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A RIEMANN-ROCH THEOREM FOR DIFFERENTIABLE MANIFOLDS

by Friedrich HIRZEBRUCH

We shall show that the Riemann-Roch-Grothendieck theorem [2] has analogies in the differentiable case. The theorems are formulated in 3.4, 3.5, 3.7. In 3.3 their motivation by the RR-Grothendieck theorem is explained. Further generalizations of these differentiable RR-theorems are possible (ATIYAH) but we shall not go into that. Paragraph 4 brings examples how one can apply the differentiable RR-theorems. In paragraph 5 we indicate the proofs which rely heavily on the Bott theory [4], [8].

The results reported upon in this exposé are mainly due to ATIYAH and can be found in his correspondence with the speaker who had conjectured theorem 3.4 and has contributed a little bit to the proof of the differentiable RR-theorems and to their applications.

1. The groups $K(X)$ and $K_0(X)$.

The base spaces X, Y of vector bundles are assumed to be finite dimensional CW-complexes except otherwise mentioned. This assumption, much too strong for many of the following definitions and results, is made for convenience.

1.1. - Let $F(X)$ be the free abelian group generated by the set of all isomorphy classes of complex vector bundles over X. (It is not assumed that a complex vector bundle has the same fibre dimension over different connectedness components of X). An element x of $F(X)$ is thus a formal finite linear combination

$$x = n_i \xi_i \ , \quad n_i \in \mathbb{Z} \ , \quad \xi_i \text{ complex vector bundles over } X .$$

Let

$$\text{(E)} \qquad 0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$

be an exact sequence of complex vector bundles, (i.e. ξ is the Whitney sum $\xi' \oplus \xi''$). To (E) we attach the element $Q(E) = \xi - \xi' - \xi''$ of $F(X)$.

DEFINITION. - $K(X)$ is the quotient group of $F(X)$ modulo the subgroup generated by the elements $Q(E)$ where E runs through all exact sequences with three terms. In the same way, starting with real vector bundles, one defines the group $K_{\underset{\sim}{0}}(X)$, (where $\underset{\sim}{0}$ stands for "orthogonal").

This definition is analogous to that of [2], paragraph 4. The group operation in $K(X)$ and $K_{\underset{\sim}{0}}(X)$ respectively is called "Whitney sum".

1.2. - A complex vector bundle is called trivial if it is the product bundle over each connectedness component of X. Two complex vector bundles ξ_1, ξ_2 (over X) are called I-equivalent if there exist trivial complex vector bundles η_1, η_2 such that $\xi_1 \oplus \eta_1$ and $\xi_2 \oplus \eta_2$ are equivalent. To each complex vector bundle α there exists a complex vector bundle β such that $\alpha \oplus \beta$ is trivial. This follows from the consideration of the universal bundle over the Grassmannian. Thus the I-equivalence classes constitute an abelian group $K'(X)$ whose group operation is induced by the Whitney sum. We have the canonical split exact sequence

$$(1) \qquad 0 \longrightarrow H^0(X, \underset{\sim}{Z}) \xrightarrow{j} K(X) \longrightarrow K'(X) \longrightarrow 0$$

where j is the isomorphism of $H^0(X, \underset{\sim}{Z})$ onto the subgroup of $K(X)$ generated by the trivial bundles. If one attaches to each vector bundle ξ the element $rk(\xi)$ of $H^0(X, \underset{\sim}{Z})$ which as a continuous map $X \to \underset{\sim}{Z}$ attaches to each $p \in X$ the dimension of the fibre of ξ over p, one gets the "rank-homomorphism" $rk : K(X) \to H^0(X, \underset{\sim}{Z})$ with $rk \circ j = Id$.

The tensor product of vector bundles induces a product in $K(X)$, but not in $K'(X)$, which is also called tensor product. $K(X)$ is a ring with respect to Whitney sum and tensor product.

1.3. - The discussions of 1.2 can also be made for real vector bundles. We have the ring $K_{\underset{\sim}{0}}(X)$ and a canonical ring homomorphism $K_{\underset{\sim}{0}}(X) \to K(X)$, the complex extension. If f is a continuous map $Y \to X$, then we have ring homomorphisms $f^* : K(X) \to K(Y)$ and $f^* : K_{\underset{\sim}{0}}(X) \to K_{\underset{\sim}{0}}(Y)$ which are induced by the lifting of bundles.

1.4. - The (total) Chern class $c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \dots$ is well defined for every $\xi \in K(X)$. This follows from the Whitney multiplication theorem

$$(2) \qquad c(\xi) = c(\xi').c(\xi'')$$

for every exact sequence (E), (see 1.1). The Chern character $ch(\xi) \in H^*(X, \underset{\sim}{Q})$ is defined as follows

$$ch(\xi) = rk(\xi) + \sum (e^{a_j} - 1), \qquad \xi \in K(X) \qquad ,$$

where the Chern class $c_i(\xi) \in H^{2i}(X, \underset{\sim}{Z})$ is regarded formally as the i-th elementary symmetric function in the a_j. If $c_i(\xi) = 0$ for $0 < i < n$, then $ch(\xi) = rk(\xi) + (-1)^{n-1} c_n(\xi)/(n-1)! +$ higher terms.

$ch : K(X) \longrightarrow H^*(X, \underset{\sim}{Q})$ _is a ring homomorphism._

The Pontrjagin classes of an element $\alpha \in K_0(X)$ are well defined [5], paragraphs 4.5. $ch(\alpha)$ is defined as the Chern character of the complex extension of α.

$ch : \underset{\sim}{K_0}(X) \longrightarrow H^*(X, \underset{\sim}{Q})$ _is a ring homomorphism._

1.5. - The theorem of Peterson [8], theorem 3 is a consequence of Bott's theory. It can be formulated as follows.

THEOREM. - _If the torsion coefficients of_ $H^{2q}(X, \underset{\sim}{Z})$, $q = 1, 2, \ldots$, _are_ 0 _or prime to_ $(q-1)!$, _then_ $ch : K(X) \longrightarrow H^*(X, \underset{\sim}{Q})$ _is a monomorphism._

Thus, under the above assumptions, $K(X)$ and the subring $ch(K(X))$ of $H^*(X, \underset{\sim}{Q})$ are isomorphic.

