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THE INDEX OF ELLIPTIC OPERATORS ON COMPACT MANIFOLDS

by Michael F. ATIYAH

Introduction. - The index of elliptic operators has been studied recently by several authors (AGRANOVIČ-DYNIN [2], SEELEY [6]). Using the theory of singular integral operators they make good progress on the analysis but rapidly get involved with difficult questions of algebraic topology. In [4], on which this talk is based, these topological aspects of the problem are dealt with, and we obtain finally an explicit formula for the index on compact manifolds.

1. Preliminaries on $K(X)$.

We recall (cf. [3]) that, for a finite complex X and subcomplex Y, we have the Grothendieck group $K(X, Y)$. This may be defined as the set of homotopy classes of maps

$$(X, Y) \to (\underset{\sim}{Z} \times B_U, \text{ base point})$$

where B_U is the classifying space of the "infinite" unitary group $U = \varinjlim U(n)$. Moreover we have a natural homomorphism (the Chern character)

$$\mathrm{ch} : K(X, Y) \to H^{\mathrm{even}}(X, Y; \underset{\sim}{Q}) .$$

For a complex vector bundle E of dimension p it is usual to write

$$\mathrm{ch}\, E = \sum_{i=1}^{p} \exp x_i$$

where the x_i have formal dimension 2. The Todd class of E is then defined by

$$\mathcal{T}(E) = \prod_{i=1}^{p} \frac{x_i}{1 - \exp(-x_i)} .$$

Consider now a sequence

$$E : \quad 0 \to E_0 \xrightarrow{\sigma_0} E_1 \to \dots \xrightarrow{\sigma_{m-1}} E_m \to 0$$

where the E_i are complex vector bundles on X, the homomorphisms σ_i are defined on Y and the sequence is exact on Y. Such a sequence is elementary, if for some j,

$$E_j = E_{j+1} , \qquad \sigma_j = 1 ,$$
$$E_i = 0 \quad \text{for } i \neq j , \; j+1 .$$

Two sequences E , E' are equivalent if there exist elementary sequences $P_1, \dots, P_r, Q_1, \dots, Q_s$ such that

$$E \oplus P_1 \oplus \dots \oplus P_r \cong E' \oplus Q_1 \oplus \dots \oplus Q_s .$$

The set of equivalences is an abelian semi-group under $\oplus$ and is denoted by $L(X, Y)$.

(1.1). PROPOSITION. - There is a unique natural homomorphism

$$\chi : L(X, Y) \to K(X, Y)$$

such that, if $Y = \emptyset$,

$$\chi(E) = \Sigma(-1)^i E_i .$$

Moreover χ is an isomorphism.

(1.2). PROPOSITION. - If E is a sequence as above and V is a vector bundle on X , then

$$\chi(V \otimes E) = V\chi(E)$$

where, on the right, we use the $K(X)$-module structure of $K(X, Y)$.

The proof of (1.1) is elementary but lengthy and will be omitted. Essentially it provides us with a "Grothendieck-type" definition of the relative group $K(X, Y)$. (1.2) is an easy consequence of (1.1).

2. Symbols of differential operators.

Let X be a compact oriented smooth (i. e. infinitely differentiable) manifold of dimension n , let E , F be smooth complex vector bundles on X and denote by $\Gamma(E)$, $\Gamma(F)$ the spaces of smooth sections. Let

$$d : \Gamma(E) \to \Gamma(F)$$

be a linear differential operator of order k , i. e. a linear operator given locally by a matrix of partial derivatives of order $\leqslant k$ with smooth coefficents. Thus d is a smooth section of a certain vector bundle $\mathrm{Diff}_k(E, F)$. Now we have an exact sequence of vector bundles :

$$0 \to \mathrm{Diff}_{k-1}(E, F) \to \mathrm{Diff}_k(E, F) \to S^k(T) \otimes_{\mathbf{R}} \mathrm{Hom}_{\mathbf{C}}(E, F) \to 0$$

where $T = T(X)$ is the tangent bundle of X , S^k denotes the k - th symmetric power and Hom is the bundle of homomorphisms. Thus d defines a section of $S^k(T) \otimes_{\mathbf{R}} \mathrm{Hom}_{\mathbf{C}}(E, F)$ and hence a bundle homomorphism

$$\sigma'(d) : \pi^* E \to \pi^* F$$

where $\pi : T^* \to X$ is the projection onto X of T^* (the dual of T). The homomorphism $\sigma(d) = i^k \sigma'(d)$ is called the <u>symbol</u> of d (the factor i^k is inserted to fit in with Fourier transforms). An operator d is <u>elliptic</u> if $\sigma(d)$ is an isomorphism on the complement T^*_o of the zero-section of T^*.

