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SOME APPLICATIONS OF INVARIANT DIFFERENTIAL OPERATORS ON A SEMISIMPLE LIE ALGEBRA

by HARISH-CHANDRA

Let R and C be the fields of real and complex numbers respectively and E_0 a vector space of finite dimension over R. We assume that there is given on E_0 a real, non-degenerate, symmetric bilinear form $\langle X , Y \rangle$ $(X , Y \in E_0)$. Let E denote the complexification of E_0 and $S(E)$ the symmetric algebra over E. By means of the above bilinear form, we can identify E with its dual. In this way any element of $S(E)$ becomes a polynomial function on E. Now let $C^{\infty}(E_0)$ denote the space of all indefinitely differentiable functions (with complex values) on E_0. For any $X \in E_0$, we define a differential operator $\partial(X)$ on E_0 as follows :

$$(\partial(X)f)(Y) = \left\{\frac{d}{dt} f(Y + t X)\right\}_{t=0} \quad (f \in C^{\infty}(E_0) , \; Y \in E_0 , \; t \in R) .$$

Let $\mathcal{E}$ be the algebra of all differential operators on E_0. The mapping $X \longrightarrow \partial(X)$ can obviously be extended uniquely to a homomorphism ∂ of $S(E)$ into $\mathcal{E}$. Thus for every $p \in S(E)$, we get a differential operator on E_0. Moreover p, being a polynomial function on E_0, is also a differential operator of order zero. Thus $S(E)$ and $\partial(S(E))$ are both subalgebras of $\mathcal{E}$. We denote by $\mathfrak{D}(E)$ the subalgebra of $\mathcal{E}$ generated by $S(E) \cup S(\partial(E))$. $\mathfrak{D}(E)$ will be called the algebra of polynomial differential operators on E.

For any two elements p , q in $S(E)$, let $\langle p , q \rangle$ denote the value of the polynomial function $\partial(p)q$ at zero. It is easy to see that in this way we get an extension of our original bilinear form to a non-degenerate bilinear form on $S(E)$.

We fix the following notation. For any open set U in E_0, $C^{\infty}(U)$ denotes the space of all indefinitely differentiable functions on U and $C_c^{\infty}(U)$ the subspace of $C^{\infty}(U)$ consisting of those functions which vanish outside some compact subset of U. Moreover $\mathcal{C}(U)$ is the space of those $f \in C^{\infty}(U)$ such that

$$\nu_D(f) = \sup_{X \in U} |(Df)(X)| < \infty$$

for every $D \in \mathfrak{D}(E)$. We topologise $\mathcal{C}(U)$ by means of the seminorms ν_D $(D \in \mathfrak{D}(E))$.

Now let $\mathfrak{g}_0$ be a semisimple Lie algebra over R. Put $\langle X , Y \rangle = tr(ad\, X\, ad\, Y)$

$(X, Y \in \mathfrak{g}_0)$, where $X \to \operatorname{ad} X$ is the adjoint representation of $\mathfrak{g}_0$. Then the above procedure is applicable to $\mathfrak{g}_0$. Let G denote the connected component of 1 in the adjoint group of $\mathfrak{g}_0$. Naturally G operates on the algebra $\mathcal{E}$ of all differential operators on $\mathfrak{g}_0$ in the obvious way. Moreover since the fondamental bilinear form is invariant under G , p^x is the function $X \to p(x^{-1} X)$ $(X \in \mathfrak{g}_0)$ and $\partial(p^x) = (\partial(p))^x$ $(p \in S(\mathfrak{g})$, $x \in G)$. It is clear that $\mathfrak{D}(\mathfrak{g})$ is stable under the operations of G . Let $\mathfrak{I}'(\mathfrak{g})$ denote the set of those elements of $\mathfrak{D}(\mathfrak{g})$ which are invariant under G . Also let $I^{\infty}(\mathfrak{g}_0)$ denote the set of invariant functions in $C^{\infty}(\mathfrak{g}_0)$ (i.e. those f for which $f(xX) = f(X)$ for all $x \in G$ and $X \in \mathfrak{g}_0)$. Then $I^{\infty}(\mathfrak{g}_0)$ is stable under any operator in $\mathfrak{I}'(\mathfrak{g})$.