1.6. - For the sphere $\underset{\sim}{S}_{2n}$ we have $K(\underset{\sim}{S}_{2n}) = ch(K(\underset{\sim}{S}_{2n})) = H^*(\underset{\sim}{S}_{2n}, \underset{\sim}{Z})$. This follows from Bott's theorem ([4] and [8] theorem 2) that the Chern class c_n of a complex vector bundle over $\underset{\sim}{S}_{2n}$ is divisible by $(n-1)!$ and that there exists a complex vector bundle over $\underset{\sim}{S}_{2n}$ with c_n equal to $(n-1)!$ times a generator of $H^{2n}(\underset{\sim}{S}_{2n}, \underset{\sim}{Z})$, see also [1], paragraph 26.5. For a sphere $\underset{\sim}{S}_{8k}$ we have

$$\underset{\sim}{K_0}(\underset{\sim}{S}_{8k}) \cong ch(\underset{\sim}{K_0}(\underset{\sim}{S}_{8k})) = H^*(\underset{\sim}{S}_{8k}, \underset{\sim}{Z})$$

whereas for the sphere $\underset{\sim}{S}_m$ with $m = 8k + 4$ we have

$$\underset{\sim}{K_0}(\underset{\sim}{S}_m) \cong ch(\underset{\sim}{K_0}(\underset{\sim}{S}_m)) = H^0(\underset{\sim}{S}_m, \underset{\sim}{Z}) + 2.H^m(\underset{\sim}{S}_m, \underset{\sim}{Z}) \qquad .$$

(See [8], lemma on p. 07).

1.7. - Let $B_{\underset{\sim}{U}}$ denote as usual the classifying space of the infinite unitary group. $B_{\underset{\sim}{U}}$ is the limit of the classifying spaces $B_{\underset{\sim}{U}(n)}$. The "universal" Whitney sum $B_{\underset{\sim}{U}(r)} \times B_{\underset{\sim}{U}(s)} \to B_{\underset{\sim}{U}(r+s)}$ induces an H-space structure on $B_{\underset{\sim}{U}}$. Let $o' \in B_{\underset{\sim}{U}}$ be a base point. <u>In the following all sets of homotopy classes are taken with base points</u>. If X is a space, we let X^+ be the disjoint union of X with a point e which plays in X^+ the role of a base point. The group $\pi(X^+, B_{\underset{\sim}{U}})$ is canonically isomorphic with $K'(X)$, (see 1.2).

Let $\underset{\sim}{K}$ be $B_{\underset{\sim}{U}} \times \underset{\sim}{Z}$ with base point $o' \times 0$. It is an H-space. It follows from 1.2 that $K(X)$ and $\pi(X^+, \underset{\sim}{K})$ are canonically isomorphic. <u>For an arbitrary space</u> X <u>we define</u> $K(X)$ <u>to be</u> $\pi(X^+, \underset{\sim}{K})$. Chern classes and Chern character for an element of $K(X)$ are well-defined. For $\xi \in K(X)$ we have $ch(\xi) \in H^{**}(X, \underset{\sim}{Q})$. Here H^{**} denotes the direct product of the H^j. $K(X)$ can also be made into a ring.

If Y is a subspace of X, we can consider the space X/Y obtained by collapsing Y to a point which is then taken as base point of X/Y. We define

$$K(X, Y) = \pi(X/Y, \underset{\sim}{K}) .$$

For $\xi \in K(X, Y)$ the Chern character $ch(\xi)$ is an element of $H^{**}(X, Y; \underset{\sim}{Q})$.

If the pair (X, Y) satisfies the homotopy extension condition, then we have the exact sequence

$$K(X, Y) \to K(X) \to K(Y) . \tag{3}$$

The suspension SA of a space A with base point $\underline{a}$ is the double cone over A with the generator through $\underline{a}$ shrunk to a point which is then taken in SA as base point. The m-fold suspension $S^m(X^+)$ is the cartesian product $X \times \underset{\sim}{S}_m$ with $X \times \{b\}$ shrunk to a point for some $b \in \underset{\sim}{S}_m$, the point corresponding to $X \times \{b\}$ being the base point. We have the split exact sequence

$$0 \to K(X \times \underset{\sim}{S}_m, X \times \{b\}) \xrightarrow{i} K(X \times \underset{\sim}{S}_m) \xrightarrow{j} K(X) \to 0 \tag{4}$$

(For (3), (4) compare D. PUPPE [10]. This author shows how (3) can be imbedded in an exact sequence unlimited to the left).

Corresponding definitions are possible for real vector bundles. We define the H-space $\mathbf{K}_0 = \mathbf{B}_0 \times \mathbf{Z}$, etc. We have the exact sequences

$$\mathbf{K}_0(X\ ,\ Y) \to \mathbf{K}_0(X) \to \mathbf{K}_0(Y) \tag{5}$$

$$0 \to \mathbf{K}_0(X \times \mathbf{S}_m\ ,\ X \times \{b\}) \to \mathbf{K}_0(X \times \mathbf{S}_m) \to \mathbf{K}_0(X) \to 0 \quad . \tag{6}$$

1.8. - Let π_1 ; π_2 be the projections of $X \times \mathbf{S}_m$ on its two factors. $\pi_1^* K(X)$ is a direct summand of $K(X \times \mathbf{S}_m)$. Let m be even, $m = 2n > 0$. Let $\eta \in K(\mathbf{S}_{2n})$ be the element whose Chern character is the canonical generator g of $H^{2n}(\mathbf{S}_{2n}\ ,\ \mathbf{Z})$, (see 1.6). It is clear that $\pi_1^* K(X) \otimes \pi_2^* \eta$ is in the *kernel* of the homomorphism j of the exact sequence (4).

THEOREM. - $\alpha \to \pi_1^* \alpha \otimes \pi_2^* \eta$ *is an additive isomorphism of* $K(X)$ *and the kernel of* j .

COROLLARY. - $K(X \times \mathbf{S}_{2n}) = K(X) \otimes K(\mathbf{S}_{2n})$. *Each element* ζ *of* $K(X \times \mathbf{S}_{2n})$ *can be written uniquely in the form*

$$\zeta = \pi_1^* \alpha + \pi_1^* \beta \otimes \pi_2^* \eta \ ; \qquad \alpha ,\ \beta \in K(X) \quad .$$

Each element $x \in ch(K(X \times \mathbf{S}_{2n}))$ *can be written in the form*

$$x = \pi_1^* a + (\pi_1^* b)(\pi_2^* g) \ ; \qquad a\ ,\ b \in ch(K(X)) \quad . \tag{7}$$

We give in the following section an analogous theorem for real vector bundles. The proofs of the two theorems 1.8 , 1.9 , will be sketched in paragraph 2.

1.9. - We consider (6) with $m = 8k$. Let g be the canonical generator of $H^{8k}(\mathbf{S}_{8k}\ ,\ \mathbf{Z})$.

