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COHERENCE OF 3-MANIFOLD FUNDAMENTAL GROUPS

by John STALLINGS

G. P. Scott [3] proved that fundamental groups of 3-dimensional manifolds, closed or not, possess a remarkable quality : every finitely generated subgroup is finitely presented. We call this quality coherence. It should be remarked that P. Shalen also discovered this result independently, and that G. A. Swarup [5] had an important partial result along these lines.

To delineate the nature of coherence, we note that free groups and fundamental groups of 2-manifolds are coherent : if A is such a group, then any finitely generated subgroup is again such a group and hence finitely presented. Incoherent groups can be constructed using Higman's embedding theorem [1].

As a more explicit example, the direct product of two non-cyclic free groups is incoherent :

Let $A = \{a,b\} \times \{c,d\}$ be the direct product of two free groups of rank two. Consider the subgroup N generated by a , c , and bd . It is easy to see that N is a normal subgroup of A , with infinite cyclic quotient group $Z = A/N$. If we consider homology with coefficients the field Q of rational numbers, it turns out that $H_2(N)$ cannot be finitely generated over Q and so N cannot be finitely presented. This is an exercise which can be performed with the help of the spectral sequence of the extension, together with the fact that $H_0(Z\,;M)$ and $H_1(Z\,;M)$ have the same rank over Q , if M is a QZ-module which is finitely generated over Q .

It follows from Scott's theorem therefore that this group A cannot be the fundamental group of any 3-manifold.

Scott's proof follows this pattern : reduce to the case of a group S which is freely indecomposable (i.e. indecomposable into a free product). Show that any indecomposable, finitely generated group is the image of an indecomposable, finitely presented group such that all intermediate groups are indecomposable. Finally, apply some topological ideas due to Swarup to finish the proof.

THEOREM.- Let M be a 3-manifold, and let S be a subgroup of $\pi_1(M)$, such that S can be generated by n elements, $0 \leq n < \infty$. Then S is finitely presentable.

(a) The proof is by induction on n. The theorem is clear if $n = 0$ or 1. If S can be decomposed into a free product $A * B$ with neither factor trivial, then Grushko's theorem [4] shows that both A and B can be generated by fewer than n elements. Inductively both A and B are finitely presented and thus so is S.

(b) The problem is now narrowed down to the case in which S is indecomposable into a free product, and in which every subgroup of S generated by fewer than n elements is finitely presentable. At this point some very interesting points of group theory come into play.

(c) Let $G = G_1 * G_2 * \dots * G_r * F$

$$H = H_1 * \dots * H_t$$

be free products ; where F is a free group ; let $\varphi : G \twoheadrightarrow H$ be an epimorphism such that for $i = 1,\dots,r$, $\varphi(G_i)$ is conjugate to a subgroup of some factor $H_{j(i)}$. Then there is a free decomposition

$$G = K_1 * \dots * K_t$$

such that $\varphi(K_j) = H_j$ for all j.

This improvement on Grushko's theorem (to which it reduces for $r = 0$) from [2] can also be proved topologically along the lines of [4].

(d) Suppose G is a finitely generated group. Then it can be decomposed as a free product

$$G = G_1 * G_2 * \dots * G_r * F_s$$

where F_s is a free group of rank s, and each G_i is neither infinite cyclic nor decomposable into a free product. The existence of such a decomposition follows from Grushko's theorem ; the Kurosh subgroup theorem implies that the G_i are unique up to order and conjugacy within G , and that s is determined by G . The pair $c(G) = (r + s , s)$ is called the complexity of G ; complexities are compared lexicographically.

If G is as above, a homomorphism $\varphi : G \to H$ will be called semi-injective if $\varphi|G_i$ is injective for all i . It is easy to see that this notion is independent of the particular decomposition of G , and that if any free decomposition of G is given such that φ is injective on all the non-free factors, then φ is semi-injective.

A major lemma can now be stated :

(e) Let $\varphi : G \twoheadrightarrow H$ be a semi-injective epimorphism, where G is as in (d). Then if φ is not an isomorphism, $c(G) > c(H)$.

To see this, factor H into indecomposable factors

$$H = H_1 * \dots * H_t * Z * \dots * Z$$

with u infinite cyclic factors Z . Thus $c(G) = (r + s , s)$ and $c(H) = (t + u , u)$. We suppose G factored as in (d). Then, since $\varphi(G_i)$ is indecomposable, not infinite cyclic, by Kurosh's theorem $\varphi(G_i)$ is conjugate to a subgroup of some H_j . Now apply (c). The result is to factor G into

$$G = K_1 * \dots * K_t * K_{t+1} * \dots * K_{t+u}$$

where each term maps onto the corresponding factor of H ; thus, each factor K_j is non-trivial, and since G can be decomposed into only $r + s$ factors, we find

$$r + s \geq t + u .$$

If $r + s = t + u$, the factors K_j have to be indecomposable. Thus all G_i occur, up to conjugacy, in the list of K_j . If K_j is conjugate to G_i , then by the semi-injective hypothesis $\varphi|K_j$ is one-to-one, and so $\varphi(K_j)$ is not Z . This shows that, if $r + s = t + u$,

$$r \leq t , \qquad \text{i.e. } s \geq u .$$

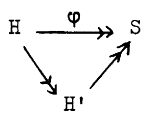
If $s = u$, then $\varphi | K_i : K_i \approx H_i$ for $i \leq t$, and for $i > t$, $\varphi(K_i)$ = one of the Z factors ; with the fact that $K_i = Z$ for $i > t$, we would have shown that φ is an isomorphism on each K factor into the corresponding H factor, and so φ would be globally an isomorphism. Since we are assuming φ is not an isomorphism, this means that if $r + s = t + u$, then $s > u$; knowing $r + s \geq t + u$, we have proved $(r + s, s) > (t + u, u)$, Q.E.D.

