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MEASURE ALGEBRAS OF A LOCALLY COMPACT ABELIAN GROUP

by Nicholas T. VAROPOULOS

0. Introduction and notations.

We shall denote, in what follows, by G a non discrete locally compact abelian group, and by $\Gamma = \hat{G}$ its non compact dual. Also we denote by $M = M(G)$ the complex *Banach algebra of bounded Radon measures on G, where multiplication is the convolution of measures and the involution is defined by $\mu \to \tilde{\mu} = \overline{\mu(-x)}$. Let then $B(\Gamma)$ denote the function algebra on Γ of Fourier transforms $\hat{\mu}$ of the elements $\mu \in M(G)$; we shall also denote by $\|\hat{\mu}\|_\infty = \sup_{\chi\in\Gamma} |\hat{\mu}(\chi)|$.

We shall denote :

$M_c = M_c(G) = \{m \in M(G) ;\ \forall x \in G ,\ \{x\}$ is an m-null set$\}$.

$M_0 = M_0(G) = \{m \in M(G) ;\ \hat{m}(\chi) \xrightarrow[\chi\to\infty]{} 0 ,\ \chi \in \Gamma\}$.

$L_1 = L_1(G) = \{m \in M(G) ;\ m$ is absolutely continuous with respect to $h_G\}$, where h_G is the Haar measure of G.

$\Delta = \Delta(G) = \{m \in M(G) ,\ m$ is atomic $\Longleftrightarrow$ singular with $M_c\}$.

M_c, M_0, L_1 are all closed *ideals of M, and we have ([3], p. 118, 5.6.9)

$$M \supseteq M_c \supseteq M_0 \supseteq L_1 . \tag{0.1}$$

Now in general for any commutative *Banach algebra R, we shall denote by $\mathfrak{M}(R)$ its maximal ideal space and

$$\mathcal{S}(R) = \{M \in \mathfrak{M}(R) ;\ \hat{x}^*(M) = \overline{\hat{x}(M)} ,\ \forall x \in R\} ,$$

the set of symmetric ideals. Also for J a closed ideal of R, we set :

$$h(J) = \{M \in \mathfrak{M}(R) ;\ M \supset J\} \subset \mathfrak{M}(R) .$$

Then it is well known that we have the following canonical identification :

$$\mathfrak{M}(J) = \mathfrak{M}(R) \setminus h(J) . \tag{0.2}$$

And if J is a *ideal we have for the above identification :

$$\mathcal{S}(J) \subset \mathcal{S}(R) . \tag{0.3}$$

We state straight away the fundamental and classical fact ([1], § 31) that we have a canonical identification via the Fourier transform :

$$\mathfrak{M}(L_1) = \mathcal{S}(L_1) = \Gamma .$$

Finally we shall denote for any algebra R

(0.4) $$R^2 = \{ \sum_{j=1}^{N} \lambda_j x_j y_j \; ; \; N \geqslant 1 , \; \lambda_j \in \underset{\sim}{C} , \; x_j , y_j \in R\} .$$

1. The Wiener-Pitt phenomenon.

Let us apply (0.2) , (0.3) and (0.4) to $L_1 \triangleleft M$ we obtain a canonical identification :

(1.1) $$\Gamma \subset \overline{\Gamma} \subseteq \mathcal{S}(M) \subseteq \mathfrak{M}(M) .$$

The problem we shall be concerned in this paragraph can be stated in vague terms as follows :

"How big is $\mathfrak{M}(M) \setminus \overline{\Gamma}$ and how big is $\mathfrak{M}(M) \setminus \mathcal{S}(M)$ the asymmetric spectrum of the algebra M(G) ".

The answer to the above problem, properly formulated is given by :

THEOREM 1.1. - For any non discrete locally compact abelian group G , we have :

(i) $\mathfrak{M}(M) \setminus \mathcal{S}(M) \neq \emptyset \iff M(G)$ is a non symmetric algebra.

(ii) There exists $\mu \in M(G)$ such that μ^{-1} does not exist in M(G) , and yet

$$\inf_{\chi\in\Gamma} |\hat{\mu}(\chi)| > 0 .$$

This theorem was proved in full generality by J. H. WILLIAMSON [14]. The particular case $G = \underset{\sim}{R}$ is due to WIENER-PITT [13]. WILLIAMSON obtained theorem 1.1 by first proving :

THEOREM 1.2. In any non discrete locally compact abelian group G there exists $\mu = \tilde{\mu} \in M_c(G)$, $\mu \geqslant 0$, $\|\mu\| = 1$ such that

(W) $r , s \in \underset{\sim}{Z}$; $r \geqslant 1$, $s \geqslant 1$; $r \neq s \implies \mu^r \perp \mu^s$ (are mutually singular measures).