PROPOSITION. - *There exists an additive isomorphism of* $\mathbf{K}_0(X)$ *and the kernel of* $\mathbf{K}_0(X \times \mathbf{S}_{8k}) \to \mathbf{K}_0(X)$ if X *is a product of spheres or the suspension of a space.*

THEOREM. - Each element $x \in ch(K_0(X \times S_{8k}))$, X arbitrary, can be written

$$x = \pi_1^* a + (\pi_1^* b)(\pi_2^* g) \quad ; \quad a , b \in ch(K_0(X)) \quad .$$

2. The Bott isomorphisms.

2.1. - Bott defines in [4] a map f_m of $G_{mm} = U(2m)/U(m) \times U(m)$ into $\Omega\, SU(2m)$ as follows. Let $s_m(t)$ be the diagonal matrix with first m entries $\exp(2\pi it)$ and last m entries $\exp(-2\pi it)$. For $u \in U(2m)$ let $g_m(u)$ be the loop in $SU(2m)$ with

$$g_m(u)(t) = u.s_m(t).u^{-1}.s_m(t)^{-1} \quad , \quad 0 \leq t \leq 1 \quad ;$$

f_m is induced by g_m. In the limit f_m gives rise to a map $B_U \rightarrow \Omega SU$ and since $\Omega U = \Omega S_1 \times \Omega SU$ we get a map

$$F : K = B_U \times Z \rightarrow \Omega U \quad .$$

One can check that F preserves the H-space structures of K (Whitney sum) and of ΩU (loops). F is, according to one of Bott's main results, a (weak) homotopy equivalence. Since we have a canonical (weak) homotopy equivalence $U \rightarrow \Omega B_U$, we get a (weak) homotopy equivalence

$$K \rightarrow \Omega^2 B_U = \Omega^2 K \quad . \tag{1}$$

This is the Bott periodicity (period 2) for the unitary group.

For any space X we have by (1) a homomorphism $\pi(X^+, K) \rightarrow \pi(X^+, \Omega^2 K)$ and the isomorphism $\pi(X^+, \Omega^2 K) \rightarrow \pi(S^2(X^+), K)$. For any finite dimensional CW-complex X the composition

$$\pi(X^+, K) = K(X) \rightarrow \pi(S^2(X^+), K) = K(X \times S_2 , X \times \{b\}) \tag{2}$$

is an isomorphism. We wish to show that this Bott isomorphism is of the form given in the theorem of 1.8.

From (1) we get a map $S^2(\mathbf{K}^+) \to \mathbf{K}$ and a homomorphism

(3) $H^{**}(\mathbf{K}, \mathbf{Q}) \to {}'H^{**}(S^2(\mathbf{K}^+), \mathbf{Q})$ ($'H$: cohomology classes with augmentation 0).

There is a canonical isomorphism of $'H^{**}(S^2(\mathbf{K}^+), \mathbf{Q})$ and $H^{**}(K, \mathbf{Q})$ lowering degrees by 2. (3) followed by this isomorphism gives a homomorphism

$$(4) \qquad H^{**}(\mathbf{K}, \mathbf{Q}) \to H^{**}(\mathbf{K}, \mathbf{Q}) .$$

It was shown in BOTT [4] - modulo certain precautions concerning the 0-dim classes which we have to take in account here, that (4) is a derivation of polynomials which leaves the universal Chern character unchanged. This proves theorem 1.8 at least if one goes over to Chern characters. But since the complex Grassmannians are universal base spaces and are free of torsion, 1.8 follows now from 1.5 (We have proved 1.8 only for $n = 1$. It follows for arbitrary n by a simple induction argument).

2.2. - The Bott periodicity (period 8) of the orthogonal group means that there exists a (weak) homotopy equivalence

$$(5) \qquad \mathbf{K}_0 = \mathbf{B}_0 \times \mathbf{Z} \to \Omega^8 \mathbf{B}_0 = \Omega^8 \mathbf{K}_0 .$$

This proves the proposition 1.9. Quite likely (5) respects the H-space structures (Whitney sum-loops). If so, the assumption on X in Proposition 1.9 could be avoided. Bott does not prove the periodicity (5) with the methods of [4] . Instead he uses for (5) his μ-sequences of symmetric spaces ([8] , and [3] detailed version to appear later). Bott constructs a μ-sequence M_i with $M_i = M_{i+1}^{\mu}$ $(i = 0 , 1 , \ldots)$ for which $M_0 = \mathbf{O}(2n)/\mathbf{O}(n) \times \mathbf{O}(n)$ and $M_8 = \mathbf{O}(32n)/\mathbf{O}(16n) \times \mathbf{O}(16n)$.
By a detailed investigation of this μ-sequence it will probably be possible to describe explicitly the isomorphism of proposition 1.9 and to avoid the assumption on X in this proposition. This was not done so far. It is however possible to give for the theorem 1.9 a more indirect proof starting from the diagram of Bott homomorphisms

$$\begin{array}{ccc} \pi_{4n}(\mathbf{B}_0) & \longrightarrow & \pi_{4n+8}(\mathbf{B}_0) \\ \downarrow & & \downarrow \\ \pi_{4n}(\mathbf{B}_U) & \longrightarrow & \pi_{4n+8}(\mathbf{B}_U) \end{array} .$$

Since the groups involved are all $\mathbb{Z}$, and the horizontal arrows are isomorphisms, while the vertical ones are either isomorphisms (n even) or multiplication by ± 2 (n odd) it follows that this diagram is commutative up to sign. Theorem 1.9 can be deduced from 1.8 and proposition 1.9. One uses that the rational Pontrjagin ring of $\mathbb{B}_0$ (the H-space structure being given by the Whitney sum) is generated by the spherical cycles and that theorem 1.9 is true if X is a product of spheres. For the last fact the diagram has to be applied (This mothod gave theorem 1.9 for $k = 1$. It follows for arbitrary k by a simple induction argument).

3. The Riemann-Roch theorems.

3.1. - Let $\{\hat{A}_j(p_1 , \ldots , p_j)\}$ be the multiplicative sequence of polynomials ([5] , paragraph 1) belonging to the power series

$$Q(z) = \frac{\frac{1}{2}\sqrt{z}}{\sinh \frac{1}{2}\sqrt{z}} \quad .$$

The the (total) $\hat{\mathfrak{A}}$-class ([1] , paragraph 23.1) of a real vector bundle ξ (with base B_ξ) is

$$(1) \qquad \hat{\mathfrak{A}}(\xi) = \sum_{j=0}^{\infty} \hat{A}_j(p_1(\xi) , \ldots , p_j(\xi)) \in H^{**}(B_\xi , \mathbb{Q})$$

where $p_i(\xi) \in H^{4i}(B_\xi , \mathbb{Z})$ is the i-th Pontrjagin class. $\hat{\mathfrak{A}}(X)$, for a differentiable manifold X , is the $\hat{\mathfrak{A}}$-class of its tangential bundle.

