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ON THE CONJECTURES OF BIRCH AND SWINNERTON-DYER AND A GEOMETRIC ANALOG

by John TATE

§ 1. The conjectures

They grew out of the attempt to apply to elliptic curves the methods used by Siegel in his work on the arithmetic of quadratic forms, these methods having been reworked and fruitfully applied to linear algebraic groups in recent years by Tamagawa, Kneser, Weil, et al., [22], [11]. Much has been written about the motivation for, and the historical development of, these conjectures [6], [2], [7], [3], [4], [18], mostly in terms of the special case of elliptic curves over $\mathbf{Q}$. We content ourselves here with the bare statements, but give these in the most general case.

Let A be an abelian variety over a number field K. Let S be a finite set of primes of K containing the archimedean ones and large enough so that A has non-degenerate reduction outside S, that is, such that A comes from an abelian scheme A_S over the ring K_S of elements of K which are integral outside S. For each prime $v \notin S$, let $\tilde{A}_v$ denote the abelian variety over the residue field $k(v)$ obtained by reducing A_S mod v. Let $Nv = \text{Card } k(v)$, and let $d = \dim A = \dim \tilde{A}_v$. According to well known results of Weil, there is a polynomial of the form

$$P_v(T) = \prod_{i=1}^{2d} (1 - \alpha_{i,v} T) = (Nv)^d T^{2d} P_v\left(\frac{1}{(Nv)T}\right) \tag{1.1}$$

of degree $2d$, with coefficients in $\mathbf{Z}$, and with complex "reciprocal roots"

$\alpha_{i,v}$ of absolute value $(Nv)^{\frac{1}{2}}$, these roots, and hence $P_v(T)$, being characterized by the fact that for all $m \geq 1$

$$(1.2) \qquad \prod_{i=1}^{2d} (1 - \alpha_{i,v}^m) = \left\{ \begin{array}{l} \text{Number of points of } \tilde{A}_v \text{ with coordinates in the} \\ \text{extension of degree } m \text{ of the finite field } k(v). \end{array} \right\}$$

The Euler product

$$(1.3) \qquad L_S(s) = \prod_{v \notin S} \frac{1}{P_v(Nv^{-s})} = \prod_{v \notin S} \prod_{i=1}^{2d} \frac{1}{(1 - \alpha_{i,v} Nv^{-s})}$$

converges for $Rs > \frac{3}{2}$ because it is dominated by the product for $(\zeta_K(s-\frac{1}{2}))^{2d}$. It is generally conjectured that L_S has an analytic continuation throughout the s-plane. This general conjecture, which in principle underlies those of Birch and Swinnerton-Dyer, has been verified in some special cases, notably for A of C.M.-type (Weil, Deuring, Shimura), in which case L_S can be identified as a product of Hecke L-series, and for some elliptic curves related to modular function fields, when L_S can be related to modular forms (Eichler, Shimura).

Let r be the rank of the group A(K) of K-rational points on A, which is finitely generated by the Mordell-Weil Theorem. Birch and Swinnerton-Dyer's first conjecture was

(A) <u>The function</u> $L_S(s)$ <u>has a zero of order</u> r <u>at</u> $s = 1$.

As explained in [19], this conjecture fits beautifully with conjectures I have made concerning the rank of the Néron-Severi group of a variety V defined over any field K of finite type over the prime field. But in this report I wish to concentrate on a refinement of (A), concerning the value of the constant C such that $L_S(s) \sim C(s-1)^r$ as $s \to 1$. While (A) is trivially independent of

our chosen set of primes S, the value of C is not. Helped by prodding from Cassels, Birch and Swinnerton-Dyer overcame this difficulty by following the Tamagawa method, which we now briefly recall.

For each prime v of K, the completion K_v of K at v is a locally compact field. We choose for each v a Haar measure μ_v on K_v, such that for almost all v the ring of integers O_v gets measure 1. (In other words, we choose a Haar measure on the adèle ring of K.) For each v let $|x|_v$ denote the normed absolute value on K_v, the norming being such that $\mu_v(xU) = |x|_v \mu_v(U)$ for $x \in K_v$ and $U \subset K_v$. The group $A(K_v)$ of points of A with coordinates in K_v is a compact analytic group over K_v. Choose a non-zero invariant exterior differential from ω of degree d on A defined over K. Then ω and μ_v determine a Haar-measure $|\omega|_v \mu_v^d$ on $A(K_v)$ in a well-known way, [22].

Let us call v <u>good for</u> ω <u>and</u> μ if v is non-archimedean and satisfies the two conditions

(i) $\mu_v(O_v) = 1$

(ii) ω is v-regular with non-zero reduction $\tilde{\omega}_v$ for Néron's minimal model A_v for A over O_v, [14].

For such good v we have

$$\int_{A(K_v)} |\omega|_v \mu_v^d = \frac{n_v}{(Nv)^d}, \tag{1.4}$$

where n_v is the number of k(v)-rational points on the fiber $\tilde{A}_v$ of Néron's minimal model ; and if v is not so good, but still non-archimedean, the value given in (1.4) has only to be corrected by multiplication by the factor

$$\left|\frac{\omega}{\omega_o}\right|_v(\mu_v(O_v))^d,$$

where ω_o is some v-regular form with $(\tilde{\omega}_o)_v \neq 0$. For archimedean v, the value of the integral (1.4) can presumably be expressed in terms of a Riemann matrix for A.