REMARKS.

(1.(i)) An equivalent formulation of (W) is :

$$\forall N \geqslant 1 \text{ and } \alpha_j \in \underset{\sim}{C} , \; 1 \leqslant j \leqslant N \implies \|\sum_{j=1}^{N} \alpha_j \mu^j\| = \sum_{j=1}^{N} |\alpha_j| .$$

Following WILLIAMSON, we prove how theorem 1.2 $\implies$ theorem 1.1.

<u>Proof</u>. – Consider $\nu = \mu^2$ and $\lambda = \nu - \nu^2$. Then we have, for all $\sigma \in \mathcal{S}(M)$, $\mu(\sigma) = \widetilde{\mu}(\sigma) = \overline{\mu(\sigma)}$, thus $\nu(\sigma) \geqslant 0$, also since $\|\mu\| = 1$, $\nu(\sigma) \leqslant 1$, therefore $0 \leqslant \hat{\lambda}(\sigma) \leqslant 1$. But on the other hand, for all $r \in \underset{\sim}{Z}$, $r \geqslant 1$, we have :

$$(1.2) \quad \lambda^r = \sum_{q=0}^{r} (-1)^{r-q} \binom{q}{r} \nu^{2r-q} \implies \|\lambda^r\| = 2^r \implies \|\lambda\|_{sp} = $$
$$= \lim_n \|\lambda^n\|^{1/n} = \sup_{h \in \mathfrak{M}(M)} |\hat{\lambda}(h)| = 2$$

which gives (i) at once. In fact comparing $\|\hat{\lambda}\|_\infty \leqslant 1$ and $\|\lambda\|_{sp} = 2$, we see that the Šilov boundary of M is not contained in $\overline{\Gamma}$.

Now from (1.2) we see that, for some $h_0 \in \mathfrak{M}(M)$, we have :

$$|\hat{\lambda}(h_0)| = 2 \implies |1 - \hat{\mu}^2(h_0)| = 2 \implies \hat{\mu}^2(h_0) = -1$$

but then if we let $\theta = \delta + \mu^2$, we have $\hat{\theta}(h_0) = 0$ while $\inf_{\chi \in \Gamma} |\hat{\theta}(\chi)| \geqslant 1$, and that gives (ii).

Now if we apply (0.2), (0.3) and (0.4) to $L_1 \lhd M_0$, we see that we have

$$(1.3) \qquad \Gamma = \overline{\Gamma} \subseteq \mathcal{S}(M_0) \subseteq \mathfrak{M}(M_0)\ .$$

The analogous problem as the one above can then be raised and answered by :

THEOREM 1.3. – <u>For any non discrete locally compact abelian group</u> $M_0(G)$ <u>is a non symmetric algebra</u>.

The special case of theorem 1.3, $G = \underset{\sim}{R}$ was obtained by W. RUDIN [2]. In fact the general result is a consequence of the following theorem which we have proved recently [9], [10], [11].

THEOREM 1.4. – <u>If</u> G <u>is a compact non discrete abelian group there exists</u> $(\mu_\rho)_{\rho \geqslant 1}$, <u>a family of measures such that</u> :

(i) $\widetilde{\mu}_\rho = \mu_\rho \in M_0(G)$; $\mu_\rho \geqslant 0$, $\|\mu_\rho\| = 1$, $(\rho \geqslant 1)$;

(ii) $\rho, \sigma \in [1, +\infty[\implies \mu_{\rho+\sigma} = \mu_\rho * \mu_\sigma$;

(iii) $\rho, \sigma \in R$; $\rho \geqslant 1$, $\sigma \geqslant 1$; $\rho \neq \sigma \implies \mu_\rho \perp \mu_\sigma$.

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(1.(ii)) Theorem 1.3 follows from theorem 1.4 the same way as we have explained above, and we see in fact that if $\lambda = \mu_1^2 - \mu_1^4$, then $\lambda = \widetilde{\lambda}$, $\|\hat{\lambda}\|_\infty \leqslant 1$, $\|\lambda\|_{sp} = 2$.