3.2. - Let Y be a differentiable manifold whose integral Stiefel-Whitney class W_3 vanishes. Then there exists an element $d \in H^2(Y , \mathbb{Z})$ whose restriction mod 2 is the Stiefel-Whitney class w_2 of Y . We have the subring ch(K(Y)) of $H^*(Y , \mathbb{Q})$. Multiplication with the element $e^{d/2}.\hat{\mathfrak{A}}(Y)$, which is inversible with respect to the cup-product, induces an additive isomorphism of K(Y) onto a subgroup R(Y) of $H^*(Y , \mathbb{Q})$.

$$(2) \qquad R(Y) = ch(K(Y)).e^{d/2}.\hat{\mathfrak{A}}(Y) \quad .$$

The group R(Y) does not depend on the choice of d . Namely, if d and $d' \in H^2(Y , \mathbb{Z})$ have w_2 as restriction mod 2 , then $d - d' = 2y$ with $y \in H^2(Y , \mathbb{Z})$. The assertion follows since $e^y.ch(K(Y)) = ch(K(Y))$. The additive group $R(Y) \subset H^*(Y , \mathbb{Q})$ is called the Riemann-Roch group of Y . It is defined for arbitrary differentiable manifolds with $W_3 = 0$. It is isomorphic to ch(K(Y)).

3.3. - The preceding definition is motivated by the RR-theorems of [5] and [2]. If Y is compact and carries the structure of a projective algebraic manifold, then we can choose for d in (2) the Chern class c_1 of Y. If η is a holomorphic complex vector bundle over Y, then the element $ch(\eta).e^{c_1/2}.\hat{\mathfrak{A}}(Y)$ of $H^*(Y, \mathbb{Q})$ plays the following role in the RR-theorem [5]. Its value on the oriented fundamental cycle of Y is the Euler number of Y with respect to the cohomology with coefficients in the sheaf of holomorphic sections of η. (When taking the value on the fundamental cycle all components of dimension less than the topological dimension of Y give 0).

GROTHENDIECK [2] has proved a more general theorem which involves the Gysin homomorphism f_*. If Y and X are compact oriented manifolds and if $f : Y \to X$ is a continuous map, then f_* is the linear map

$$f_* : H^*(Y, \mathbb{Q}) \longrightarrow H^*(X, \mathbb{Q})$$

obtained by the Poincaré isomorphisms $H^*(Y, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$ and $H^*(X, \mathbb{Q}) \cong H_*(X, \mathbb{Q})$ from the homology homomorphism $H_*(Y, \mathbb{Q}) \to H_*(X, \mathbb{Q})$ induced by f. GROTHENDIECK shows for projective algebraic manifolds Y, X, a holomorphic map $f : Y \to X$ and a holomorphic complex vector bundle η over Y that

$$f_*(ch(\eta).e^{c_1(Y)/2}.\hat{\mathfrak{A}}(Y)) \in R(X) .$$

<u>More precisely, he finds an explicit element</u> $f_!(\eta) \in K(X)$, <u>given by the alternating sum of the direct images of the sheaf of holomorphic sections of</u> η, <u>such that</u>

$$f_*(ch(\eta).e^{c_1(Y)/2}.\hat{\mathfrak{A}}(Y)) = ch(f_!(\eta)).e^{c_1(X)/2}.\hat{\mathfrak{A}}(X) .$$

The RR-theorem of [5] is Grothendieck's theorem for the map of Y onto a point. The RR-theorem of [5] gave rise to the question whether for every compact oriented differentiable manifold Y (with $W_3(Y) = 0$) the value of any element of $R(Y)$ on the fundamental cycle of Y is an integer. This question was answered in affirmative (except the prime 2) in [1], theorem 25.5. The prime 2 was also settled (see [1], Part III and [6]), using new results of MILNOR (compare THOM'S talk in this seminar). These integrality theorems can be generalized in a way which parallelizes the Grothendieck generalization of the RR-theorem of [5]. Needless to say that these generalizations, which we call audaciously

RR-theorems for differentiable manifolds, do not contain the Grothendieck theorem as a special case.

The old proofs of the integrality theorems and the new proofs of their Grothendieck type generalizations are completely different. Bott's theory was not used in the old proofs.

In the remaining sections of this paragraph we shall formulate the differentiable RR-theorems. In paragraph 4 we will give applications and in paragraph 5 we shall sketch the proof.

3.4. - THEOREM. - Let Y and X be compact oriented differentiable manifolds whose integral 3-dimensional Stiefel-Whitney classes vanish. Assume that $\dim Y \equiv \dim X \pmod 2$. Let $f : Y \to X$ be a continuous map. Then

$$f_*(R(Y)) \subset R(X) \qquad (3)$$

Fixing elements $b \in H^2(Y, \mathbb{Z})$ and $a \in H^2(X, \mathbb{Z})$ whose restrictions mod 2 are $w_2(Y)$ and $w_2(X)$ respectively, (3) means that given $\eta \in K(Y)$, there exists $\xi \in K(X)$ such that

$$f_*(\mathrm{ch}(\eta).e^{b/2}.\hat{\mathfrak{A}}(Y)) = \mathrm{ch}(\xi).e^{a/2}.\hat{\mathfrak{A}}(X) \qquad (4)$$

This equation is equivalent with

$$f_*(\mathrm{ch}(\eta).e^{(b-f^*a)/2}.\hat{\mathfrak{A}}(Y)) = \mathrm{ch}(\xi).\hat{\mathfrak{A}}(X) \qquad (5)$$

which motivates the following generalization of the preceding theorem.

