positive (effective) if it is non-negative and not zero. If D_1 and D_2 are two divisors, then $D_1 D_2$ will be the intersection number of their divisor classes (or homology classes, if you wish).

Script letters : $\mathcal{F}$, $\mathcal{L}$,... will denote vector bundles or their locally free sheafs of sections. $P(\mathcal{F})$ will be the projective bundle, associated to $\mathcal{F}$, $\hat{\mathcal{F}}$ will be the dual bundle of $\mathcal{F}$, $S^n\mathcal{F}$ the n-fold symmetric product of $\mathcal{F}$. Finally, $c_i(\mathcal{F})$ will be the i-th Chern class of $\mathcal{F}$. Ω_X^1 is the sheaf of holomorphic 1-forms on X , i.e. the cotangent bundle of X .

The dimension of $H^i(X, \mathcal{F}) = H^i(\mathcal{F})$ will be denoted by $h^i(\mathcal{F})$.

Given a divisor D , there is one line bundle, to be denoted by $O_X(D)$ which has a rational section with divisor D . In this way there arises an isomorphism between the group of divisor classes and the group of line bundles. The divisors, corresponding to $\mathcal{K}_X^{\otimes n}$ are the n-canonical divisors, to be written as nK_X .

PROPOSITION 2.1.- If on the surface X there are two linearly independent holomorphic 1-forms ω_1 , ω_2 , such that $\omega_1 \wedge \omega_2 \equiv 0$, then there exists a curve Y , *a connected holomorphic map* $f : X \to Y$, *and holomorphic* 1-*forms* θ_1 , θ_2 *on* Y *with* $\omega_i = f^*(\theta_i)$, $i = 1 , 2$.

For a proof see [16], p. 286.

PROPOSITION 2.2 (Bogomolov).- *If on the algebraic surface* X *there is a line bundle* $\mathcal{L}$ *with* $h^0(\mathrm{Hom}(\mathcal{L} , \Omega^1_X)) \neq 0$, *then there is a constant* c , *such that* $h^0(\mathcal{L}^{\otimes k}) \leq ck$ *for all* $k \geq 1$.

Proof. If $h^0(\mathcal{L}^{\otimes k}) \leq 1$ for all $k \geq 1$, the result is obvious. So we may assume that $h^0(\mathcal{L}^{\otimes k_0}) \geq 2$ for some $k_0 \geq 1$. We start with the case $k_0 = 1$. Let s_1 and s_2 be linearly independent sections of $\mathcal{L}$, and let h be a homomorphism from $\mathcal{L}$ into Ω^1_X , $h \neq 0$. Then $h(s_1)$ and $h(s_2)$ are linearly independent 1-forms on X with $h(s_1) \wedge h(s_2) \equiv 0$, so we can apply Proposition 2.1. It follows that if s_1 vanishes on a curve, then this curve is contained in some fibre of f . Consequently, $\mathcal{L} = \mathcal{O}_X(D)$, where every component of the non-negative divisor D is contained in some fibre of f . Let F be a fibre of f . Then there exists a natural number c such that on X there are no non-negative divisors homologous to $k(D - cF)$ for all $k \geq 1$. Let the divisor F_k consist of ck general (hence non-singular) fibres of f . Then from the standard exact sequence

$$0 \to \mathcal{O}_X(kD - F_k) \to \mathcal{O}_X(kD) \to \mathcal{O}_X(kD) \otimes \mathcal{O}_{F_k} \to 0$$

we find : $h^0(\mathcal{L}^k) = h^0(\mathcal{O}_X(kD)) \leq h^0(\mathcal{O}_X(kD) \otimes \mathcal{O}_{F_k}) = ck$ for all $k \geq 1$. As to the general case, by the "branched covering trick" there exists an algebraic surface Y and a regular surjective map $f : Y \to X$, such that $f^*(\mathcal{L})$ has two independent sections. Since $h^0(\mathrm{Hom}(\mathcal{L} , \Omega^1_X)) \neq 0$ implies $h^0(\mathrm{Hom}(f^*(\mathcal{L}) , \Omega^1_Y) \neq 0$, we can apply to Y and $f^*(\mathcal{L})$ the result just proved for $k_0 = 1$. Thus there exists a constant c , such that for all $k \geq 1$ the inequality $h^0(f^*(\mathcal{L})^{\otimes k}) \leq ck$ holds. But $h^0(\mathcal{L}^{\otimes k}) \leq h^0(f^*(\mathcal{L})^{\otimes k})$, and the proposition is proved.