(1.(iii)) It might be interesting to note that for some groups (e. g. $G = \prod_{n=1}^{\infty} \underset{\sim}{Z}(p_n)$ for p_n prime numbers) we can obtain a larger semigroup $(\mu_\rho)_{\rho>0}$ satisfying conditions (i), (ii), (iii) as in theorem 1.4. That is implicit in [9] although not explicitly stated there. We do not know whether that can be done in general.

(1.(iv)) The measures $(\mu_\rho)_{\rho\geqslant 1}$ of theorem 1.4 can be given simple constructive descriptions ; e. g. for $G = \underset{\sim}{T} = \underset{\sim}{R}/\underset{\sim}{Z}$, they can be given by Riesz products, more explicitly : let $\{q_n \in \underset{\sim}{Z}^+\}_{n=1}^{\infty}$ and $\{\varepsilon_n > 0\}_{n=1}^{\infty}$ such that :

$$q_{n+1} - q_n \geqslant 10n \quad \text{for} \quad n \geqslant 1 \ ; \quad \varepsilon_n < 1/8 \ , \quad \varepsilon_n \xrightarrow[n\to\infty]{} 0 \ ,$$
$$\sum_{n=1}^{\infty} \varepsilon_n^{\sigma} = +\infty \quad \text{for all} \quad \sigma \in \underset{\sim}{R} \ , \quad \sigma > 0 \ ;$$

then we have all the conditions of theorem 1.4 satisfied if we set :

$$\mu_\rho = \lim_N \left\{\left(\prod_{n=1}^{N} \left[1 + 2\varepsilon_n^{\rho} \cos(2^{q_n} t)\right]\right) h_{\underset{\sim}{T}}\right\} ,$$

$t \in \underset{\sim}{T}$ an integration variable and the limit taken for the vague topology of measures, and $h_{\underset{\sim}{T}}$ the normalised Haar measure of $\underset{\sim}{T}$.

2. Symbolic calculus.

We shall, in this paragraph, assume that G is an infinite compact group since the corresponding results for any other group can be deduced from the compact case by using simple and classical devices.

DEFINITION. - Let $L_1(G) \subset A \subset M(G)$ be any algebra, not necessarily complete, we shall say that Φ a complex function defined in $(-1\ ,\ 1)$ operates in A if

$$\alpha \in A \ , \qquad \hat{\alpha}(\chi) \in (-1\ ,\ 1) \implies \Phi[\hat{\alpha}] \in B(\Gamma) \ .$$

We then denote by $\Phi[\alpha]$ the element of $M(G)$ such that $(\Phi[\alpha])^\wedge = \Phi[\hat{\alpha}]$.

The following results on the problem are by now classical ([3], Chapter 6, due to HELSON-KAHANE-KATZNELSON-RUDIN).

THEOREM 2.1. - Φ <u>operates in</u> $L_1(G) \iff \Phi$ <u>is analytic in some neighbourhood of</u> 0 .

THEOREM 2.2. - Φ <u>operates in</u> $M(G) \iff \Phi$ <u>is the restriction of an entire function to</u> $(-1\ ,\ 1)$.

REMARKS.

(2.(i)) Theorem 2.2 provides a link of this paragraph with the problems considered in § 1, for, it implies that $\Gamma \subset \mathfrak{M}(M)$ is a "small" subset.

We have recently proved [10], [11].

THEOREM 2.3. - Φ operates in $M_0(G)$ $\Longleftrightarrow$ Φ coincides with an entire function at some neighbourhood of 0.

To illustrate some of our methods, we give the proof of theorem 2.3 for the special case $G = H \times K$ for H, K non discrete compact subgroups. The consideration of that particular case is justified by the fact that its proof contains the key idea required to solve the general problem.

Proof of theorem 2.3 for $G = H \times K$. - We only prove the result that, if Φ operates in $M_0(G)$, then Φ coincides with an entire function at some neighbourhood of 0, for the result in the other direction is trivial. We split the proof in 10 steps.

(A) Using theorem 1.4, we see that there exists $\mu \in M_0(K)$ such that $\mu = \tilde{\mu}$, $\mu \geqslant 0$, $\|\mu\| = 1$ and $\delta_0 = \mu^0, \mu, \mu^2, \dots, \mu^n, \dots$ are mutually singular ($\mu^r \perp \mu^s$, $r \neq s$, $r, s \geqslant 0$).