More generally we shall consider a complex of operators, i. e. a sequence of differential operators (all of the same order (1))

$$\mathcal{E} : 0 \to \Gamma(E_o) \overset{d}{\to} \Gamma(E_1) \to \dots \overset{d}{\to} \Gamma(E_m) \to 0$$

with $d^2 = 0$. The <u>symbol</u> $\sigma(\mathcal{E})$ will be the sequence on $T^*(X)$

$$0 \to \pi^* E_o \xrightarrow{\sigma(d)} \pi^* E_1 \to \dots \xrightarrow{\sigma(d)} \pi^* E_m \to 0 \quad .$$

The complex $\mathcal{E}$ is <u>elliptic</u> if $\sigma(\mathcal{E})$ is <u>exact</u> on $T^*_o(X)$.

Suppose now that X has a Riemannian metric and let $B(X)$, $S(X)$ denote the unit ball and unit sphere bundles of $T^*(X)$. Then the symbol $\sigma(\mathcal{E})$ of an elliptic complex defines an element (still denoted by $\sigma(\mathcal{E})$) of $L(B(X), S(X))$. Using the homomorphism of (1.1) and the Thom isomorphism

$$\Phi_* : H^*(X ; \mathbb{Q}) \to H^*(B(X), S(X) ; \mathbb{Q})$$

we shall define a <u>cohomological invariant</u> $\operatorname{ch} \mathcal{E}$ by

$$\operatorname{ch} \mathcal{E} = \Phi_*^{-1} \operatorname{ch} \chi\sigma(\mathcal{E}) \in H^*(X ; \mathbb{Q}) \quad .$$

If $\mathcal{E}$ has only two terms, i. e. if we are dealing with an elliptic operator $d : \Gamma(E) \to \Gamma(F)$, we write $\operatorname{ch} d$ for $\operatorname{ch} \mathcal{E}$.

For applications it is important to be able to calculate $\operatorname{ch} \mathcal{E}$. This can be done if X has a G-structure and $\sigma(\mathcal{E})$ <u>is associated to the G-structure</u> (G being a compact Lie group). This means that we are given : a principal G-bundle P over X, a real oriented G-module V, an isomorphism

$$P \times_G V \cong T(X) \quad ,$$

complex G-modules M_i and G-homomorphisms $s_i : S^k(V^*) \to \operatorname{Hom}_{\mathbb{C}}(M_i, M_{i+1})$ so that $E_i = P \times_G M_i$ and σ_i is induced by s_i.

(2.1). PROPOSITION. – <u>Suppose $\mathcal{E}$ is an elliptic complex associated to a G-structure as above and assume further that</u> $\dim X = 2 \operatorname{rank} G_V$, <u>where G_V is the image of G in $\operatorname{Aut} V$. Then $\operatorname{ch} \mathcal{E}$ is the characteristic class of the G-structure associated to the universal class</u>

(1) This restriction can be relaxed.

$$\frac{\Sigma(-1)^i \operatorname{ch} M_i}{\prod \omega_j} \in H^{**}(B_G ; \mathbb{Q})$$

where ch M_i is the character of the G-module M_i, ω_j are the negative [2] weights of the real G-module V and we use the Borel-Hirzebruch method of describing the cohomology of B_G.

Proof. – By considering the bundle over B_G with fibre V^* : $\prod \omega_j$ is the Euler classe of this bundle.

In another direction one can also prove

(2.2). PROPOSITION. – ch $\mathcal{E} = 0$ if dim X is odd.

Proof. – By considering the real projective space bundle associated to $S(V^*)$.

3. The main theorems.

(3.1). PROPOSITION. – The cohomology of an elliptic complex is finite-dimensional.