PROPOSITION 2.3.- *Let* X *be an algebraic surface,* $\mathcal{O}_X(D)$ *a line bundle on* X , *and* $\mathfrak{F}$ *a locally free, rank-two subsheaf of* Ω^1_X , *such that*

(i) *for some* $n_0 > 0$ *the line bundle* $(\det \mathfrak{F})^{\otimes n_0}$ *is generated by global sections* ;

(ii) $h^0(\mathrm{Hom}(\mathcal{O}_X(D) , \mathfrak{F})) \neq 0$.

Then $c_1(\mathfrak{F})D \leq \max(c_2(\mathfrak{F}) , 0)$.

<u>Remark</u>.- Equivalently, $\mathcal{F}$ can be seen as a rank-two vector bundle, admitting a homomorphism into Ω^1_X , which is generically an isomorphism on the fibres.

<u>Proof</u>. Let s be a section of $\mathrm{Hom}(O_X(D), \mathcal{F})$, $s \neq 0$. There is a non-negative divisor S on X , such that $\mathcal{F} \otimes O_X(-D-S)$ admits a section with isolated zeros only.

Hence

$$c_2(\mathcal{F} \otimes O_X(-D-S)) = (D+S)^2 - (D+S)c_1(\mathcal{F}) + c_2(\mathcal{F}) \geq 0 \text{ , i.e.}$$

$$c_1(\mathcal{F})D \leq (D+S)^2 - c_1(\mathcal{F})S + c_2(\mathcal{F}) \text{ .}$$

Since $c_1(\mathcal{F})^{\otimes n_o}$ is generated by global sections we have $c_1(\mathcal{F})S \geq 0$. Consequently, the proposition is already clear in the case that $(D+S)^2 \leq 0$. On the other hand, if $(D+S)^2 > 0$, then application of the Riemann Roch theorem to $O_X(n(D+S))$ yields :

$$h^o(O_X(n(D+S))) + h^2(O_X(n(D+S))) > dn^2$$

for some constant $d > 0$, provided that n is large enough. By Serre duality, $h^2(O_X(n(D+S))) = h^o(O_X(K_X - n(D+S)))$, and we find that either for an infinite number of values of n we have that $h^o(O_X(n(D+S))) > \frac{1}{2}dn^2$ or that for an infinite number of values of n we have that $h^o(O_X(K_X - n(D+S))) > \frac{1}{2}d^2$. The first possibility is excluded by Proposition 2.2 $(h^o(\mathrm{Hom}(O_X(D+S),\mathcal{F})) \neq 0)$ and in the second case we have $c_1(\mathcal{F})(K_X - n(D+S)) \geq 0$ for arbitrarely large values of n , that is, $c_1(\mathcal{F})(D+S) \leq 0$, i.e. $c_1(\mathcal{F})D \leq -c_1(\mathcal{F})S \leq 0$.

PROPOSITION 2.4.- <u>Let</u> X <u>be an algebraic surface</u>, $O_X(D)$ <u>a line bundle on</u> X , <u>and</u> $\mathcal{F}$ <u>a locally free</u>, <u>rank-two subsheaf of</u> Ω^1_X , <u>such that</u>

(i) <u>for some</u> $n_o > 0$ <u>the line bundle</u> $(\det \mathcal{F})^{\otimes n_o}$ <u>is generated by global sections</u> ;

(ii) $h^o(\mathrm{Hom}(O_X(D), S^n\mathcal{F})) \neq 0$.

<u>Then</u> $c_1(\mathcal{F})D \leq \max(nc_2(\mathcal{F}), 0)$.

<u>Proof</u>. Let $p : P(\mathcal{F}) \to X$ be the projection. Then ([5], p. 68) there is a divisor class $H = H_{\mathcal{F}}$ on $P(\mathcal{F})$ such that for any divisor E on X there is a canonical isomorphism between $\Gamma(O_{P(\mathcal{F})}(nH + p^*(E)))$ and $\Gamma(O_X(S^n\mathcal{F} \otimes O_X(E)))$. So in our case there is a positive divisor G on $P(\mathcal{F})$ with $O_{P(\mathcal{F})}(G) = O_{P(\mathcal{F})}(nH - p^*(D))$. By the "branched covering trick" there is a surface Y , together with a surjective map $f : Y \to X$ (of degree k , say) such that under the induced bundle map from $P(f^*(\mathcal{F}))$ onto $P(\mathcal{F})$ the pull-back of G