Also using remark (1.(ii)), we see that there exists $\theta \in M_0(H)$ such

$$\theta \text{ real} ; \quad \theta = \tilde{\theta} ; \quad \|\hat{\theta}\|_\infty = \sup_{\chi\in\Gamma} |\hat{\theta}(\chi)| \leqslant 1 ; \quad \|\theta\|_{sp} = 2 .$$

(B) We know, from theorem 2.1, that as soon as Φ operates in $M_0(G)$, since it also operates in $L_1(G)$, that there exists some $0 < \delta \leqslant 1$ such that

$$\Phi(\xi) = \sum_{k=0}^{\infty} \alpha_k \xi^k \quad \text{for} \quad \xi \in (-\delta, \delta) .$$

(C) Let now $R > 0$ be arbitrary, we proceede to prove that $\alpha_j = O(R^{-j})$ which will prove the result. Towards that, using (A) and raising θ to an appropriate positive power and then multiplying that by an appropriate positive constant, we see that there exists $\nu \in M_0(H)$ such that $\nu = \tilde{\nu} \in M_0(H)$; $\|\hat{\nu}\|_\infty \leqslant \frac{\delta}{2}$ and $\|\nu\|_{sp} \geqslant 2R$. Thus, there exists $j_0 \geqslant 1$ such that $j \geqslant j_0$ implies $\|\nu^j\| \geqslant (\frac{3}{2} R)^j$.

(D) Now, from (C) for $j \geqslant j_0$, we can find $g \in \underset{\sim}{C}_R(H)$ such that $\|g\|_\infty \leqslant 1$ and $|\langle \nu^j, g\rangle| \geqslant (\frac{5}{4} R)^j$, and then, approximating g uniformly on H by real Fourier transforms of elements of $L_1(\hat{H})$, we can find $f_j = \tilde{f}_j \in L_1(\hat{H})$ such that

$$\|\hat{f}_j\|_\infty \leqslant 1 \quad \text{and} \quad |\langle \nu^j, \hat{f}_j\rangle| \geqslant R^j .$$

(For $\alpha \in M(G)$ and $f \in \underset{\sim}{C}(G)$, we denote $\langle \alpha, f\rangle = \int f(g)\, d\alpha(g)$.)

(E) For j and f_j as in (D), we can find $M > j$ such that

$$\sum_{p\geqslant M} |\alpha_p \langle \nu^p, \hat{f}_j\rangle| = \sum_{p\geqslant M} |\alpha_p \langle \hat{\nu}^p, f_j\rangle| \leqslant \|f_j\|_{L_1} \left(\sum_{p\geqslant M} |\alpha_p| (\tfrac{\delta}{2})^p\right) \leqslant 1 .$$

(F) Now since δ_0, μ, μ^2, ..., μ^{M-1}, are orthogonal (mutually singular) measures of K, we can find disjoint compact sets of K, $C_0 = 0_K$, C_1, C_2, ..., C_{M-1} such that C_j supports the "best" part of μ^j for $0 \leqslant j \leqslant M-1$.

Thus, we can find $\varphi_j \in L_1(\hat{K})$ such that

$$\tilde{\varphi}_j = \varphi_j \ ; \quad \|\hat{\varphi}_j\|'_\infty \leqslant 1 \ ; \quad \left|\sum_{p=0}^{M-1} \alpha_p \langle \mu^p , \hat{\varphi}_j \rangle \langle \nu^p , \hat{f}_j \rangle - \alpha_j \langle \nu^j , \hat{f}_j \rangle \right| \leqslant 1 .$$

(G) Now, since $\|\hat{\varphi}_j\|_\infty \leqslant 1$ and $\|\mu\| = 1$ it follows that for all $p \geqslant 0$, $|\langle \mu^p , \hat{\varphi}_j \rangle| \leqslant 1$ thus also by (E)

$$\sum_{p \geqslant M} |\alpha_p \langle \mu^p , \hat{\varphi}_j \rangle \langle \nu^p , \hat{f}_j \rangle| \leqslant 1 .$$

(H) Putting (F) and (G) together, we see that

$$\left|\alpha_j \langle \nu^j , \hat{f}_j \rangle - \sum_{p=0}^{\infty} \alpha_p \langle \mu^p , \hat{\varphi}_j \rangle \langle \nu^p , \hat{f}_j \rangle \right| \leqslant 2 .$$