We now suppose that S contains all primes which are not good for our chosen ω and μ, in addition to the primes where A has degenerate reduction, and we put

$$L_S^*(s) = \frac{|\mu|^d}{\prod_{v\in S}\left(\int_{A(K_v)} |\omega|_v \mu_v^d\right) \prod_{v\notin S} P_v(Nv^{-s})} \tag{1.5}$$

where $|\mu|$ denotes the measure of the compact quotient of the adèle ring of K by the discrete subfield K, relative to the measure $\mu = \prod_v \mu_v$. For example

$$|\mu| = \frac{|d_K|^{\frac{1}{2}}}{2^{r_2}}, \text{ if } \begin{cases} \mu_v = \text{Lebesgue measure for archimedean } v \\ \mu_v(O_v) = 1 \text{ for non-archimedean } v, \end{cases}$$

where d_K is the absolute discriminant of K, and r_2 is the number of complex primes of K. For our "sufficiently large" S, the function L_S^* is independent of the choice of μ and ω, by homogeneity and the product formula (Tamagawa principle !). By (1.1), (1.2), and (1.4) we have

$$\int_{A(K_v)} |\omega|_v \mu_v^d = P_v(Nv^{-1}), \text{ for } v\notin S,$$

and consequently, the asymptotic behavior of $L_S^*(s)$ as $s \to 1$, is independent of S. Since it is this behavior which interests us, we shall from now on write simply $L^*(s)$ for any such function, or more generally for any Euler product L^*,

all but a finite number of whose factors coincide with those of L_S^*, and such that the elementary function L^*/L_S^* takes the value 1 at $s = 1$. Presumably there is a best such L^*, satisfying an especially simple functional equation relating $L^*(s)$ and $L^*(2-s)$, but we do not enter into this question here.

The second conjecture of Birch and Swinnerton-Dyer, which refines the first, is

$$(B) \qquad L^*(s) \sim \frac{[Ш]\ |\det < a_i', a_j >|}{[A(K)_{tors}][A'(K)_{tors}]} (s-1)^r \ , \quad \text{as} \quad s \to 1 \ ,$$

where the quantities on the right are now to be explained.

We write $[X]$ for the cardinal number of a set X, and X_{tors} for the torsion subgroup of an abelian group X. The dual abelian variety to A is denoted by A'. The groups $A'(K)$ and $A(K)$ have the same rank r, because A' is isogenous to A over K, and $(a_i)_{1 \leqslant i \leqslant r}$, resp. $(a_i')_{1 \leqslant i \leqslant r}$, is a base for $A(K)$, resp. $A'(K)$, mod torsion.

The symbol $< a', a >$ has to do with canonical heights on abelian varieties. If $x = (x_0, \ldots, x_m)$ is a point of projective m-space with coordinates in K, its "logarithmic height" is defined by

$$(1.6) \qquad h(x) = \log \prod_{\text{all } v} \ \max_{0 \leqslant i \leqslant m} \{|x_i|_v\} \ .$$

Let f be a K-morphism of A into a projective space, and let D be the inverse image under f of a hyperplane defined over K. We call f <u>symmetric</u> if D is linearly equivalent to D^-. Let $\varphi_f(a)$ denote the point on A' representing the divisor $D_a - D$. Then $< a', a >$ is the unique biadditive real-valued function on $A'(K) \times A(K)$ such that for every symmetric f the function $< \varphi_f(a), a > + 2h(f(a))$ is bounded for $a \in A(K)$. The existence of such a function

is due to Néron [15]. I have given a simpler construction, based solely on the functorial properties of heights and of divisors on abelian varieties due to Weil (cf. [12], and [13]), but the approach of Néron through the local symbols $(X,a)_v$ is undoubtedly essential both for the finer theory, and also for computational purposes.

It remains to discuss the Šafarevič group, $Ш$. This can be defined by the exactness of the sequence

$$0 \to Ш \to H^1(K,A) \to \coprod_{\text{all } v} H^1(K_v,A), \tag{1.7}$$

where the cohomology in the Galois cohomology of commutative algebraic groups, cf. [16]. It is known that $Ш$ is a torsion group whose p-primary component $Ш(p)$ is of finite corank for each prime p. Another deep conjecture underlying (B) is that $Ш$ is finite. As far as I know, this has not been proved for a single $A \neq 0$, although the finiteness of $Ш(2)$ or $Ш(3)$ has been shown for hundreds of elliptic curves over $\mathbb{Q}$. Cassels [5, IV] for $d = 1$ and I in general [20] have constructed a canonical pairing $Ш \times Ш' \to \mathbb{Q}/\mathbb{Z}$ which is non-degenerate if $Ш$ is finite and is alternating in case $Ш'$ can be identified with $Ш$ by means of an isomorphism $A \xrightarrow{\sim} A'$ coming from a polarization of degree 1. Hence in this latter case, and in particular if A is a Jacobian, $[Ш]$ is a square. In general (assuming finiteness) we have $[Ш] = [Ш']$, so the right hand side of (B) is invariant when A and A' are interchanged.

§ 2. The evidence

The numerical evidence for the conjectures is very impressive. Most of it is contained in [4], where Birch and Swinnerton-Dyer discuss the case $K = \mathbb{Q}$, and A an elliptic curve of the form $y^2 = x^3 - Dx$. In that case, as Weil [22] has shown, the function $L^*(s) = L_D^*(s)$ is, essentially, a Hecke L-series associated with the Gaussian field $\mathbb{Q}(i)$ of complex multiplications of A, and Birch and Swinnerton-Dyer were able to find a finite expression for $L_D^*(1)$. This expression is a sum of Δ^2 terms, where Δ is the product of the odd primes dividing D, each term involving a quartic residue symbol and a division value of a Weierstrass $\wp$-function associated with A. Their electronic computer could compute $L_D^*(1)$ for all D's corresponding to a given Δ (D is fourth-power free) in about $\Delta^2/20$ seconds. It computed the quantity

$$\gamma = [A(K)_{tors}]^2 L_D^*(1)$$

for 1348 values of D (all those for $|\Delta| < 108$ and a few more). For each of these D's it also tried to compute the rank r, together with the order of $Ш(2)$, succeeding in all but about 200 cases. Now, according to conjecture (B), one should have $\gamma = 0$ if $r > 0$, and $\gamma = [Ш]$ if $r = 0$. In each of the more than 1000 cases where r was determined, the machine found $\gamma = 0$ whenever $r > 0$, and found γ to be a non-zero square whose 2-component was equal to $[Ш(2)]$ whenever $r = 0$. Moreover even in the cases where r and $Ш(2)$ were not determined by the given program, there were various consistencies ; in particular, γ was always a square. The non-zero values of γ which turned up, i.e. the various conjectural orders of $Ш$ for $r = 0$, were 1, 4, 9, 16, 25, 36, 49, and 81.