Thus we can define the Euler characteristic

$$\chi(\mathcal{E}) = \Sigma(-1)^i \dim H^i(\mathcal{E}) \quad .$$

For one operator d this is called the index, thus

$$\text{index } (d) = \dim (\operatorname{Ker} d) - \dim (\operatorname{Coker} d) \quad .$$

Define the Todd class of X by

$$\mathcal{T}(X) = \mathcal{T}(T(X) \otimes_{\mathbb{R}} \mathbb{C}) \quad ,$$

and, for any $\alpha \in H^*(X ; \mathbb{Q})$, denote by $\alpha[X]$ the value of the top-dimensional component of α on the fundamental homology class of X. Then the main result is the following formula for the Euler characteristic of an elliptic complex :

(3.2). THEOREM. – Let $\mathcal{E}$ be an elliptic complex, then

$$\chi(\mathcal{E}) = \{\operatorname{ch} (\mathcal{E})\, \mathcal{T}(X)\} [X] \quad .$$

In particular for one operator this becomes :

(3.3) $$\text{index } (d) = \{\operatorname{ch} d \times \mathcal{T}(X)\} [X] \quad .$$

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[2] The "negative" weights ω_j depend on some choices but $\prod \omega_j$ depends only on the orientation.

Applying (3.2) and (2.1) to the usual $\bar{\partial}$ complex, and then using the Dolbeault isomorphism, one gets :

(3.4). THEOREM (HIRZEBRUCH-RIEMANN-ROCH). - Let X be a compact complex manifold, V a holomorphic vector bundle on X, $\chi(X, V)$ the Euler characteristic of the cohomology of the sheaf of germs of holomorphic sections of V, then

$$\chi(X, V) = \{\mathrm{ch}\, V \times \mathfrak{T}(X)\}[X] ,$$

where $\mathfrak{T}(X) = \mathfrak{T}(T(X))$.

Remark. - One must not confuse $\mathfrak{T}(X)$ for a complex manifold X with $\mathfrak{T}(X_r)$, where X_r is the underlying differentiable manifold : in fact $\mathfrak{T}(X_r) = \mathfrak{T}(X)\,\bar{\mathfrak{T}}(X)$, where $\bar{\mathfrak{T}}(X) = \mathfrak{T}(T(X)^*)$. Thus (3.4) follows from (3.3) because (2.1) gives, in this case,

$$\mathrm{ch}\, \mathcal{E} = \mathrm{ch}\, V \times \bar{\mathfrak{T}}(X)^{-1} .$$

Now let X be Riemannian with $\dim X = 2\ell$, and let

$$\mathfrak{T}^* : \Lambda^p(T^*) \to \Lambda^{2\ell-p}(T^*)$$

be the usual operator on differential forms. Let α be the involution of $E = \bigoplus_p \Lambda^p(T^*)$ defined on $\Lambda^p(T^*)$ by $\alpha = i^{p(p+1)-\ell} \star$, and denote by E^+, E^- the sub-bundles of E corresponding to the $+1$ (-1) eigenvalues of α. Let d be the exterior derivative, δ its formal adjoint, then one easily verifies that

$$D_o = d + \delta : \Gamma(E^+) \to \Gamma(E^-)$$

is elliptic and, if ℓ is even,

index D_o = Hirzebruch index of X (i. e. the index of the quadratic form on $H^\ell(X ; \mathbb{R})$).

(3.3) then becomes the Hirzebruch index theorem, expressing the index of X in terms of Pontrjagin numbers.

Remark. - All the various integrality theorems of Borel-Hirzebruch are included in (3.2).