3.5. - THEOREM. - Let Y and X be compact oriented differentiable manifolds with $\dim Y \equiv \dim X \pmod 2$. Let $f : Y \to X$ be a continuous map satisfying

$$f^*(W_3(X)) = W_3(Y) \qquad (6)$$

Let $d \in H^2(Y, \mathbb{Z})$ have $w_2(Y) - f^* w_2(X)$ as restriction mod 2. Such a d exists by (6). Then for given $\eta \in K(Y)$ there exists $\xi \in K(X)$ such that

$$f_*(\mathrm{ch}(\eta).e^{d/2}.\hat{\mathfrak{A}}(Y)) = \mathrm{ch}(\xi).\hat{\mathfrak{A}}(X) \qquad (7)$$

Let us call f above a c_1-map if a definite element $c_1(f) \in H^2(Y, \mathbb{Z})$ whose restriction mod 2 is $w_2(Y) - f^* w_2(X)$ has been chosen. Let us further define

$$\hat{\mathfrak{A}}(f) = \hat{\mathfrak{A}}(Y)\, f^*(\hat{\mathfrak{A}}(X))^{-1} \tag{8}$$

and

$$f_!(y) = f_*(y . e^{c_1(f)/2} . \hat{\mathfrak{A}}(f)) \quad \text{for} \quad y \in H^*(Y, \mathbb{Q}) \quad . \tag{9}$$

We have thus defined a linear map

$$f_! : H^*(Y, \mathbb{Q}) \longrightarrow H^*(X, \mathbb{Q}) \quad .$$

Theorem 3.5 is then equivalent with

$$f_!(\mathrm{ch}(K(Y))) \subset \mathrm{ch}(K(X)) \quad , \tag{10}$$

and (7) takes the form

$$f_*(\mathrm{ch}(\eta) . e^{c_1(f)/2} . \hat{\mathfrak{A}}(Y)) = f_! \mathrm{ch}(\eta) . \hat{\mathfrak{A}}(X) \quad . \tag{11}$$

Notice that $f_!$ is only defined for a c_1-map f. The composition $f \circ g$ of two c_1-maps $g : Z \longrightarrow Y$ and $f : Y \longrightarrow X$ (where Z, Y, X are compact oriented differentiable) is in a natural way a c_1-map. One chooses

$$c_1(f \circ g) = c_1(g) + g^*(c_1(f)) \quad .$$

Since

$$\hat{\mathfrak{A}}(f \circ g) = \hat{\mathfrak{A}}(g) . g^*(\hat{\mathfrak{A}}(f)) \quad ,$$

we obtain the functorial property

$$(f \circ g)_! = f_! \circ g_! \quad . \tag{12}$$

Thus, if (10) is proved for the c_1-maps g and f, it is proved also for their composition. This fact is, of course, useful for the proof of theorem 3.5, or equivalently of (10). Since every continuous map is homotopic with a differentiable one, we need to give the proof only for a differentiable f.

REMARK. - It arises the question whether one can define for a c_1-map $f : Y \to X$, $(\dim Y \equiv \dim X \pmod 2)$, a homomorphism $f_? : K(Y) \to K(X)$ (such that ? is functorial like ! in (12)) which satisfies the "Riemann-Roch-Grothendieck equation"

$$f_*(\mathrm{ch}(\eta).e^{c_1(f)/2}.\hat{\mathfrak{A}}(Y)) = \mathrm{ch}(f_? \eta).\hat{\mathfrak{A}}(X) \quad .$$

Since $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Q})$ is a monomorphism if X has no torsion (1.5), $f_?$ is canonically defined if X has no torsion. ATIYAH has developped a RR-theory for almost-complex manifolds and almost-complex maps with a functorial $f_?$. He uses essentially the fact that the classifying space $B_{U(n)}$ has no torsion, whereas in the proof of theorem 3.5 the classifying space $B_{\mathrm{Spin}(n)}$ will occur which has torsion.

3.6. - For an $SO(n)$-bundle η the Stiefel-Whitney class $w_2(\eta)$ vanishes if and only if the structural group can be reduced to $\mathrm{Spin}(n)$ with respect to the covering map $\mathrm{Spin}(n) \to SO(n)$. This fact motivates the following terminology. $f : Y \to X$ is called a Spin-map if $w_2(Y) = f^* w_2(X)$. The Spin-maps may be identified with the c_1-maps f for which $c_1(f) = 0$. For a Spin-map we have theorem 3.5 with $d = 0$. We have moreover the following result for orthogonal bundles.

3.7. - THEOREM. - Let Y and X be compact oriented differentiable manifolds with $\dim Y = \dim X \pmod 8$. Let $f : Y \to X$ be a Spin-map. Then for given $\eta \in K_0(Y)$ there exists $\xi \in K_0(X)$ such that

$$f_*(\mathrm{ch}(\eta).\hat{\mathfrak{A}}(Y)) = \mathrm{ch}(\xi).\hat{\mathfrak{A}}(X) \quad . \tag{13}$$

4. Applications.

4.1. - If one applies 3.4 or 3.5 to the case where Y is mapped onto X = point, one gets the integrality theorems mentioned in 3.3. For example, we get the integrality of the Todd genus of an almost-complex manifold and also the integrality of the $\hat{A}$-genus [1], paragraph 23.1 of a compact oriented differentiable manifold whose second Stiefel-Whitney class vanishes. This last fact means for a 4-dimensional manifold X with $w_2(X) = 0$ that $p_1(X) = 0 \pmod{24}$, whereas ROHLIN has shown in this case that $p_1(X) = 0 \pmod{48}$. This factor 2 could not be obtained by the old methods. But Rohlin's theorem will follow from 3.7 (as special case of 4.2).

Let Y be a manifold with $\dim Y \equiv 4 \pmod 8$. It is assumed to be compact oriented differentiable. Assume moreover that $w_2(Y) = 0$. Let f be the map $Y \to \underset{\sim}{S}_4$ which sends Y onto a single point of $\underset{\sim}{S}_4$. Then f is a Spin-map satisfying the assumptions of 3.7. Let $\eta \in \underset{\sim}{K}_0(Y)$. It follows that the value of $ch(\eta).\hat{\mathfrak{A}}(Y)$ on the fundamental cycle of Y is an even integer. In fact, this value which was denoted in [1, paragraph 25.5] by $\hat{A}(Y, 0, \eta)$ equals the value of $ch(\xi)$ on the fundamental cycle of $\underset{\sim}{S}_4$ since $\hat{\mathfrak{A}}(\underset{\sim}{S}_4) = 1$. Here ξ has the meaning of 3.7. Since $p_1(\xi) \equiv w_2(\xi)^2 = 0 \mod 2$ and since $ch(\xi)$ equals its 0-dim term plus $p_1(\xi)$, we have that $ch(\xi)[\underset{\sim}{S}_4]$ is even. We have proved the following theorem which was conjectured in [1], paragraph 25.6.

4.2. – THEOREM. – Let Y be a compact oriented differentiable manifold with $\dim Y \equiv 4 \pmod 8$ and $w_2(Y) = 0$. Let η be an element of $\underset{\sim}{K}_0(Y)$. Then $\hat{A}(Y, 0, \eta)$ is an even integer ; in particular, the $\hat{A}$-genus of Y is an even integer which for $\dim Y = 4$ is Rohlin's theorem : $p_1(Y) \equiv 0 \pmod{48}$.