I understand that more recently N.M.Stephens, in his as-yet-unpublished 1965 Manchester Ph. D. thesis has given a similar discussion of curves of the form $x^3 + y^3 = D$, but including some computations of the values of the higher derivatives of $L^*(s)$ at $s = 1$, getting numerical evidence for (B) in hundreds of cases with $r = 0$ and 1 for that type of curve, and also in four cases with $r = 2$ and in one case with $r = 3$!

In addition to all this numerical evidence, there are strong theoretical indications that (B) is right. In [4] it is proved in general, i.e. for all D, that $L_D^*(1)$ is a rational number, whose denominator is explicitly bounded (in the sense of divisibility) in terms of D.

It should be an easy job to check that (B) is consistent with Weil's "restriction of the ground field" functor R_{K/K_0} for a subfield K_0 of K, that is, that (B) holds for A over K if and only if it holds for $A_0 = R_{K/K_0} A$ over K_0. Indeed if this is not so, then I have presumably made a mistake in the normalization of the measures, or of the height pairing $< a', a >$.

For a given field K, the conjecture (B) is trivially compatible with products ; indeed each individual quantity entering in (B) is easily seen to be multiplicative.

Much more interesting in the question of compatibility with isogenies. It is true, but not at all trivial, that

THEOREM 2.1.- The truth of conjecture (B) depends only on the K-isogeny class of A.

For elliptic curves this was proved by Cassels [5,VIII]. In trying to prove it in general I was led to the following basic theorem on global Euler-

Poincaré characteristics of finite Galois modules over number fields :

THEOREM 2.2.- Let S be a finite set of primes of K and let M be a finite module for the Galois group of the maximal unramified-outside-S extension of K, whose order, $[M]$, is invertible in the ring K_S of S-integers in K. Let $M' = \mathrm{Hom}(M, \mathbf{G}_m)$ be the Cartier dual of M. Then

$$(2.1) \qquad \frac{[H^0(K_S,M)][H^2(K_S,M]}{[H^1(K_S,M)]} = \prod_{v \text{ arch}} \frac{[H^0(K_v,M)]}{|[M]|_v} = \prod_{v \text{ arch}} \frac{[\hat{H}^0(K_v,M')]}{[H^0(K_v,M')]}$$

where $\hat{H}^0$ is the reduced 0-dimensional cohomology group (which was denoted by H^0 in the too-fancy notation of [20]).

Indeed using Theorems 2.1, 2.3, 2.5, and 3.1 of [20], one proves that, assuming $Ш$ finite, the compatibility of (B) with a K-isogeny f is equivalent to Theorem 2.1 for $M' = \mathrm{Ker}\, f$ and $M = \mathrm{Ker}\, f'$. Cassels lemma 6.1 of [5, VIII] seems to be a variant of theorem 2.2 for the case M is of prime order. The equality of the middle and right terms in (2.1) is a triviality, but both expressions are useful. Defining $\chi(K_S,M)$ = left/middle, one checks the multiplicativity of χ by means of Theorem 3.1(C) of [20]. But I was at a loss in trying to prove $\chi = 1$ until Serre suggested I use his methods of [16, p.II-34 to II-37], which work beautifully. One is reduced to proving (2.1) holds (in the form extreme right = extreme left) in case $M = \mu_p$, $M' = \mathbf{Z}/p\mathbf{Z}$, for a prime p, but where $[X]$ now denotes not the order of X, but the class of X in the Grothendieck group of finite modules over the group algebra $(\mathbf{Z}/p\mathbf{Z})[G]$, where G is a group of automorphisms of K of order prime to p, such that S is stable under G. This can be done by considering the cohomology sequence of the

exact sequence $0 \to \mu_p \to G_m \xrightarrow{p} G_m \to 0$, together with the knowledge of the cohomology of G_m furnished by class field theory.

§ 3. Relation between Ш and Br

It was Mike Artin who first remarked that the finiteness of Ш was equivalent to the finiteness of a Brauer group in certain cases, and the rest of this talk is a report on joint work with him.

In this section V will denote either (i) an open subset of the spectrum of the ring of integers in a number field, or (ii) an irreducible curve-scheme smooth over a perfect field k. We let K denote the field of rational functions on V, and V^o the set of closed points on V. For $v \in V^o$ we let K_v denote the completion of K with respect to the valuation associated to v. If A is an abelian variety defined over K, we let $Ш(V,A)$ denote the kernel of the map

$$H^1(K,A) \to \coprod_{v \in V^o} H^1(K_v,A).$$

Notice that in the arithmetic case the Ш of § 1 (cf.(1.7)) is a subgroup of $Ш(V,A)$ and is not in general equal to it unless K is totally imaginary and V the spectrum of the ring of integers of K.

THEOREM 3.1.- Let $f : X \to V$ be a proper morphism with fibers of dimension 1 and X regular of dimension 2. Suppose that the geometric fibers of f are connected, and the generic fiber smooth. If f admits a section, then there is an exact sequence

$$0 \to Br(V) \xrightarrow{Br(f)} Br(X) \to Ш(V,A) \to 0,$$

where A denotes the Jacobian of the generic fiber of f, and where Br denotes Brauer group. Moreover if V is a complete curve, then $Br(V) = 0$.

