

# SÉMINAIRE N. BOURBAKI

STEPHEN SMALE

## **Stability and genericity in dynamical systems**

*Séminaire N. Bourbaki*, 1971, exp. n° 374, p. 177-185

[http://www.numdam.org/item?id=SB\\_1969-1970\\_\\_12\\_\\_177\\_0](http://www.numdam.org/item?id=SB_1969-1970__12__177_0)

© Association des collaborateurs de Nicolas Bourbaki, 1971, tous droits réservés.

L'accès aux archives du séminaire Bourbaki (<http://www.bourbaki.ens.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## STABILITY AND GENERICITY IN DYNAMICAL SYSTEMS

par Stephen SMALE

A general reference to this subject, with examples, written about the summer of 1967 is [7], (reported in a recent Bourbaki Seminar by C. Godbillon). Here I will try to emphasize developments since. An important source of much of this more recent work should appear in the immediate future [1].

For simplicity we restrict ourselves to a dynamical system defined by a diffeomorphism  $f$  of a compact manifold  $M$  into itself. This is the case of a discrete differentiable dynamical system with time represented by the number of times  $f$  is iterated or the  $n$  in  $f^n$ . Most of the results discussed here are valid also in the case of a dynamical system defined by a 1st order ordinary differential equation.

The space of all dynamical systems will be denoted by  $\text{Dyn}(M)$ , topologized by putting the  $C^r$  uniform topology on the corresponding diffeomorphism  $f$ ,  $1 \leq r \leq \infty$ . The study of the dynamical system is the study of the orbits  $O(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$  of  $f$  especially from the global point of view. Thus a natural equivalence relation is topological conjugacy, i.e.,  $f, g \in \text{Dyn}(M)$  are topologically conjugate if there is a homeomorphism  $h : M \rightarrow M$  such that  $fh = hg$ . Clearly such an  $h$  sends the orbits of  $f$  onto the orbits of  $g$ .

More than 10 years ago I posed the problem of finding a dense open set  $U$  (or at least a Baire set) of  $\text{Dyn}(M)$  such that the elements of  $U$  could somehow

be described qualitatively by discrete numerical and algebraic invariants. Since this problem has been often quoted since, I would like to take this opportunity to revise the problem in the light of what has been learned in these 10 years.

The problem posed in this way is too simple, too rough and too centralized. I believe now that the main problems of dynamical systems can't be unified so elegantly. The above problem however can be split apart so that it makes good sense and in my opinion gives some perspective to the subject. This goes as follows.

One should search for a sequence of subsets  $U_i$  of  $\text{Dyn}(M)$ ,  $U_1 \subset U_2 \subset \dots \subset U_k \subset \text{Dyn}(M)$ ,  $k$  not too large,  $U_i$  open (or at least say a Baire subset of an open set) with  $U_k$  dense in  $\text{Dyn}(M)$ . One main feature of the  $U_i$  is that as  $i$  increases,  $U_i$  includes a substantially bigger class of dynamical system, but as  $i$  decreases one has a deeper understanding and the elements of  $U_i$  have greater regularity (or stability properties). So  $U_1$  should consist of the simplest best-behaved non-trivial class of dynamical systems, and  $U_k$  cannot be expected to have very strong stability properties at all.

To give a better idea of what I am saying, I will give a schema of the  $U_i$  now which to some extent illustrates our state of knowledge of dynamical systems. (There will always be some arbitrariness in the exact choice of the  $U_i$ .) We will first state briefly the choice of the  $U_i$ 's and the rest of the talk will give some justification for our choice, defining the necessary terms as we proceed. In each of the following  $U_i$  there is a large class of examples not in the preceding  $U_{i-1}$ . The reader may consult the literature cited for many of these.

$$\begin{aligned}
U_1 &= \{f \in \text{Dyn}(M) \mid f \text{ satisfies Axiom A, } \Omega(f) \text{ is finite and } f \text{ satisfies the} \\
&\quad \text{transversality condition}\} \\
U_2 &= \{f \in \text{Dyn}(M) \mid f \text{ satisfies Axiom A and the strong transversality condition}\} \\
U_3 &= \{f \in \text{Dyn}(M) \mid f \text{ satisfies Axiom A and the no cycle condition}\} \\
U_4 &= \{f \in \text{Dyn}(M) \mid f \text{ satisfies the main known generic properties}\}
\end{aligned}$$

We have used the unifying language of Axiom A which can be stated as follows:  $f \in \text{Dyn}(M)$  satisfies Axiom A if the non-wandering set  $\Omega = \Omega(f)$  has a hyperbolic structure and the periodic points of  $f$  are dense in  $\Omega$ . We recall that  $\Omega$  is the closed invariant set of  $x \in M$  such that for any neighborhood  $U$  of  $x$  there is some  $n > 0$  with  $f^n(U) \cap U \neq \emptyset$ . A hyperbolic structure on  $\Omega$  is a continuous splitting of the tangent bundle  $T_\Omega(M)$  of  $M$  restricted to  $\Omega$ ,  $T_\Omega(M) = E^u + E^s$ , invariant under the derivative,  $Df$ , such that  $Df$  is contracting on  $E^s$  and  $D(f^{-1})$  is contracting on  $E^u$ . Finally  $Df : E^s \rightarrow E^s$  is said to be contracting if given a Riemannian metric on  $M$ , there is  $c > 0$ ,  $\mu$ ,  $0 < \mu < 1$  with  $\|Df^m(x)(v)\| \leq c\mu^m\|v\|$  for all  $v \in E^s$ .