The integrality of the $\hat{A}$-genus (if $w_2(Y) = 0$) together with the sharpened result of 4.2 on this genus implies by MILNOR-KERVAIRE [8], [9] that, in the stable range, the image of $J : \pi_{4k-1}(\underset{\sim}{SO}(n)) \to \pi_{n+4k-1}(\underset{\sim}{S}_n)$ is cyclic of an order divisible by the denominator of the rational number $B_k/4k$. Here B_k is the k-th Bernoulli number ($B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, ...). We shall give later a different proof of this result using homotopy invariance properties of Pontrjagin classes.

4.3. – REMARK. – It is not astonishing that 4.2 contains Bott's theorem (1.6) that the Pontrjagin class p_{2r+1} of an orthogonal bundle over $\underset{\sim}{S}_{8r+4}$ is divisible by $(4r+1)!.2$. If one maps $\underset{\sim}{S}_{8k}$ onto a point, 3.7 gives that the Pontrjagin class p_{2k} of an orthogonal bundle over $\underset{\sim}{S}_{8k}$ is divisible by $(4k-1)!$.

4.4. *Homotopy invariance properties of Pontrjagin classes.* – As always we consider compact oriented differentiable manifolds. Let $f : Y \to X$ be a homotopy equivalence and $g : X \to Y$ its homotopy inverse. We can use 3.7 for the map f and put $ch(\eta) = 1$. This yields

$$(1) \qquad g^*(\hat{\mathfrak{A}}(Y))/\hat{\mathfrak{A}}(X) \in ch(K_0(X)) \subset ch(K(X)) \ .$$

Using 3.7 for the map g yields

$$(2) \qquad \hat{\mathfrak{A}}(X)/g^*(\hat{\mathfrak{A}}(Y)) \in ch(K_0(X)) \subset ch(K(X)) \ .$$

Equations (1), (2) imply *for manifolds with vanishing third Stiefel-Whitney class that the Riemann-Roch subgroup of the rational cohomology group is a homotopy type invariant*, (see 3.2 , 3.4).

To be more precise, let us denote by $p_i^!$ the image of p_i in the quotient of $H^{4i}(X, \mathbb{Z})$ modulo its torsion subgroup. Actually in all RR-theorems only the $p_i^!$ are relevant. (1) , (2) contain information on the behaviour of the Pontrjagin classes $p_i^!$ under homotopy equivalences. We give only an example :

4.5. – THEOREM. – *The first Pontrjagin class* $p_1^!$ *of a compact oriented differentiable manifold* X *is a homotopy type invariant* mod 24. *If* $H^2(X, \mathbb{Z}_2) = 0$, *then it is a homotopy type invariant* mod 48.

Proof : Under a homotopy equivalence g of X , we have according to (1) using the notations of 4.4

$$g^*(1 - p_1(Y)/24)/(1 - p_1(X)/24) = 1 + p_1(\xi) \quad \text{with} \quad \xi \in K_0(X) \ .$$

Thus

$$(p_1(X) - g^* p_1(Y))/24 = p_1(\xi)$$

which proves the first assertion. If $H^2(X, \mathbb{Z}_2) = 0$, then $p_1(\xi) \equiv w_2(\xi)^2 = 0$ mod 2 . This proves the second assertion.

REMARK. – 4.4 and 4.5 are also true for non-orientable manifolds because the RR-theorems 3.5 , 3.7 are true (also if the manifolds are non-orientable) if the map $f : Y \to X$ is orientable $(f^* W_1(X) = W_1(Y))$. But we do not consider these generalizations of RR .

4.6. – We consider the stable groups $\pi_{4k-1}(\underset{\sim}{SO}(q))$ and $\pi_{4k-1+q}(\underset{\sim}{S}_q)$, $(q \geq 4k+1)$. Every element α of the first group may be identified with a principal $\underset{\sim}{SO}(q)$-bundle over $\underset{\sim}{S}_{4k}$ to which a sphere bundle (fibre $\underset{\sim}{S}_{q-1}$) is associated. We denote the total space of this sphere bundle by $B(\alpha)$. Let $\pi : B(\alpha) \to \underset{\sim}{S}_{4k}$ be the projection. π^* is a monomorphism. It is easy to show that the total Pontrjagin class of the manifold $B(\alpha)$ is given by

$$p(B(\alpha)) = 1 + \pi^* p_k(\alpha) . \tag{3}$$

Next we consider the homomorphism $J : \pi_{4k-1}(\underset{\sim}{SO}(q)) \to \pi_{4k-1+q}(\underset{\sim}{S}_q)$. <u>Let α be an element of the first group and assume</u> $J\alpha = 0$. Then $B(\alpha)$ and the product $\underset{\sim}{S}_{4k} \times \underset{\sim}{S}_{q-1} = B(0)$ are of the same homotopy type. ([7], who proves also that $J\alpha = 0$ implies the homotopy equivalence of $B(\alpha)$ and $B(0)$). We shall now apply (1) : Let g be the homotopy equivalence $B(0) \to B(\alpha)$. We observe that $\hat{\mathfrak{A}}(B(0))$ equals 1 and that

$$\hat{\mathfrak{A}}(B(\alpha)) = 1 - \frac{B_k}{(2k)!.2} . \pi^* p_k(\alpha) . \tag{4}$$

(Here we use the explicit formula for the coefficient of p_k in the polynomial $\hat{A}_k$; compare [5], paragraph 1). By (1) there exists an element $\eta \in \underset{\sim}{K}_0(B(0))$ such that

$$g^* \hat{\mathfrak{A}}(B(\alpha)) = ch(\eta) . \tag{5}$$

We multiply this equation with $(2k-1)!$ and get by (4)

$$-(B_k/4k).g^* \pi^*(p_k(\alpha)) = \pm p_k(\eta) . \tag{6}$$

We restrict η to $(\underset{\sim}{S}_{4k} \times \text{point}) \subset \underset{\sim}{S}_{4k} \times \underset{\sim}{S}_{q-1} = B(0)$. This gives a stable element say $\beta \in \pi_{4k-1}(\underset{\sim}{SO}(q))$. It follows easily that the restriction of $g^* \pi^*(\alpha)$ to $\underset{\sim}{S}_{4k} \times$ point is $\pm \alpha$. If $B_k/4k$ is expressed in lowest terms as a quotient of integers, $(B_k/4k = N_k/M_k)$, then (6) yields

$$p_k(N_k \alpha \pm M_k \beta) = 0 . \tag{7}$$

$p_k : \pi_{4k-1}(\underset{\sim}{SO}(q)) \to H^{4k}(\underset{\sim}{S}_{4k}, \underset{\sim}{Z})$ is a homomorphism of two infinite cyclic group which has kernel 0, (in fact it is known (1.6) to be multiplication by

$(2k-1)!$ or $(2k-1)!.2$ resp., depending whether k is even or odd). Thus (7) implies that α is divisible by M_k which is exactly the result of MILNOR-KERVAIRE [8], [9] that the image of J in $\pi_{4k-1+q}(S_q)$ has an order divisible by the denominator of $B_k/4k$. (This is actually a little improvement of [9]; compare 4.2).