Concerning the definition and basic properties of Br we refer to Grothendieck's talks in the last two Bourbaki seminars [8]. Notice that since the dimensions are ≤ 2 and X is regular, both Brauer groups in question are isomorphic to the cohomological Brauer groups. We do not prove Theorem 3.1 here, because of space limitations, and because of Grothendieck plans to give a third exposé [9] on the Brauer group, which will contain a proof of a somewhat more general statement. Suffice it to say that the proof is based on the Leray spectral sequence of étale cohomology for the map f and the sheaf $\underline{G}_{m,X}$ of multiplicative groups on X, the starting point being

THEOREM 3.2 (Mike Artin).- If $f : X \to V$ is a proper morphism with fibers of dimension 1 and X regular of dimension 2, then $R^q f_* \underline{G}_{m,X} = 0$ for $q \geq 2$.

It might be interesting to try to find a direct connection between $Ш(V,A)$ and Azumaya algebras over X.

§ 4. The geometric analog

Let k be a finite field with q elements and V an irreducible algebraic curve proper and smooth over k with function field K. If A is an abelian variety defined over K, then the conjecture (B) of § 1 makes good sense for A over K, by the usual analogies between number fields and function fields in one variable over finite fields. Moreover, the Ш occurring in (B) is identical with the $Ш(V,A)$ of § 3, because V is complete. The proof of compatibility with isogenies given in § 2 carries over to this geometric case, so long as the

degree of the isogeny is prime to the characteristic p. The proof of compatibility with p-isogenies looks like an interesting problem.

Let us now abandon our curve V temporarily and consider an algebraic surface X projective and smooth over k such that $\bar{X} = X \times_k \bar{k}$ is connected. By the étale Lefschetz fixed point theorem and Poincaré duality [1], [10], the zeta function of X is of the form.

$$\zeta(X,s) = \frac{P_1(X,q^{-s})P_1(X,q^{1-s})}{(1-q^{-s})P_2(X,q^{-s})(1-q^{2-s})} \quad , \tag{4.1}$$

where $P_i(X,T) = \det(1-\varphi_{i,\ell}\, T)$ is the characteristic polynomial of the endomorphism $\varphi_{i,\ell}$ of the étale $H^i(\bar{X},\mathbb{Q}_\ell)$ induced by the Frobenius endomorphism φ of X. From Weil's work on abelian varieties and the relation between the étale H^1 and the Picard variety of X, it is known that P_1 has integral coefficients and is independent of ℓ, and it follows that the same is true for P_2. It is also known (Weil) that the complex "reciprocal roots" of P_1 have absolute value $q^{\frac{1}{2}}$; it is conjectured, but not known, that those of P_2 have absolute value q.

Inspired by the work of Birch and Swinnerton-Dyer, in the way explained below, Mike Artin and I conjecture :

(C) <u>The Brauer group</u> Br(X) <u>is finite, and</u>

$$P_2(X,q^{-s}) \sim \frac{[Br(X)]\,|\det(D_i.D_j)|}{q^{\alpha(X)}\,[NS(X)_{tors}]^2}\,(1-q^{1-s})^{\rho(X)}, \ \underline{\text{as}} \ \ s \to 1 \ ,$$

<u>where the quantities on the right are defined as follows</u> :

(4.2) $\alpha(X) = \chi(X,\underline{O}_X) - 1 + \dim\,(\text{Pic Var}(X)) = p_g(X) - \delta(X)$,

where $\delta(X) = \dim H^1(X,\underline{O}_X) - \dim\,(\text{Pic Var}(X))$ is the "defect of smoothness" of the Picard scheme of X. It is known that $\alpha(X) \geqslant 0$.

$NS(X)$ denotes the Néron-Severi group of X, which we define to be the image of $\text{Pic}(X)$ in $NS(\bar{X})$, where $\text{Pic}(X) = H^1(X,\underline{O}^*_X)$ and $NS(\bar{X})$ is the group of algebraic equivalence classes of divisors on $\bar{X}$.

$\rho(X)$ denotes the rank of $NS(X)$, and $(D_i)_{1 \leqslant i \leqslant \rho}$ is a base for $NS(X)$ mod torsion. The symbol $(D_i \cdot D_j)$ denotes the total intersection multiplicity of D_i and D_j .

We also conjecture

(d) Suppose that $f : X \to V$ is a k-morphism of the surface X onto the curve V, with connected geometric fibers and smooth generic fiber. Let A be the Jacobian of the generic fiber. Then (B) holds for A over K if and only if (C) holds for X.

This conjecture gets only a small letter (d) as label, because it is of a much more elementary nature than (B) and (C). We have checked that (d) is true in case f is smooth and has a section, and we are 99 % sure it is true in case the generic fiber of f is an elliptic curve with a K-rational point. Indeed it was in trying to translate (B) into a statement about the surface X in the latter case that we arrived at (C).

We have not yet made a serious attempt to prove (d) in the general case, and will here just briefly indicate the explicit and plausible equality to which it reduces. For each $v \in V^o$ let $X_v = f^{-1}(v)$ be the fiber over v , and let m_v

denote the number of irreducible components of X_v. Define $P_v(T)$ by

$$P_v(Nv^{-s}) = \zeta(X_v,s)(1-Nv^{-s})(1-Nv^{1-s})^{m_v} .$$

(Concerning the general formalism of zeta functions of schemes, see Serre [17].) For the good v, where f is smooth and A has non-degenerate reduction, $P_v(T)$ is the polynomial associated to A at v as in (1.1). We have

$$(4.3)\quad \zeta(X,s) = \prod_{v\in V^o} \zeta(X_v,s) = \frac{\zeta(V,s)\zeta(V,s-1)}{L(s)} \prod_{v\in V^o} (1-Nv^{1-s})^{1-m_v} ,$$

where $L(s) = \prod_{v\in V^o} P_v(Nv^{-s})^{-1}$. Substituting the expression (4.1) for $\zeta(X,s)$ and the analogous formula for $\zeta(V,s)$ in (4.3) one finds (ultimately) that (d) is equivalent to the equality

$$(4.4)\quad \frac{[Ш(V,A)]}{[Br(X)]} \left(\frac{[NS(X)_{tors}][B(k)]}{[A(K)_{tors}]}\right)^2 \frac{\left|\det\left(\frac{<a_i,a_j>}{\log q}\right)\right|}{|\det(D_i.D_j)|} \prod_{v\in S} \frac{(\deg v)^{m_v-1} n_v}{(Nv)^d P_v(Nv^{-1})} = q^c$$

(sauf erreur), where :

B = Coker (Pic Var$(V) \hookrightarrow$ Pic Var(X)) = (K/k)-trace of A .