A generic property is a property that is true for some Baire set in  $\text{Dyn}(M)$ .

A basic notion for the study of dynamical systems is the notion of stable manifold. Given  $f \in \text{Dyn}(M)$  and some fixed metric on  $M$ , we say that  $x \sim_s y$  if  $d(f^m(x), f^m(y)) \rightarrow 0$  as  $m \rightarrow \infty$ . This is an equivalence relation; the equivalence class of  $x$  is denoted by  $W^s(x)$  and called the stable manifold of  $x$ . The following theorem is a consequence of the work of a number of mathematicians, see especially [1], [5].

**THEOREM 1.-** If  $f \in \text{Dyn}(M)$  satisfies Axiom A, then for each  $x \in M$ ,  $W^s(x)$  is a smoothly, injectively immersed open cell.

It is an outstanding question as to whether the conclusion of Theorem 1 is a generic property.

The unstable manifolds of  $f$  are the stable manifolds of  $f^{-1}$  and are denoted by  $W^u(x)$ .

With this behind us consider the dynamical systems in  $U_1$ . That  $\Omega$  is finite implies that  $\Omega$  consists of the periodic points of  $f$ , and Axiom A amounts to saying that if  $x \in \Omega$  has period  $m$ , then  $Df^m(x) : T_x \rightarrow T_x$  has no eigenvalues of absolute value 1. The transversality condition of  $U_1$  means that if  $x \in M$ , then  $W^s(x)$  and  $W^u(x)$  meet transversally at  $x$ . (Stated in this manner, this transversality condition coincides with the strong transversality condition of  $U_2$ .)

If  $f \in U_1$ , then  $f$  has indeed very strong stability properties. Say that  $f \in \text{Dyn}(M)$  is structurally stable if it possesses a neighborhood of diffeomorphisms, each topologically conjugate to  $f$ .

**THEOREM 2 (Palis-Smale [1]).** If  $f \in \text{Dyn}(M)$  with  $\Omega(f)$  finite, then  $f$  is structurally stable if and only if  $f \in U_1$ .

It was known for sometime via gradient dynamical systems that for any  $M$ ,  $U_1$  is not empty and more recently  $U_1$  was shown to be open [6].

Structural Stability is a very strong regularity property (now known to be not generic) and largely via the preceding theorem,  $U_1$  can be considered to consist of very well understood dynamical systems of relatively simple character. On the other hand, the proof of this theorem is not altogether simple because of

the fact that structurally stable is such a strong (and subtle) property.

To define the remaining terms used in describing  $U_2$ , say that for  $f \in \text{Dyn}(M)$  satisfying Axiom A,  $f$  has the strong transversality property if for any  $x \in M$ ,  $W^s(x)$  and  $W^u(x)$  meet transversally at  $x$ .

It has been conjectured that a necessary and sufficient condition for  $f \in \text{Dyn}(M)$  to be structurally stable is that  $f \in U_2$ . It is known that there exists  $f \in U_2$ ,  $f \notin U_1$  which is structurally stable. Among the rather complicated example of  $f \notin U_1$ , some have the property that a neighborhood doesn't even intersect  $U_2$  because of lacking the strong transversality property. Also for  $f$  satisfying Axiom A, it can be seen that the strong transversality property is necessary for  $f$  to be structurally stable. Via this route it was first found that structurally stable was not a generic property. Proving that  $U_2 =$  structurally stable dynamical systems would cement  $U_2$  into our hierarchy.

We pass to  $U_3$ . To understand the no cycle property, we recall the spectral decomposition theorem which states that if  $f$  satisfies Axiom A, then  $\Omega(f)$  can be written canonically as the finite union of closed invariant disjoint subsets  $\Omega_1, \dots, \Omega_k$  on each of which  $f$  has a dense orbit.

Define  $W^s(\Omega_1) = \bigcup_{x \in \Omega_1} W^s(x)$  and  $W^u(\Omega_1) = \bigcup_{x \in \Omega_1} W^u(x)$ .