REMARK. - The second statement of theorem 4.5 is best possible : Take in the above discussions $k = 1$ and α to be 24 times a generator of $\pi_3(SO(q))$. Then $B(0)$ and $B(\alpha)$ are homotopy equivalent, but the first Pontrjagin class is 0 resp. 48 times a generator of $H^4(B(0), Z)$ resp. $H^4(B(\alpha), Z)$. This type of examples is due to THOM.

5. The proof of the RR-theorems.

5.1. - We are going to prove the RR-theorem 3.5 for a differentiable c_1-map $f : Y \to X$, i.e. we have to prove (10) of 3.5. Imbed Y in $X \times Y$ by its graph :

$$\gamma_f : Y \to X \times Y, \quad \gamma_f(y) = (f(y), y) \quad .$$

Let i be an imbedding of Y into a sphere of even dimension say $2n$. Then $f \times i$ is an imbedding j of Y in $X \times S_{2n}$

$$j = f \times i : Y \longrightarrow X \times S_{2n} \quad .$$

We can make j to a c_1-map by letting $c_1(j)$ equal $c_1(f)$. If π_1 is the projection of $X \times S_{2n}$ on X, we have $f = \pi_1 \circ j$. The projection π_1 becomes a c_1-map by putting $c_1(\pi_1) = 0$. Then the c_1-map f is the composition of the c_1-maps j and π_1. Thus we have to prove (10) of 3.5 only for the case of an imbedding and for the projection π_1.

5.2. - Riemann-Roch for $\pi_1 : X \times S_{2n} \to X$ with $c_1(\pi_1) = 0$. We wish to prove (10) of 3.5 for this projection. Since $\hat{\mathfrak{A}}(\pi_1) = 1$, we have only to prove that $(\pi_1)_*$ maps $ch(K(X \times S_{2n}))$ in $ch(K(X))$. But this is an immediate consequence of (7) in 1.8.

We remark here already that theorem 3.7 follows in the case of the projection $X \times S_{8k} \to X$ from the theorem in 1.9.

5.3. - We still have to prove (10) of 3.5 for the case where Y is a differentaible submanifold of X and $f : Y \to X$ is the injection. Let ν be the normal bundle of Y in X. Then $\hat{\mathfrak{A}}(f) = \hat{\mathfrak{A}}(\nu)^{-1}$, and $c_1(f)$ is an element of $H^2(Y, \mathbb{Z})$ whose restriction mod 2 is $w_2(\nu)$. Let A be a closed tubular neighborhood of Y whose boundary E is a sphere bundle over Y associated to ν. Let

$$\varphi_* : H^*(Y, \mathbb{Q}) \to H^*(A, E; \mathbb{Q})$$

be the Gysin-Thom isomorphism ([11]). Assume that we have found $\beta \in K(A, E)$ such that

$$(1) \qquad \varphi_*(e^{c_1(f)/2} \cdot \hat{\mathfrak{A}}(\nu)^{-1}) = ch(\beta) \quad .$$

Then (10) of 3.5 is proved for f. Namely let $\eta \in K(Y)$. Since $K(Y)$ and $K(A)$ are canonically isomorphic, we have an element $\eta' \in K(A)$. Via tensor product, $K(A, E)$ is a $K(A)$-module and $\eta' \otimes \beta$ an element of $K(A, E)$ with Chern character $ch(\eta').ch(\beta) \in H^*(A, E; \mathbb{Q})$. Standard rules ([11], théorème I.4) and (1) yield

$$(2) \qquad \varphi_*(ch(\eta).e^{c_1(f)/2} \cdot \hat{\mathfrak{A}}(\nu)^{-1}) = ch(\eta' \otimes \beta) \in ch(K(A, E)) \quad .$$

Since $K(A, E) = K(X, X - (A - E))$, we get by the canonical homomorphism

$$K(X, X - (A - E)) \to K(X)$$

from $\eta' \otimes \beta$ an element γ of $K(X)$ and (2) implies

$$(3) \qquad f_*(ch(\eta).e^{c_1(f)/2} \cdot \hat{\mathfrak{A}}(\nu)^{-1}) = ch(\gamma) \in K(X)$$

which was to be proved. It remains to find an element β satisfying (1). This is done by a universal contruction in the next section.

5.4. - A general reference for the terminology of this section is [1]. Let π_q be the covering map $\mathrm{Spin}(q + 2) \to SO(q + 2)$. Put $G_q = \pi_q^{-1}(SO(2) \times SO(q))$. It is connected and a twofold covering of $SO(2) \times SO(q)$. We have

(4) $$G_{2q}/G_{2q-1} = \underset{\sim}{S}_{2q-1} \quad .$$

Let B_q be the classifying space for G_q. Then we have a fibration

(5) $$\rho : \; B_{2q-1} \to B_{2q} \, , \qquad \text{fibre } \underset{\sim}{S}_{2q-1}$$

which is associated to the universal principal G_{2q}-bundle ξ over B_{2q}. Let T be the standard maximal torus of $\underset{\sim}{SO}(2) \times \underset{\sim}{SO}(2q)$ and $y, x_1, \ldots, x_q$ the standard base of $H^1(T, \underset{\sim}{Z})$. Put $T' = \pi_{2q}^{-1}(T)$. It is a maximal torus of G_{2q} and covers T twofold. $(\pi_{2q}|T')^* : H^1(T, \underset{\sim}{Z}) \to H^1(T', \underset{\sim}{Z})$ is a monomorphism. We regard $H^1(T, \underset{\sim}{Z})$ as subgroup of $H^1(T', \underset{\sim}{Z})$. We have

$$(\pm y \pm x_1 \pm x_2 \pm \ldots \pm x_q)/2 \in H^1(T', \underset{\sim}{Z}) \, .$$