decomposes into a sum of n positive divisors, of divisor class $H_{f^*(\mathfrak{F})} - q^*(D_i)$, $i = 1,..,n$ respectively.
(q denotes the projection from $P(f^*(\mathfrak{F}))$ onto Y). Hence for each i we have that $h^o(\mathrm{Hom}(O_Y(D_i), f^*(\mathfrak{F}))) \neq 0$. Since $f^*(\mathfrak{F})$ is a subsheaf of Ω^1_Y, Proposition 2.3 yields :

$$\begin{aligned}
c_1(f^*(\mathfrak{F}))D_i &\leq \max(c_2(f^*(\mathfrak{F})),0)\\
f^*(c_1(\mathfrak{F}).D) &\leq \max(nc_2(f^*(\mathfrak{F})),0)\\
kc_1(\mathfrak{F}).D &\leq k\max(c_2(\mathfrak{F}),0)\\
c_1(\mathfrak{F}).D &\leq \max(nc_2(\mathfrak{F}),0) .
\end{aligned}$$

Proof of the conjecture. We may assume that X is minimal. As was already mentioned before (property 2) of representable pairs) we then have $c_1^2(X)$, $c_2(X) > 0$.

We shall derive a contradiction from the assumption that

$$\alpha = \frac{c_2(X)}{c_1^2(X)} < \frac{1}{3} .$$

Let $\beta = \frac{1}{4}(1 - 3\alpha)$, and let n be a natural number such that $n(\alpha + \beta) \in \mathbb{Z}$. We consider the vector bundle $S^n\Omega^1_X \otimes O_X(-n(\alpha+\beta)K_X)$. Then the cohomology of this bundle vanishes in dimensions 0 and 2, provided n is sufficiently large. In fact, the vanishing of $h^o(S^n\Omega^1_X \otimes O_X(-n(\alpha+\beta)K_X))$ is an immediate consequence of Proposition 2.4 : you take $\mathfrak{F} = \Omega^1_X$ and $D = n(\alpha+\beta)K_X$, and you use the fact that $(\det \Omega^1_X)^{\otimes n} = \mathcal{K}_X^{\otimes n}$ is generated by global sections for $n \geq 4$ (Theorem 1.1). And to see that also $h^2(S^n\Omega^1_X \otimes O_X(-n(\alpha+\beta)K_X))$ vanishes for n sufficiently large you first use Serre duality and the fact that $\hat{\Omega}^1_X = \Omega^1_X \otimes O_X(-K_X)$; this gives $h^2(S^n\Omega^1_X \otimes O_X(-n(\alpha+\beta)K_X)) = h^o(S^n\hat{\Omega}^1_X \otimes O_X((n(\alpha+\beta)+1)K_X)) =$
$h^o(S^n\Omega^1_X \otimes O_X((n(\alpha+\beta-1)+1)K_X))$, and then you observe that this last dimension vanishes because of Proposition 2.4, provided n is large enough.

We conclude that

$$\chi(X, S^n\Omega^1_X \otimes O_X(-n(\alpha+\beta)K_X)) \leq 0$$

for n sufficiently large.

But, on the other hand the Riemann Roch theorem expresses this Euler characteristic as a polynomial of degree 3 in n :

$$\chi(X, S^n\Omega^1_X \otimes O_X(-n(\alpha+\beta)K_X)) = \frac{1}{6}[3(\alpha+\beta)^2 - 3(\alpha+\beta) - \alpha + 1]K_X^2 n^3 + O(n^2)$$

with stricly positive leading coefficient. This gives the contradiction.

As an application I mention the following

THEOREM 2.5.- Let X be a surface with a positive definite topological intersection form. Then $b_2(X) = 1$, and $b_1(X) = 0$.

Proof. Firstly we observe that $c_2(X) \geq 0$, for the only surfaces with $c_2(X) < 0$ are certain ruled surfaces ([2], p. 214) and these cannot have a positive intersection form. We then conclude from the index formula $\tau(X) = \frac{1}{3}\,(c_1^2(X) - 2c_2(X))$ that $c_1^2(X) > 0$. Hence ([9], p. 958) X is an algebraic surface with $c_2(X) \geq 0$. It is known ([15], p. 1625) that for all these surfaces the inequality $c_1^2(X) \leq 3c_2(X)$ holds. So we have :

$b_2(X) = \tau(X) = \frac{1}{3}\,(c_1^2(X) - 2c_2(X)) \leq \frac{1}{3}\,c_2(X) = \frac{1}{3}\,(2 - 2b_1(X) + b_2(X))$, i.e.

$b_1(X) = 0$ and $b_2(X) = 1$.