4. Integral operators.

We discuss first elliptic operators in a Hilbert-space framework. Introducing metrics we consider the Hilbert-space $L^2(E)$ of square-integrable sections of E. We denote by $L^2_k(E)$ the subspace of $L^2(E)$ consisting of all u such that

$du \in L^2(1)$ for all $d \in \mathrm{Diff}_k(E, 1) = \Gamma\ \underset{\sim}{\mathrm{Diff}}_k(E, 1)$ (where 1 is the trivial bundle $X \times C$ and du is taken in the distribution sense). If $D \in \mathrm{Diff}_k(E, F)$ we denote by $\hat{D}$ the extension of D to an operator

$$\hat{D} : L_k^2(E) \to L^2(F) \quad .$$

Then the main facts about elliptic differential operators (on compact manifolds) may be summarized as follows :

(4.1). PROPOSITION. - <u>If</u> $D ; \Gamma(E) \to \Gamma(F)$ <u>is elliptic, then</u>

(i) $\hat{D}$ <u>is a closed operator (from</u> $L^2(E)$ <u>to</u> $L^2(F)$) <u>with closed range,</u>

(ii) <u>if</u> $D^* : \Gamma(F) \to \Gamma(E)$ <u>is the formal adjoint of</u> D , <u>then</u> $\widehat{D^*}$ <u>is the Hilbert-space adjoint of</u> $\hat{D}$,

(iii) $\mathrm{Ker}\ D = \mathrm{Ker}\ \hat{D}$ <u>and is finite-dimensional,</u>

(iv) $\mathrm{Coker}\ D \cong \mathrm{Coker}\ \hat{D}$.

<u>Remarks</u>.

a. These results are not unfortunately, to be found in this form in the literature, mainly because P. D. E. experts do not like manifolds, bundles, etc. However, they are easily deduced from what is available in published form.

b. (4.1) implies that $\mathrm{index}\ D = \mathrm{index}\ \hat{D}$.

c. If we have an elliptic complex $\mathcal{E}$

$$\to \Gamma(E_i) \overset{d}{\to} \Gamma(E_{i+1}) \to \cdots$$

then we can introduce its adjoint $\mathcal{E}^*$

$$\leftarrow \Gamma(E_i) \overset{d^*}{\longleftarrow} \Gamma(E_{i+1}) \leftarrow \cdots$$

which will also be elliptic. Moreover the operator

$$D = d + d^* : \bigoplus_k \Gamma(E_{2k}) \to \bigoplus_k \Gamma(E_{2k+1})$$

is elliptic. Using this one can deduce (3.1) easily from (4.1), on the lines of the Hodge theory, and one finds that

$$\chi(\mathcal{E}) = \mathrm{index}\ D , \qquad \chi\sigma(\mathcal{E}) = \chi\sigma(D) \quad .$$

Thus (3.3) follows from (3.2), so that we need only consider a single elliptic operator.

Now let $d_E : \Gamma(E) \to \Gamma(E \otimes T^*(X))$ be the covariant derivative with respect to some connection, and put $\Delta_E = d_E^* d_E$ (the Laplacian of E). Then $1 + \hat{\Delta}_E$ is, by (4.1), a positive definite self-adjoint operator. Thus we can define a bounded (in fact compact) operator J_E on $L^2(E)$ such that

$$J_E^2 = (1 + \hat{\Delta}_E)^{-1} \quad .$$

Moreover one can show [5] that, for all k, J_E maps $L_k^2(E)$ isomorphically (as vector space) onto $L_{k+1}^2(E)$.

Next, generalizing Calderon-Zygmund from R^n to manifolds, one can define the space $\mathrm{Int}(E, F)$ of singular integral operators [5]. These are bounded operators $L^2(E) \to L^2(F)$, and by Fourier transforms one can define a symbol σ which is a homomorphism

$$\sigma : \ \mathrm{Int}(E , F) \to \mathrm{Hom}_{S(X)}(\pi^* E , \pi^* F)$$

(where $\mathrm{Hom}_{S(X)}$ denotes all continuous homomorphisms on $S(X)$). This symbol has the following properties

(i) σ is an epimorphism and $\mathrm{Ker}\, \sigma$ consists of all compact operators,

(ii) $\sigma(T \oplus T') = \sigma(T) + \sigma(T')$, $\quad \sigma(TT') = \sigma(T)\, \sigma(T')$

where T, T' belong to appropriate bundles,

(iii) if $D \in \mathrm{Diff}_k(E , F)$ then

$$DJ_E^k \in \mathrm{Int}(E , F) \quad \text{and} \quad \sigma(D) = \sigma(DJ_E^k) \quad \text{on } S(X) \quad .$$