S is a finite set of points of V outside of which f is smooth.

n_v is the number of $k(v)$-rational points on the fiber $\tilde{A}_v$ of Nérons's minimum model for A.

$c = \deg \underline{\omega} + (g-1)(d-1) - \chi(X,\underline{O}_X)$,

where

$\underline{\omega}$ is the line-bundle on V whose fiber at v is the space of exterior differential forms of degree d on $\tilde{A}_v$.

The abelian variety B enters because $P_1(X,T)/P_1(V,T) = P_1(B,T)$, and

$P_1(B,1)P_1(B,q^{-1}) = [B(k)]^2/q^{\dim B}$. The constant c has several sources : (4.2), with $\dim(\mathrm{Pic}\ \mathrm{Var}(X)) = g + \dim B$, the fact that $|\mu| = q^{g-1}$ if $\mu_v(O_v) = 1$ for all v, and the use of a rational section ω of the bundle $\underline{\omega}$ as in (1.4) and the remarks following to get $L^*(s)$ in terms of $L(s)$. The n_v also arise from (1.4).

In deriving (4.4) one also uses $\rho = r + 2 + \Sigma(m_v - 1)$, which results from an unscrewing of $NS(X)$ of the type

$$(4.5) \qquad \frac{NS(X)_{\text{deg 0 on fiber}}}{f^*(NS(V))} \sim A(K) + \sum_v (NS)_v \ , \ \text{up to torsion,}$$

where $(NS)_v$ denotes the quotient of the free group generated by the irreducible components of $f^{-1}(v)$ by the subgroup generated by the cycle $f^{-1}(v)$. This unscrewing will lead to a factorization of $\det(D_i.D_j)$ into factors $\det_A$ and $\det_v$'s, and in (4.4) the $\det_v$'s should be moved over into the product of local terms (note that $(\deg v)^{m_v - 1}$ divides $\det_v$), and the $\det_A$ should essentially cancel with $\det \langle a_i, a_j \rangle$. Indeed, if D and E are divisors on X which intersect every irreducible component of every fiber with multiplicity 0, and if a and b are the points in $A(K)$ representing the intersection cycles of D and E on the general fiber, then we have $\langle a,b \rangle = (D.E) \log q$, cf. [15]. The factor $\log q$ appears here because $\log(|x|_v) = -(\mathrm{ord}_v x)(\deg v)(\log q)$; the source of $\log q$ in (4.4) is simply $1 - q^{1-s} \sim (\log q)(s-1)$ as $s \to 1$.

If f has a section, then $Ш = Br$ by Theorem 3.1, and also $c = 0$, for in that case, as Grothendieck explained to me, $R^1 f_*(\underline{O}_X)$ is a locally free sheaf on V of rank d, the d-th exterior power of whose dual is $\underline{\omega}$, and on the other hand, $\chi(X,\underline{O}_X) = \chi(V,\underline{O}_V) - \chi(V,R^1 f_*(\underline{O}_X))$; now use Riemann-Roch on V.

If in addition f is smooth, then all individual quotients in (4.4) are equal to 1. In the general case, there may be some spill-over from one quotient to another, but it looks hopeful to reduce (d) to a purely local relationship among n_v, $P_v(Nv^{-1})$, and the $\det_v$ mentioned above.

§ 5. The main theorem

We consider now a surface X of the type described in the second paragraph of § 4, without reference to a fibration $X \to V$. After proving a self-duality theorem for Br(X) analogous to Cassels' self-duality for the Ш of elliptic curves, we prove our main theorem 5.2. If we grant (d), this theorem shows that, for a Jacobian A over a function field K of a curve over k, the finiteness of $Ш(\ell)$ for one prime $\ell \neq p = \mathrm{char}(k)$ is equivalent to conjecture (A) of Birch and Swinnerton-Dyer for A over K, and implies the finiteness of the part of Ш prime to p and the truth of their conjecture (B) up to a factor $\pm p^{\nu}$. Our proof uses the étale cohomology theory over $\bar{X} = X \times_k \bar{k}$, and therefore does not offer much hope of adaptation to the number field case at the moment.

Let G denote the Galois group of $\bar{k}$ over k, and consider the following exact commutative diagram of finite groups :

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathrm{Pic}(X)/m\mathrm{Pic}(X) & \to & (\mathrm{NS}(\bar{X})/m\mathrm{NS}(\bar{X}))^G & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^1(\bar{X},\mu_m)_G & \to & H^2(X,\mu_m) & \to & H^2(\bar{X},\mu_m)^G & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\mathrm{Pic}(\bar{X})_m)_G & & \mathrm{Br}(X)_m & \to & \mathrm{Br}(\bar{X})_m^G & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{5.1}
$$