Let Δ^+_{q+1}, resp. Δ^-_{q+1}, be the right, resp. left, spinor representation of $\underset{\sim}{\mathrm{Spin}}(2q+2)$. If one restricts these representations to G_{2q} they split into irreducible components. Let $\tilde{\Delta}^+_q$ be the component of the restricted Δ^+_{q+1} which has highest weight $(y + x_1 + \ldots + x_q)/2$. Correspondingly let $\tilde{\Delta}^-_q$ be the component of the restricted Δ^-_{q+1} which has highest weight $(y + x_1 + \ldots + x_{q-1} - x_q)/2$. The characters of $\tilde{\Delta}^+_q$ and $\tilde{\Delta}^-_q$ (in the sense of [1], paragraph 10.2) can easily be computed. We obtain

(6)
$$\mathrm{ch}(\tilde{\Delta}^+_q) - \mathrm{ch}(\tilde{\Delta}^-_q) = e^{y/2} \cdot \prod_{i=1}^{q} (e^{x_i/2} - e^{-x_i/2}) \, ,$$

$$= x_1 \, x_2 \ldots x_q \cdot e^{y/2} \cdot \prod_{i=1}^{q} \frac{\sinh x_i/2}{x_i/2} \quad .$$

The universal G_{2q}-bundle ξ can be extended by $\tilde{\Delta}^+_q$ and $\tilde{\Delta}^-_q$. We get elements

$$\Delta^+_q \xi \, , \; \Delta^-_q \xi \in K(B_{2q}) \quad .$$

Let A_{2q} be the mapping cylinder of ρ (see (5)). Then we have the commutative diagram of exact sequences

$$\begin{array}{ccccccc} K(A_{2q}, B_{2q-1}) & \xrightarrow{j} & K(B_{2q}) & \xrightarrow{\rho^*} & K(B_{2q-1}) & , & (K(B_{2q}) = K(A_{2q})) \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \\ 0 \longrightarrow H^{**}(A_{2q}, B_{2q-1}) & \longrightarrow & H^{**}(B_{2q}) & \longrightarrow & H^{**}(B_{2q-1}) \longrightarrow 0 & & \end{array}$$

(rational cohomology) .

$\widetilde{\Delta}_q^+ \xi - \widetilde{\Delta}_q^- \xi$ is in the kernel of ρ^* since by (6) the characters of the two representations are equal when restricted to G_{2q-1}. Thus there exists a $\beta_q \in K(A_{2q}, B_{2q-1})$ with $j(\beta_q) = \widetilde{\Delta}_q^+ \xi - \widetilde{\Delta}_q^- \xi$. This β_q is not uniquely determined, however, by the diagram, $ch(\beta_q)$ is uniquely determined. Under the negative transgression $H^1(T', \underset{\sim}{Z})$ is mapped into $H^2(B_{T'}, \underset{\sim}{Q})$; corresponding elements are denoted by the same symbol. As usual $H^{**}(B_{2q}, \underset{\sim}{Q})$ is regarded as subring of $H^{**}(B_{T'}, \underset{\sim}{Q})$. Over B_{2q} we have canonically a principal $\underset{\sim}{SO}(2q)$-bundle ν_q and a principal $\underset{\sim}{SO}(2)$-bundle δ_1 which is an $\underset{\sim}{U}(1)$-bundle. Let c_1 be its first Chern class. Let W_{2q} be the Euler class of ν_q. Then (6) goes over in

$$ch(\widetilde{\Delta}_q^+ \xi) - ch(\widetilde{\Delta}_q^- \xi) = W_{2q} . e^{c_1/2} . \hat{\mathfrak{A}}(\nu_q)^{-1} . \tag{7}$$

We observe that (5) is also associated to ν_q and that W_{2q} is the Euler class of (5) and get by standard rules

$$\varphi_*(e^{c_1/2} . \hat{\mathfrak{A}}(\nu_q)^{-1}) = ch(\beta_q) \tag{8}$$

where $\varphi_* : H^{**}(B_{2q}, \underset{\sim}{Q}) \to H^{**}(A_{2q}, B_{2q-1}; \underset{\sim}{Q})$ is the Thom-Gysin homomorphism

5.5. - In 5.3 (1) we looked for a certain $\beta \in K(A, E)$. This can now be found as follows. Let δ be a principal $\underset{\sim}{SO}(2)$-bundle with characteristic class $c_1(f)$, i.e. $c_1(f)$ is the first Chern class of δ if δ is considered as $\underset{\sim}{U}(1)$-bundle. The Whitney sum $\delta \oplus \nu$ has vanishing second Stiefel-Whitney class and its structure group can therefore be lifted to G_{2q}, (where $2q$ is the codimension of Y in X). If we induce this G_{2q}-bundle from the universal one, we get β from β_q of 5.4. The proof of the RR-theorem 3.5 is now completed.

5.6. - We still have to prove the RR-theorem 3.7. By the remark in 5.2 it suffices to prove it for an imbedding $f : Y \to X$, Y having in X a codimension divisible by 8 , say codim = 8k . By assumption the second Stiefel-Whitney class of the normal bundle of Y vanishes. Its structure group can therefore be lifted to $\mathbf{Spin}(8k)$. One makes a universal construction similar to that in 5.4. If $x_1 , \dots , x_{4k}$ is a standard base of $H^1(T , \mathbf{Z})$, T standard maximal torus of $\mathbf{SO}(8k)$, then we have for the spinor representations the formula

$$\mathrm{ch}(\Delta^+_{4k}) - \mathrm{ch}(\Delta^-_{4k}) = x_1\, x_2 \dots x_{4k} \cdot \prod_{i=1}^{4k} \frac{\sinh x_i/2}{x_i/2} \quad . \tag{9}$$

By a theorem of E. Cartan-Malcev (compare [1] , paragraph 26.5 end), the two spinor representations of $\mathbf{Spin}(8k)$ in $\mathbf{U}(2^{4k-1})$ factor through $\mathbf{SO}(2^{4k-1})$. Now the procedure is as in 5.4. Let ν_{4k} denote here the canonical $\mathbf{SO}(8k)$-bundle over $B_{\mathbf{Spin}(8k)}$. Let A_{4k} be the mapping cylinder of

$$B_{\mathbf{Spin}(8k-1)} \longrightarrow B_{\mathbf{Spin}(8k)} \ .$$

Using (9) we arrive at a certain element $\beta_{4k} \in \mathbf{K}_0(A_{4k} , B_{\mathbf{Spin}(8k-1)})$ whose Chern character is under the Thom-Gysin homomorphism the image of $\hat{\mathfrak{A}}(\nu_{4k})^{-1}$. This completes the proof of the RR-theorem 3.7.

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