An operator $T \in \mathrm{Int}(E , F)$ is elliptic if $\sigma(T)$ is an isomorphism. It follows from (i) and (ii) that $T \in \mathrm{Int}(E , F)$ is elliptic if and only if $\exists\ T' \in \mathrm{Int}(F, E)$ with TT' and $T'T$ both equal to the identity plus a compact operator. This implies that T is a "Fredholm-operator", i. e. has closed range, and $\mathrm{Ker}\, T$, $\mathrm{Coker}\, T$ both finite-dimensional. Thus index T is defined. Since

$$J_E^k : \ L^2(E) \to L_k^2(E)$$

is an isomorphism it follows that, if $D \in \mathrm{Diff}_k(E , F)$ is elliptic, then

$$\mathrm{index}\ \hat{D} = \mathrm{index}\ (DJ_E^k) \quad .$$

Thus, in view of remark (b) after (4.1), (3.3) will follow from :

(4.2). THEOREM. - Let $T \in \mathrm{Int}(E , F)$ be elliptic then

$$\mathrm{index}\ T = \{\mathrm{ch}\ T \times \mathcal{C}(X)\}\, [X] \quad .$$

(ch T is defined by the same formula as ch D). Now the following facts about Fredholm-operators have been proved by various authors (cf. for example [6]).

(4.3). If T is Fredholm and K is compact then $T + K$ is Fredholm and index T = index $(T + K)$.

(4.4). *If* T *is Fredholm and* $\|T - T'\|$ *is sufficiently small then* T' *is Fredholm and* index T = index T' .

From these it follows that :

(4.5). *If* $T \in \mathrm{Int}(E, F)$ *is elliptic,* index T *depends only on the homotopy class of* $\sigma(T)$ *in* $\mathrm{Iso}_{S(X)}(\pi^* E, \pi^* F)$.

Now if T is an operator induced by $T_0 \in \mathrm{Iso}(E, F)$ then $\sigma(T) = \pi^* T_0$ and Ker T = Coker $T = 0$, so that index $T = 0$. From (4.5) therefore we deduce :

(4.6). *If* $T \in \mathrm{Int}(E, F)$ *and* $\sigma(T)$ *extends to* $B(X)$ *(as an isomorphism) then* index $T = 0$.

This, together with additivity of the index under $\oplus$ and (1.1), implies that the index is essentially a homomorphism

$$\text{index} : K(B(X), S(X)) \to \mathbb{Z} \quad .$$

5. Main steps in proof of (3.3).

Let $\dim X = 2\ell$, and let D_0 be the operator described in § 3. Put $\sigma_0 = \sigma(D_0)$,

$$\chi_0 = \chi(\sigma_0) \in K(B(X), S(X)) \quad .$$

(5.1). PROPOSITION. – $K(B(X), S(X)) \otimes \mathbb{Q}$ *is a free* $K(X) \otimes \mathbb{Q}$*-module with* χ_0 *as a generator.*

Proof. – Since $\mathrm{ch} : K(X) \otimes \mathbb{Q} \to H^{even}(X ; \mathbb{Q})$ is an isomorphism compatible with products [3] and since $H^*(B(X), S(X) ; \mathbb{Q})$ is a free $H^*(X ; \mathbb{Q})$-module with generator $\Phi_*(1)$, it is only necessary to check that

$$\mathrm{ch}\, \chi_0 = \varepsilon \Phi_*(1)$$

with ε a unit of $H^*(X ; \mathbb{Q})$ (i. e. $\varepsilon_0 \neq 0$ where ε_0 is the zero-dimensional component of ε). But (2.1) gives

$$\varepsilon_0 = 2^\ell \quad .$$

(5.2). PROPOSITION. – *Let* X *be the boundary of a* $(2\ell + 1)$*-manifold* Y *and let* V *be a vector bundle on* X *which is the restriction of a bundle* W *on* Y . *Then*

$$\text{index}(V\chi_0) = 0 \quad .$$

Proof (sketch). – Let $E = \bigoplus_k \Lambda^{2k}(T^*(Y))$ and let

$$D : \Gamma(E \otimes W) \to \Gamma(E \otimes W)$$

be the operator on Y defined by $D = \star d + d \star$ (using a metric on Y and a connection in W). On the boundary X the bundle E decomposes into E^+, E^- (notation of § 3). Consider the subspace B of $L^2_X(E \otimes V)$ given by boundary values of $\operatorname{Ker} D$. Using Stokes' formula :