Now we return to the proof of the theorem, recalling the situation at (b).

(f) Let S be generated by n elements and be indecomposable into a free product. Suppose that every subgroup of S generated by fewer than n elements is finitely presented. Then there is an indecomposable finitely presented group H and an epimorphism

$$\varphi : H \twoheadrightarrow S$$

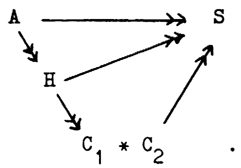
such that if H' is any intermediate group, i.e.



then H' is also indecomposable.

To see this, consider the class of all finitely presented groups generated by n elements, which can be mapped onto S by a semi-injective epimorphism. The free group of rank n is in this class, and their complexities constitute a well-ordered set. Let then $A \twoheadrightarrow S$ be a semi-injective epimorphism with A generated by n elements and having minimal complexity. In the uninteresting case that $A \twoheadrightarrow S$ is an isomorphism, we take $H = A \approx S$.

Otherwise let H be A factored by one additional relation ρ in the kernel of $A \twoheadrightarrow S$. Suppose H' is an intermediate group ; we shall show that H' is indecomposable. If, to the contrary, $H' = B_1 * B_2$ non-trivially, since H' has n generators, Grushko's theorem implies that B_1 and B_2 have fewer generators. Then the images C_i of B_i in S are, by hypothesis, finitely presented groups, and $H \twoheadrightarrow S$ factors through $C_1 * C_2$. Note that $C_1 * C_2$ is finitely presented, and that $C_1 * C_2 \twoheadrightarrow S$, being injective on each factor, is semi-injective :



The top arrow is semi-injective, and so $A \twoheadrightarrow C_1 * C_2$ is semi-injective ; and in the case we are considering, the kernel of $A \twoheadrightarrow C_1 * C_2$ contains a non-trivial element ρ . Thus by (e), $c(C_1 * C_2) < c(A)$ contradicting the choice of A . Thus, H' is indecomposable.

(g) If S is a subgroup of $\pi_1(M^3)$, by passing to a covering space, we can arrange to have $S = \pi_1(M)$. By parts (a), (b) and (f), the proof of the theorem will be finished if we can show this :

<u>Let</u> $\pi_1(M) = S$ (<u>a non-cyclic group</u>), <u>and suppose there is a finitely presented group</u> H <u>and an epimorphism</u> $\varphi : H \twoheadrightarrow S$ <u>such that any group intermediate to</u> H <u>and</u> S <u>is freely indecomposable</u>. <u>Then there is a compact submanifold</u> N <u>of</u> M <u>with</u> $\pi_1(N) \approx \pi_1(M)$. (The fundamental group of a compact manifold is always finitely presented.)

To see this, first find a finite 2-dimensional complex K with $\pi_1(K) = H$ and a map $f : K \to M$ inducing φ . A neighborhood of $f(K)$ in M can be taken to be a compact 3-manifold L with boundary $\text{Bd}\, L$. If H' is the image of $\pi_1(K)$ in $\pi_1(L)$, then H' is intermediate to H and S and therefore freely indecomposable.

We shall change L by surgery on its boundary within M . Throughout, there will be a subgroup H' of $\pi_1(L)$ which maps by inclusion <u>onto</u> $\pi_1(M) = S$, and which is intermediate to H and S .

<u>Exterior surgery</u> : a 2-cell $D \subset M$ with $D \cap L = \text{Bd}\, D$ non-contractible in $\text{Bd}\, L$. Change L to $L \cup$ (thickened D) . H' is changed to the image of the old H' in the new $\pi_1(L)$.

Interior surgery : a 2-cell $D \subset L$, with $\mathrm{Bd}\, D = D \cap \mathrm{Bd}\, L$ non-contractible in Bd L . Change L to $L' = L \setminus (\text{thickened } D)$. This removes a 1-handle from L , and thus $\pi_1(L) = \pi_1(L') * Z$ (*). The old H' is an indecomposable subgroup of $\pi_1(L)$, not cyclic, and so is conjugate to a subgroup of $\pi_1(L')$ which becomes the new H' .

Eventually, Bd L becomes "incompressible" and so, by the theorems of Papakyriakopoulos $\pi_1(L) \to \pi_1(M)$ is a monomorphism. It is also an epimorphism since H' in $\pi_1(L)$ maps onto $\pi_1(M)$. Let N be this final L . That completes the proof.

(*) Or, $\pi_1(L) = \pi_1(L') * \pi_1(L'')$.

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