There are many examples of minimal surfaces of general type X with $\frac{c_1^2(X)}{c_2(X)}$ small, but it seems more difficult to produce examples with $\frac{c_1^2(X)}{c_2(X)}$ nearer to 3 . In fact, I can only mention the following two types of examples with $\frac{c_1^2(X)}{c_2(X)} \geq 2$ (i.e. minimal surfaces of general type with non-negative index) :

1) Kodaira surfaces : surfaces X with a connected map $f : X \to Y$ onto a curve Y , which is everywhere of maximal rank, but such that the complex structure of the fibres $f^{-1}(y)$ varies with $y \in Y$ (see for example [8]). These surfaces have $\frac{c_1^2(X)}{c_2(X)}$ somewhere between 2 and $2\frac{1}{2}$, depending on the surface.

2) Quotients of bounded domains : quotients of a 2-dimensional bounded symmetric domain D by a discontinuous transformation group, acting without fixpoints and with compact quotient. Hirzebuch ([6]) proved that their Chern numbers are proportional to those of the compact symmetric space D' dual to D . There are two cases :

(i) D is the polydisk $\{(z_1,z_2) \in \mathbb{C}^2 \; , \; |z_1| < 1 \; , \; |z_2| < 1\}$. Here $D' = P_1 \times P_1$, hence for all these examples we have $c_1^2(X) = 2c_2(X)$. In this class we find the product of two curves of genus ≥ 2 , and furthermore the examples recently constructed by Kuga ([4]), among which there is a "false quadric" $(c_1^2(X) = 8$, $c_2(X) = 4$,

$b_1(X) = 0)$.

(ii) D is the unit ball $\{(z_1,z_2) \in \mathbb{C}^2 , |z_1| + |z_2| < 1\}$, with $D' = P_2$. Here we have only the Borel-Hirzebruch examples, mentioned before. It is known that an infinity of values of $c_1^2(X) = 3c_2(X)$ occurs, but not which ones.

All these examples have infinite fundamental group ; in fact there is the

CONJECTURE (Bogomolov).- Every surface of general type with $c_1^2(X) \geq 2c_2(X)$ has an infinite fundamental group.

The bound $c_1^2(X) \geq 2c_2(X)$ is sharp in any case, for there are simply-connected minimal surfaces of general type with $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily near to 2 , e.g. complete intersections or Hilbert modular surfaces.

The conjecture implies that a surface, homeomorphic to P_2 , is also isomorphic to P_2 . This famous special case has recently been solved by Yau who has proved that on a surface with ample canonical bundle there exists a Kähler-Einstein metric [17]. This implies (use [3]) that surfaces X with $c_1^2(X) = 3c_2(X)$ with ample canonical bundle have the unit ball as universal covering. Question : is the universal covering of a minimal surface of general type with $c_1^2 \geq 2c_2$ topologically the unit ball ?

3. Some other results

In this section I give a few examples of other recent results, which illustrate the preceding sections. Needless to say that many other important results remain unmentioned.

1) In section 1. the inequality $c_2(X) \leq 5c_1^2(X) + 36$ was stated for minimal surfaces of general type. It follows from Noethers inequalities ([2], p. 208) :

$$p_g(X) \leq \tfrac{1}{2}\, c_1^2(X) + 2 \qquad (c_1^2(X) \text{ even})$$

$$p_g(X) \leq \tfrac{1}{2}\, c_1^2(X) + \tfrac{3}{2} \qquad (c_1^2(X) \text{ odd}).$$

In two recent papers ([7 I], [7 II]) Horikawa has greatly extended our knowledge about surfaces with equality. For c_1^2 even, he proves that all these surfaces can be obtained as desingularisations of 2-fold branched coverings of P_2 or Hirzebruch surfaces Σ^n , showing exactly which curves in which divisor classes occur as branch loci. He then goes on to study the moduli and deformation types of these surfaces, obtaining as a corollary that all these surfaces are simply-connected. The results for c_1^2 odd are of similar nature.

2) Another type of "extreme" surfaces are those with $c_1^2(X) = 1$, $c_2(X) = 11$. It was known ([2], p. 212) that for all these surfaces $q(X) = p_g(X) = 0$, and that the order of $H_1(X,Z)$ was at most 6 . Until recently, only one type was known, the Godeaux surfaces. At the moment, the situation is as follows. Miyaoka proved ([10]) that the order of $H_1(X,Z)$ is at most 5 , and that if $H_1(X,Z) \cong Z_5$, then X is a Godeaux surface. Reid ([13]) not only showed the existence of surfaces X with $H_1(X,Z) = Z_3$ or Z_4 , but also the case $Z_2 + Z_2$ does not occur. Finally, it is not yet known whether $H_1(X,Z)$ can be Z_2 or 0 .

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