Here μ_m denotes the étale sheaf of m-th roots of unity, m being a number prime to the characteristic p. We use the symbol W_m to denote the kernel of multiplication by m in a commutative group W. The columns in (5.1) come from the étale cohomology sequence associated with the exact sequence of sheaves

$$0 \to \mu_m \to \underline{G}_m \to \underline{G}_m \to 0$$

(cf. [8,II, § 3]), over X and over $\bar{X} = X \times_k \bar{k}$. The 0 in the northwest corner comes from the fact that $H^0(\bar{X},\underline{G}_m) = \bar{k}^*$ is divisible, and the replacement of Pic by NS in the northeast corner is allowed by the divisibility of Pic Var(X)(k). The group $G = Gal(\bar{k}/k) \approx \hat{Z}$ has a canonical topological generator σ = Frobenius automorphism, and for any topological G-module M we denote by M^G (resp. M_G) the kernel (resp. cokernel) of the homomorphism $(\sigma-1) : M \to M$. Thus $H^0(G,M) = M^G$, and if M is a torsion module, then $H^1(G,M) = M_G$, and $H^i(G,M) = 0$ for $i > 2$. If F is a finite sheaf killed by m on $X_{ét}$, and $\bar{F}$ its restriction to $\bar{X}_{ét}$, then the spectral sequence of Hochschild-Serre [1,VIII 8.4] breaks up into exact sequences :

$$(5.2) \qquad 0 \to H^{i-1}(\bar{X},\bar{F})_G \to H^i(X,F) \to H^i(\bar{X},\bar{F})^G \to 0 .$$

The middle row of (5.1) is this exact sequence for $F = \mu_m$ and $i = 2$. The top and bottom rows have arrows omitted simply in order that we may say the diagram is exact.

The Poincaré duality theorem over $\bar{X}$ (cf.[1]) shows that the cup product pairing

$$(5.3) \qquad H^i(\bar{X},\mu_m) \times H^{4-i}(\bar{X},\mu_m) \to H^4(\bar{X},\mu_m^{\otimes 2}) \approx Z/mZ$$

is a perfect duality of finite G-modules for all i, with G operating

trivially on $\mathbf{Z}/m\mathbf{Z}$. From (5.2) and (5.3) one derives an "arithmetic" Poincaré duality theorem over X, which asserts that the cup product pairing

$$(5.4) \qquad H^i(\bar{X},\mu_m) \times H^{5-i}(\bar{X},\mu_m) \to H^5(\bar{X},\mu_m^{\otimes 2}) \approx \mathbf{Z}/m\mathbf{Z}$$

is a perfect duality of finite groups, for all i.

Now consider the exact sequence $0 \to \mu_m \xrightarrow{i} \mu_{m^2} \xrightarrow{j} \mu_m \to 0$. From the functorality of arithmetic Poincaré duality one finds that the two groups

$$\mathrm{Coker}(H^2(X,\mu_{m^2}) \xrightarrow{j} H^2(X,\mu_m)), \quad \text{and} \quad \mathrm{Ker}(H^3(X,\mu_m) \xrightarrow{i} H^3(X,\mu_{m^2}))$$

are dual. On the other hand these two groups are isomorphic via the connecting homomorphism $\delta: H^2(X,\mu_m) \to H^3(X,\mu_m)$, and the first of them, i.e. the Coker, is isomorphic to $\mathrm{Br}(X)_m/m(\mathrm{Br}(X)_{m^2}) = \mathrm{Br}(X)_m/(\mathrm{Br}(X)_m \cap m\mathrm{Br}(X))$, as one sees by applying the serpent lemma to a diagram involving the middle column of (5.1) and the same column with m replaced by m^2. It follows that $\mathrm{Br}(X)_m/m(\mathrm{Br}(X)_{m^2})$ is self-dual, its self-duality being induced by the pairing $(x,y) \mapsto x.\delta y$ of $H^2(X,\mu_m)$ with itself into $H^5(X,\mu_m) \approx \mathbf{Z}/m\mathbf{Z}$. This latter pairing is skew-symmetric because $x.(\delta y) + (\delta x).y = \delta(xy)$, and $\delta(xy) = 0$ because $H^5(X,\mu_m) \to H^5(X,\mu_{m^2})$ is injective. The skew-symmetric form on $\mathrm{Br}(X)_m$ which we have constructed is compatible with the inclusions $\mathrm{Br}(X)_m \subset \mathrm{Br}(X)_{mm}$, if we view its values in $\mathbf{Q}/\mathbf{Z}$. Passage to the limit gives

THEOREM 5.1.- <u>There is a canonical skew-symmetric form on</u> $\mathrm{Br}(X)$ (non p) <u>whose kernel consists exactly of the divisible elements. In particular, if</u> $\mathrm{Br}(X)$ (non p) <u>is finite, then it is self-dual and its order is a square or twice a square</u>.

In the situation of Theorem 3.1, when $\mathrm{Br}(\bar{X}) = Ш$, this form ought to correspond to the one of Cassels discussed after (1.7). We therefore suspect

that the form in theorem 5.1 is not only skew-symmetric, but alternating, and that the order of $Br(X)$ is a square, if finite. The proof of alternatingness, or equivalently of $x.\delta x = 0$ for $x \in H^2(X,\mu_m)$ in the notation of the preceding paragraph, looks like an interesting cohomological problem. Comparison with the methods of Cassels suggests that another description of the pairing, involving divisors, might be useful. It would also be interesting to give a description of it in terms of Azumaya algebras on X. On the other hand, the methods used here suggest that there should exist a "truly arithmetic" Poincaré duality theorem for schemes of finite type over number fields, which in the case of dimension 2 should yield a new construction of Cassels' pairing.

In the proof of the main theorem below we use a counting process involving homomorphisms $f : A \to B$ of Z_ℓ-modules. We will call f a quasi-isomorphism if $\mathrm{Ker}\ f$ and $\mathrm{Coker}\ f$ are finite, in which case we put

$$(5.5) \qquad z(f) = \frac{|[\mathrm{Coker}\ f]|_\ell}{|[\mathrm{Ker}\ f]|_\ell} = \ell^{\mathrm{length}(\mathrm{Ker}\ f) - \mathrm{length}(\mathrm{Coker}\ f)} .$$

We list without proof the elementary properties of $z(f)$ which we will need.