(I) Now denote by $\lambda = \nu \otimes \mu \in M_0(G)$, we have $\lambda = \tilde{\lambda}$, $\|\hat{\lambda}\|_\infty \leqslant \|\hat{\mu}\|'_\infty \|\hat{\nu}\|_\infty \leqslant \frac{\delta}{2}$, so we see that

$$\Theta = \sum_{p=0}^{\infty} \alpha_p \langle \mu^p , \hat{\varphi}_j \rangle \langle \nu^p , \hat{f}_j \rangle = \sum \alpha_p \langle \lambda^p , (f_j \otimes \varphi_j)^\wedge \rangle = \sum \alpha_p \langle \hat{\lambda}^p , f_j \otimes \varphi_j \rangle$$

$$= \langle \Phi[\hat{\lambda}] , f_j \otimes \varphi_j \rangle = \langle (\Phi[\lambda])^\wedge , f_j \otimes \varphi_j \rangle = \langle \Phi[\lambda] , (f_j \otimes \varphi_j)^\wedge \rangle .$$

(J) From (D), (F) and (I), it follows that

$$|\Theta| \leqslant \|\Phi[\lambda]\| \, \|\hat{f}_j \otimes \hat{\varphi}_j\|_\infty \leqslant \|\Phi[\lambda]\| \, \|\hat{f}_j\|_\infty \|\hat{\varphi}_j\|'_\infty \leqslant \|\Phi[\lambda]\| \ ;$$

and from that and (H), it follows that

$$|\alpha_j \langle \nu^j , \hat{f}_j \rangle| \leqslant 2 + \|\Phi[\lambda]\|' \ ;$$

and from that and (D), it follows that

$$|\alpha_j R^j| \leqslant 2 + \|\Phi[\lambda]\|$$

which implies the required result

$$\alpha_j = O(R^{-j})$$

Q. E. D.

We finish this paragraph by observing the fact that we know of no "universal proof" of theorem 2.3. The way we prove it is, by first obtaining the result for some special cases (like the one above), and then deducing the general result using structure theorems for compact abelian groups.

3. The symmetric ideals and positive forms.

We would like in this third paragraph to discribe without proofs some results which we have obtained very recently on the algebras $M(G)$, $M_c(G)$ and $M_0(G)$.

We have seen in § 1 that $\overline{\Gamma} \subseteq \mathcal{S}(M) \neq \mathfrak{M}(M)$ so the following theorem answers a very natural question :

THEOREM 3.1. - For every non discrete locally compact abelian group , $\Gamma \not\subseteq \mathcal{S}(M_0)$, and a fortiori $\overline{\Gamma} \not\subseteq \mathcal{S}(M)$.

This theorem is a very easy corollary of the following :

THEOREM 3.2. - In a very non discrete locally compact abelian group G , there exists $0 \neq \mu \in M_0(G)$, $\mu \geqslant 0$ such that $\operatorname{supp} \mu$ is compact and the subgroup generated in G by $\operatorname{supp} \mu$, $\operatorname{Gp}(\operatorname{supp} \mu) = \operatorname{Gp}\{x \ ; \ x \in \operatorname{supp} \mu\}$ is of Haar measure zero.

Theorem 3.2 was obtained for $G = \underset{\sim}{R}$ by R. SALEM [4]. A. B. SIMON [5] obtained theorem 3.1 for $G = \underset{\sim}{R}$. We have recently obtained [12] the proof of theorem 3.2 in general, and from that, we can deduce theorem 3.1 at once.

Let us now introduce the :

DEFINITION. - Let $P \subset G$ be a subset of the abelian group G , we shall say that P is strongly independent, if, for any $N \in \underset{\sim}{Z}$, $N \geqslant 1$, and any family of N distinct points of P $(p_j \in P)_{j=1}^{N}$, and any family of N integers $(n_j \in \underset{\sim}{Z})_{j=1}^{N}$ such that $\sum_{j=1}^{N} n_j \, p_j = 0_G$, we have $\{n_j \, x \ ; \ x \in P\} = 0_G$, $1 \leqslant j \leqslant N$. We can now state:

THEOREM 3.3. - If G is any non discrete metrisable locally compact abelian group, then G contains P a perfect, strongly independent subset, such that for some $0 \neq \mu \in M_0(G)$, we have $\operatorname{supp} \mu \subset P$.

That theorem, for $G = \underset{\sim}{R}$, was deduced from theorem 3.2 by W. RUDIN [2]. We were able to prove it in general [12] using our general theorem 3.2. We remark that the condition of metrisability is essential and that the theorem becomes false without it.

It might be worth noting that theorem 3.3 implies a weaker version of theorem 1.4 , [2]. But what, in our view, is the most interesting application of theorem 3.3 is the following [12].