A cycle is a sequence of distinct  $\Omega_1, \dots, \Omega_k$ ,  $k > 1$ , with the property  $W^s(\Omega_i) \cap W^u(\Omega_{i-1}) \neq \emptyset$ ,  $i = 2, \dots, k$ , and  $W^s(\Omega_1) \cap W^u(\Omega_k) \neq \emptyset$ . Then  $f$  (supposed to satisfy Axiom A) has the no cycle property if there are no cycles. Dynamical systems in  $U_3$  have the important regularity property known as  $\Omega$ -stability which is defined as follows. First say that  $f, g \in \text{Dyn}(M)$  are  $\Omega$ -conjugate or

conjugate on  $\Omega$  if there is a homeomorphism  $h : \Omega(f) \rightarrow \Omega(g)$  such that  $hf(x) = gh(x)$  for all  $x \in \Omega(f)$ . Then  $f$  is  $\Omega$ -stable if it has a neighborhood  $N$  in  $\text{Dyn}(M)$  of diffeomorphisms which are  $\Omega$ -conjugate to  $f$ . Clearly structurally stable implies  $\Omega$ -stable.

THEOREM 3 (The  $\Omega$ -stability Theorem).- If  $f \in U_3$ , then  $f$  is  $\Omega$ -stable.

The converse of Theorem 3 is an open problem. More generally one can ask what stability properties of dynamical systems are valid outside of  $U_3$ . Can even some dynamical system not in  $U_3$  be structurally stable? Another version of these questions is: Does structural stability imply Axiom A? or even does  $\Omega$ -stability imply Axiom A? My feeling is that the questions of this paragraph are very hard and important to settle.

Some other regularity properties are true of  $f \in U_3$ . For example one can define the "zeta function"  $\zeta(f) = \sum_{n=1}^{\infty} N_n t^n$  where  $N_n$  is the number of fixed points of  $f^n$ . It is an open question whether  $\zeta(f)$  having a positive radius of convergence is a generic property. But on the other hand, the following theorem was very recently proved by J. Guckenheimer [4].

THEOREM 4.- If  $f \in U_3$ , then  $\zeta(f)$  not only has a positive radius of convergence, but it is a rational function whose zeros and poles are algebraic numbers.

R. Bowen [1], [2], [3] has studied dynamical systems satisfying Axiom A in the direction of ergodic theory and has obtained the following rather striking results.

THEOREM 5.- Let  $f$  satisfy Axiom A and  $\Omega_i$  be one of the subsets given by the spectral decomposition theorem. Then there exists an invariant ergodic measure  $\mu_f$  on  $\Omega_i$ , positive on open sets, zero on points (unless  $\Omega_i$  is finite) which is the unique invariant normalized Borel measure on  $\Omega_i$  maximizing entropy. The (measure theoretic) entropy coincides with the topological entropy and this entropy is equal to the log of the radius of convergence of the zeta function of  $f$ .

Also Bowen gets good information on the distribution of periodic points in  $\Omega_i$  and shows that  $(f|_{\Omega_i}, \mu_f)$  is a  $K$ -automorphism in the "C-dense" case, a mild condition which is met for example in the case  $\Omega_i$  is connected.

Let me emphasize again that indeed these last theorems cover situations which are very rich in examples and that I am not giving them here.

What happens outside of  $U_3$ ? At the present time, there are a large number of examples outside of  $U_3$  whose import is that one cannot obtain any dense open  $U_k \subset \text{Dyn}(M)$  with very strong regularity or stability properties. Some of them are as follows:

Abraham-Smale [1] show that  $\Omega$ -stability and Axiom A are not generic properties. Shub has an example of an open set in  $\text{Dyn}(M)$  where  $\Omega = M$ , where Axiom A and  $\Omega$ -stability fail. Newhouse [1] shows that if  $r > 1$ , with the  $C^r$  topology, even on  $S^2$ , Axiom A and  $\Omega$ -stability are not generic properties. The earlier examples, with  $r$  arbitrary in the range  $1 \leq r \leq \infty$ , where on higher dimensional manifolds. C. Simon has recently shown that the zeta function being rational is



not a generic property.

Under perturbation, some features of these examples seem to be preserved. On the other hand the above examples and others need much study to get some understanding of the area beyond  $U_3$ .

As far as I know, there has been not much progress in the finding of new generic properties (or study of  $U_4$ ) since [7] was written.

## REFERENCES

- [1] "Global Analysis", Proceedings of the 1968 AMS Institute on global Analysis at Berkeley. Vol. 1 (of 3 vols) is devoted to dynamical systems.
- [2] R. BOWEN - Periodic points and measures for Axiom A diffeomorphisms, to appear Trans. AMS.
- [3] R. BOWEN - Markov partitions for Axiom A diffeomorphisms, to appear.
- [4] J. GUCKENHEIMER - Axiom A + No cycles  $\Rightarrow$   $\zeta_f(t)$  rational, to appear.
- [5] M. HIRSCH, J. PALIS, C. PUGH, M. SHUB - Neighborhoods of hyperbolic sets, to appear, Inventiones Math.
- [6] J. PALIS - On Morse-Smale dynamical systems, Topology, 8 (1969), 385-405.
- [7] S. SMALE - Differentiable Dynamical Systems, Bull. AMS, 73 (1967), 747-817.