LEMMA z.1.- Suppose A and B are finitely generated Z_ℓ-modules of the same rank and that (a_i), resp.(b_i) is a base for A, resp.B, mod torsion. Let $f : A \to B$, with $f(a_i) = \Sigma z_{ij} b_j$ mod torsion. Then f is a quasi-isomorphism if and only if $\det(z_{ij}) \neq 0$, in which case

$$z(f) = |\det(z_{ij})[B_{tors}]/[A_{tors}]|_\ell .$$

LEMMA z.2.- Suppose $f : A \to B$ and $g : B \to C$. If any two of the three maps

f,g, and gf are quasi-isomorphisms the third is also, and $z(gf) = z(g)z(f)$.

LEMMA z.3.- Let $A^* = \mathrm{Hom}_{Z_\ell}(A, Q_\ell/Z_\ell)$. Then $f : A \to B$ is a quasi-isomorphism if and only if $f^* : B^* \to A^*$ is, in which case $z(f)z(f^*) = 1$.

LEMMA z.4.- Suppose θ is an endomorphism of a finitely generated Z_ℓ-module A. Let $\theta \otimes 1$ denote the corresponding endomorphism of $A \otimes_{Z_\ell} Q_\ell$. Let $f : \mathrm{Ker}\ \theta \to \mathrm{Coker}\ \theta$ be the map induced by the identity $A \to A$. Then f is a quasi-isomorphism if and only if $\det(T - \theta \otimes 1) = T^\rho R(T)$, with $\rho = \mathrm{rk}_{Z_\ell}(\mathrm{Ker}\ \theta)$ and $R(0) \neq 0$, in which case $z(f) = |R(0)|_\ell$.

To prove this last, let θ_1 denote the restriction of θ to $\mathrm{Im}\ \theta$, and note that $\mathrm{Ker}\ f = \mathrm{Ker}\ \theta \cap \mathrm{Im}\ \theta = \mathrm{Ker}\ \theta_1$, and $\mathrm{Coker}\ f = A/(\mathrm{Ker}\ \theta + \mathrm{Im}\ \theta) \approx \mathrm{Coker}\ \theta_1$. Now apply lemma z.1 to $\theta_1 : \mathrm{Im}\ \theta \to \mathrm{Im}\ \theta$.

Let ℓ be a prime number different from the characteristic p. Passing to the inductive limit with $m = \ell^n$ in (5.1) we obtain an exact commutative diagram of Z_ℓ torsion modules of finite corank :

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & NS(X) \otimes (Q_\ell/Z_\ell) & \to & (NS(\bar{X}) \otimes (Q_\ell/Z_\ell))^G & & \\
 & & & & \downarrow & \searrow^{g} & \downarrow & & \\
(5.6) & 0 & \to & H^1(\bar{X},\mu(\ell))_G & \to\ H^2(X,\mu(\ell)) & \to & H^2(\bar{X},\mu(\ell))^G & \to & 0 \\
 & & & \wr\!\downarrow & \downarrow & & & & \\
 & & & (\mathrm{Pic}(\bar{X})(\ell))_G & Br(X)(\ell) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

The replacement of Pic by NS at the top of the middle column is justified because $\mathrm{Pic}(X) \to \mathrm{NS}(X)$ is surjective with finite kernel. Passing to the projective limit with (5.1) we get an exact commutative diagram of finitely generated $\mathbf{Z}_\ell$-modules :

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathrm{Pic}(X)\otimes \mathbf{Z}_\ell & \to & (\mathrm{NS}(\bar{X})\otimes \mathbf{Z}_\ell)^G \simeq \mathrm{NS}(X)\otimes \mathbf{Z}_\ell & \to & 0 \\ & & & & \downarrow & & \downarrow h & & \\ (5.7) \quad 0 \to & H^1(\bar{X},T_\ell(\mu))_G & \to & H^2(X,T_\ell(\mu)) & & \to & H^2(\bar{X},T_\ell(\mu))^G & \to & 0 \\ & \wr\wr & & \downarrow & & & & & \\ & (T_\ell(\mathrm{Pic}(\bar{X})))_G & & T_\ell(\mathrm{Br}(X)) & & & & & \\ & & & & 0 & & & & \end{array}$$

where $T_\ell(U) = \varprojlim(U_{\ell^n}) = \mathrm{Hom}(\mathbf{Q}_\ell/\mathbf{Z}_\ell, U)$ for an abelian group U. The isomorphism in the northeast corner follows from the definition of $\mathrm{NS}(X)$ as the image of $\mathrm{Pic}(X)$ in $\mathrm{NS}(\bar{X}) \simeq \mathrm{Pic}(\bar{X})/(\mathrm{Pic\ Var}(X)(\bar{k}))$, the fact that $\mathbf{Z}_\ell$ is $\mathbf{Z}$-flat, and the surjectivity of the maps $\mathrm{Pic}(X) \to \mathrm{Pic}(\bar{X})^G$ and $\mathrm{Pic}(\bar{X})^G \to \mathrm{NS}(\bar{X})^G$, these surjectivities resulting from $H^1(G,\bar{k}^*) = 0$ and $H^1(G,\mathrm{Pic\ Var}(X)(\bar{k})) = 0$ (Lang's theorem). This last vanishing also shows that $(\mathrm{Pic}(\bar{X})(\ell))_G \simeq (\mathrm{NS}(\bar{X})_{\ell\text{-tors}})_G$, and consequently

$$(5.8) \qquad [\mathrm{Pic}(\bar{X})(\ell)_G] = [\mathrm{NS}(\bar{X})^G_{\ell\text{-tors}}] = [\mathrm{NS}(X)_{\ell\text{-tors}}],$$

because $[A_G] = [A^G]$ for finite A.

We are now ready to prove

THEOREM 5.2.- The following statements are equivalent

(i) $Br(X)(\ell)$ is finite.