$$\langle Du , v\rangle_Y - \langle u , Dv\rangle_Y = \text{Const.} \{\langle u^+ , v^+\rangle_X - \langle u^- , v^-\rangle_X\}$$

it follows that $\overline{B}$ is the graph of an isometric operator

$$T : L^2_X(E^+ \otimes V) \to L^2_X(E^- \otimes V) \quad .$$

If one assumes that there is uniqueness in the Cauchy problem (3) for D^2 then one can show that

$$\text{range } T = L^2 , \quad \text{domain } T = L^2 \quad ,$$

so that $\operatorname{index} T = 0$. On the other hand by the methods of [1] one can show that $T \in \operatorname{Int}(E^+ \otimes V , E^- \otimes V)$ and that

$$\sigma(T) = \sigma_0 \otimes 1_V \quad .$$

The result now follows from (1.2).

Remark. - If $V = 1$ then (5.2) has of course a simple topological proof and this is a key step in Hirzebruch's proof of his index theorem. Having now established (5.2) we proceed to imitate the rest of Hirzebruch's proof.

Let Σ denote the set of all pairs (X , V) where X is a smooth compact oriented manifold of even dimension and V is a complex vector bundle on X. Then we have

(5.3). PROPOSITION. - *There is a unique function* $f : \Sigma \to \mathbb{Q}$ *such that*

(i) $f(X , V) = 0$ *if* $\exists$ (Y , W) *with* $\partial Y = X$, $W|X \cong V$,

(ii) $f(X_1 + X_2 , V_1 + V_2) = f(X_1 , V_1) + f(X_2 , V_2)$ (+ *denoting disjoint sum*),

(iii) $f(X , V_1 \oplus V_2) = f(X , V_1) \oplus f(X , V_2)$,

(iv) $f(X_1 \times X_2 , V_1 \otimes V_2) = f(X_1 , V_1) f(X_2 , V_2)$,

(v) $f(S^{2\ell} , V) = 2^\ell$ *if* $S^{2\ell}$ *is the* 2ℓ*-sphere and* 1, V *are a basis of* $K(S^{2\ell})$.

(vi) $f(P_{2n}(\mathbb{C}) , 1) = 1$, *where* $P_{2n}(\mathbb{C})$ *is complex projective space.*

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(3) We can take everything analytic without loss. However there are ways of avoiding the Cauchy problem.

Proof (sketch). - Let A_n denote the cobordism group of pairs (X, V) where $\dim V = n$ and the cobordism relation is that in (i). Following THOM one can easily determine $A_n \otimes \underset{\sim}{Q}$. Conditions (i) and (ii) assert that, for each n, f induces a homomorphism

$$f_n : A_n \otimes \underset{\sim}{Q} \to \underset{\sim}{Q} \quad .$$

Properties (iii) and (iv) relate the different f_n and give information about the general form of f. Properties (v) and (vi) are then sufficient to fix f uniquely.

(5.4). PROPOSITION. - The functions

$$f_1(X, V) = \text{index}\,(V\chi_0)$$

$$f_2(X, V) = \{\text{ch } V \times \text{ch } D_0 \times \mathcal{T}(X)\}[X]$$

both satisfy (i)-(vi) of (5.3).

Proof. - The properties of f_2 are easily verified. For f_1 property (i) is (5.2), (ii) and (iii) are immediate while (v) and (vi) are special cases which can be checked. The proof of (iv) presents a few technical difficulties which we omit.

(5.1), (5.3) and (5.4) together establish the index theorem (4.2) for even-dimensional X. The odd case follows by considering $X \times S^1$.

Final remarks.

1° There should be a generalization of (3.2) which will yield the Grothendieck-Riemann-Roch theorem.

2° Theorem (3.3) should hold also, on a manifold X with boundary ∂X, for an elliptic operator D with "well-posed" boundary conditions B. One can (with much work!) define an invariant

$$\text{ch}(D, B) \in H^*(X, \partial X ; \underset{\sim}{Q})$$

and it seems that the same formula for the index holds.

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