THEOREM 3.4. - If G is a non discrete locally compact abelian group then :

(i) $M_c/\overline{M_c^2}$ is an infinite dimensional Banach space ;

(ii) $M_0/\overline{M_0^2}$ is an infinite dimensional Banach space ;

(iii) If in addition G is metrisable, then

$$M_0 \not\subset \overline{M_c^2} .$$

While not attempting in any sense to give proofs of theorems 3.2, 3.3 and 3.4 (which incidentally are extremely technical), we would like to justify the title of this paragraph, by pointing out straight away, that theorem 3.4 implies that M_c and M_0 qua *Banach algebras, have plenty of discontinuous positive forms. For a closer analysis of the positive forms of M_c and M_0, via theorem 3.4, we can refer the interested reader to [8].

Finally, we would like to give some indication how theorem 3.4 follows from 3.3.

Towards that we proceed to explain without proofs and details a general construction on $M(G)$.

Let P be a perfect strongly independent subset of G which, for simplicity, is assumed compact ; and let us denote by

$$B = M_c(P) = \{m \in M_c(G) ;\ \text{supp } m \subset P\}$$

and by $T_n = B \hat{\otimes} B \hat{\otimes} \dots \hat{\otimes} B$ the tensor product of B with itself n times ; let us also denote by

$$T = \bigoplus_{n\geq 1} T_n = \{x = (x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} T_n ;\ \sum_{n=1}^{\infty} \|x_n\|_{T_n} < +\infty\} ;\ \|x\|_T = \sum_{n=1}^{\infty} \|x_n\|_{T_n}$$

T can be turned into a Banach algebra canonically, the "Tensor Banach algebra on B ".

Now the canonical injection

$$M_c(P) \to M(G)$$

induces a canonical map

$$T \to M(G) ,$$

which turns out to be a norm decreasing Banach algebra homomorphism. If we tensor that homomorphism with the canonical injection $\Delta(G) \to M(G)$, we obtain the Banach algebra homomorphism

$$\pi :\ \Delta(G) \hat{\otimes} T \to M(G) .$$

The kernel of that homomorphism can be studied and can be given an explicit and agreeable form, but what is more to the point in our case, are the facts listed below without proofs :

(a) We have

$$\Delta(G)\ \hat{\otimes}\ T = \bigoplus_{n} (\Delta(G)\ \hat{\otimes}\ T_n)$$

and if

$$\pi_n = \pi|_{\Delta(G)\hat{\otimes}T_n} \quad \text{then for} \quad r \neq s \ ,$$

$\mathrm{Im}\,\pi_r$ and $\mathrm{Im}\,\pi_s$ are (complex) bands in $M(G)$ and are mutually orthogonal (singular) $\mathrm{Im}\,\pi_r \perp \mathrm{Im}\,\pi_s$.

(b) From (a) it follows that

$$\Pi = \mathrm{Im}\,\pi = \bigoplus_{n\geq 1} \mathrm{Im}\,\pi_n$$

is a (complex) band in $M(G)$ and a closed subspace.

(c) $\Pi \subset M_c(G)$ and if $I \subset M_c(G)$ is the orthogonal complement in $M_c(G)$ of Π for the Riesz-Lebesgue decomposition, then I is an ideal in $M(G)$.

From those facts, taken for granted, we have from (c) the direct decomposition :

$$M_c(G) = \Pi \oplus I \ , \tag{3.1}$$

and then it follows from (a), (c) and (3.1) that :

$$\overline{M_c^2} \cap \mathrm{Im}\,\pi_1 = \{0\} \tag{3.2}$$

and (3.2), if combined with theorem 3.3, implies with no difficulty at all theorem 3.4.

Another application of the above decomposition (3.1) is the identification of $\mathfrak{M}(\Pi \oplus \Delta) = h(I) \subset \mathfrak{M}(M)$; that identification is well worth doing since $\mathfrak{M}(\Pi \oplus \Delta)$ has a very simple description in terms of B'_1 , the unit ball of $B' = (M_c(P))'$ and $\check{\Gamma}$ the Bohr compactification of Γ , namely we can identify canonically

$$\mathfrak{M}(\Pi \oplus \Delta) = \check{\Gamma} \times B'_1 \ .$$

It is with that last application in mind, that A. B. SIMON [13] , [14], has worked out some of the details of the above construction (namely the above stated fact (c)) for a special class of groups (I-groups) and a more restrictive definition of strongly independent sets.

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