(ii) The map $h : NS(X) \otimes Z_\ell \to H^2(\bar{X},T_\ell(\mu))^G$ is bijective.

(iii) $\rho(X) = rk_{Z_\ell} H^2(\bar{X},T_\ell(\mu))^G$.

(iv) $\rho(X)$ is the multiplicity of q as reciprocal root of the polynomial $P_2(X,T)$.

If these statements are true for one ℓ, then $Br(X)(\text{non } p) = \coprod_{\ell \neq p} Br(X)(\ell)$ is finite, and conjecture (C) is true up to a factor of $\pm p^\nu$; in other words, putting $R(T) = P_2(X,T)/(1-qT)^{\rho(X)}$, we have

$$R(q^{-1}) = \pm p^\nu \frac{[Br(X)(\text{non } p)] \det(D_i.D_j)}{[NS(X)_{tors}]^2} \tag{5.9}$$

for some integer ν.

It is easy to see that $(T_\ell(Pic(\bar{X})))_G$ is isomorphic to the group of k-rational points on Pic Var(X) and is therefore finite. Since $T_\ell(Br(X))$ is torsion-free, we get from (5.7) an exact sequence

$$0 \to NS(X) \otimes Z_\ell \xrightarrow{h} H^2(\bar{X},T_\ell(\mu))^G \to T_\ell(Br(X)) \to 0 \tag{5.10}$$

which proves the equivalence of (i), (ii), and (iii), because $Br(X)(\ell)$ is finite if and only if $T_\ell(Br(X)) = 0$.

As explained in [19], we have

$$\det(1 - \sigma_{2,\ell}T) = P_2(X,q^{-1}T) , \tag{5.11}$$

where $\sigma_{2,\ell}$ denotes the automorphism of $H^2(\bar{X},T_\ell(\mu)) \otimes_{Z_\ell} Q_\ell$ which is induced by the Frobenius automorphism σ. Hence the multiplicity of q as reciprocal root of P_2 is the same as the multiplicity of 1 as eigenvalue of $\sigma_{2,\ell}$, or

of 0 as eigenvalue of $\sigma_{2,\ell} - 1$. This multiplicity is clearly at least as great as the Z_ℓ-rank of $H^2(\bar{X},T_\ell(\mu))^G$. Therefore (iv) implies (iii), in view of the injectivity of h in (5.9).

Assume now that the equivalent conditions (i), (ii) and (iii) hold and consider the diagram

$$\begin{array}{ccccc} NS(X) \otimes Z_\ell & \xrightarrow{e} & Hom(NS(X),Z_\ell) & \approx & Hom(NS(X) \otimes Q_\ell/Z_\ell, Q_\ell/Z_\ell) \\ h \downarrow & & & & g^* \uparrow \\ H^2(\bar{X},T_\ell(\mu))^G & \xrightarrow{f} & H^2(\bar{X},T_\ell(\mu))_G & \approx & Hom(H^2(\bar{X},\mu(\ell))^G, Q_\ell/Z_\ell) \end{array} \tag{5.12}$$

Here h is the isomorphism of (ii). The map e is that induced by the intersection pairing $NS(X) \times NS(X) \to Z$. The non-degeneracy of this pairing over X follows from that over $\bar{X}$. By lemma z.1 we conclude that e is a quasi-isomorphism, with

$$z(e) = \frac{|\det(D_i.D_j)|_\ell}{|[NS(X)_{\ell\text{-tors}}]|_\ell}$$

The isomorphism in the top now of (5.12) is trivial, and the one on the bottom row comes from Poincaré duality on $\bar{X}$. The map g^* is the adjoint of the map g in diagram (5.6). From the exactness of (5.6) and lemmas z.2 and z.3 we conclude using (5.8) that g^* is a quasi-isomorphism with

$$z(g^*) = \frac{|[NS(X)_{\ell\text{-tors}}]|_\ell}{|[Br(X)(\ell)]|_\ell}$$

The map f is that induced by the identity on $H^2(\bar{X},T_\ell(\mu))$. By (5.11) and lemma z.4 with $\theta = \sigma - 1$, the map f is a quasi-isomorphism if and only if (iv) holds, in which case

$$z(f) = |R(q^{-1})|_\ell ,$$

where R is as in (5.9), and is directly related to, but not the same as, the R in lemma z.4.

The diagram (5.12) is commutative, i.e. $e = g^*fh$. To see this just replace the X's in the upper row by $\bar{X}$ and remove the G's in the bottom row, and use the compatibility of intersection of cycles with cup products, on $\bar{X}$.

By lemma z.2 we conclude that f is a quasi-isomorphism, hence (iv) holds, and

$$(5.13) \qquad |R(q^{-1})|_\ell = z(f) = z(e)z(g^*)^{-1} = \left| \frac{[Br(X)(\ell)] \det(D_i \cdot D_j)}{[NS(X)_{\ell\text{-tors}}]^2} \right|_\ell$$

But (iv) is independent of ℓ (because $P_2(X,T)$ is, as explained at the beginning of § 4), and consequently (5.13) holds for all primes $\ell \neq p$. Since $NS(X)_{tors}$ is finite, it follows that $Br(X)(\text{non } p)$ is finite, and (5.9) holds. This concludes the proof of theorem 5.2.

The problem of proving the analogs of theorems 5.1 and 5.2 for $\ell = p$ should furnish a good test for any p-adic cohomology theory, and might well serve as a guide for sorting out and unifying the various constructions which have been suggested and used ; Serre's Witt vectors, Dwork's banach spaces, the raisings via special affines of Washnitzer-Monsky, and Grothendieck's flat cohomology of μ_{p^n}. In view of theorem 5.2 we have especial confidence in the "non-p" part of Conjecture (C) ; nevertheless, some computations in the special case X a product of elliptic curves have furnished fragmentary evidence for the p-part as well.

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