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. For any $f \in I^{\infty}(\mathfrak{g}_0)$, let $\bar{f}$ denote the restriction of f on $\mathfrak{h}_0$. Then for a fixed $D \in \mathfrak{I}'(\mathfrak{g})$, we seek the relation between the two functions $\bar{f}$ and $\overline{Df}$ $(f \in I^{\infty}(\mathfrak{g}_0))$.

Let ℓ be the rank of $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is called regular if $\operatorname{ad} X$ takes the eigenvalue zero exactly with the multiplicity ℓ . Let $\mathfrak{g}_0'$ denote the set of all regular elements in $\mathfrak{g}_0$ and put $\mathfrak{h}_0' = \mathfrak{g}_0' \cap \mathfrak{h}_0$. Then $\mathfrak{g}_0'$ and $\mathfrak{h}_0'$ are both open and dense subsets of $\mathfrak{g}_0$ and $\mathfrak{h}_0$ respectively.

LEMMA 1. - <u>For each</u> $D \in \mathfrak{I}'(\mathfrak{g})$ <u>there exists a unique differential operator</u> $\delta'(D)$ <u>on</u> $\mathfrak{h}_0'$ <u>such that</u>

$$\overline{Df} = \delta'(D)\bar{f} \qquad \text{on } \mathfrak{h}_0$$

<u>for every</u> $f \in I^{\infty}(\mathfrak{g}_0)$. <u>Moreover</u> $D \to \delta'(D)$ <u>is a homomorphism of</u> $\mathfrak{I}'(\mathfrak{g})$ <u>into the algebra of all differential operators on</u> $\mathfrak{h}_0'$

So now we have to determine the operator $\delta'(D)$. Let $I(\mathfrak{g})$ denote the algebra of invariant elements in $S(\mathfrak{g})$ so that $I(\mathfrak{g}) = S(\mathfrak{g}) \cap \mathfrak{I}'(\mathfrak{g})$. Then $I(\mathfrak{g})$ and $\partial(I(\mathfrak{g}))$ are both subalgebras of $\mathfrak{I}'(\mathfrak{g})$. Denote by $\mathfrak{I}(\mathfrak{g})$ the subalgebra of $\mathfrak{I}'(\mathfrak{g})$ generated by $I(\mathfrak{g}) \cup \partial(I(\mathfrak{g}))$. We intend to give an explicit formula for $\delta'(D)$ in case $D \in \mathfrak{I}(\mathfrak{g})$. First of all notice that if $p \in I(\mathfrak{g})$, then $\overline{pf} = \bar{p}\,\bar{f}$. Hence $\delta'(p) = \bar{p}$. In view of the fact that δ' is a homomorphism and $\mathfrak{I}(\mathfrak{g})$ is generated by $I(\mathfrak{g})$ and $\partial(I(\mathfrak{g}))$, it is sufficient to determine $\delta'(\partial(p))$ for $p \in I(\mathfrak{g})$.

The restriction of our fundamental bilinear form on $\mathfrak{h}_0$ is also non-degenerate. Hence we can take $E_0 = \mathfrak{h}_0$ in our earlier set up. Then for any $q \in S(\mathfrak{h})$, $\partial(q)$ is a differential operator on $\mathfrak{h}_0$. Also $\mathfrak{D}(\mathfrak{h})$ is the algebra of all polynomial differential operators on $\mathfrak{h}_0$. Let W denote the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then W operates on $\mathfrak{h}$ and therefore also on $S(\mathfrak{h})$ and $\mathfrak{D}(\mathfrak{h})$. Moreover our bilinear form on $\mathfrak{h}$ is invariant under W. Let $\mathfrak{I}'(\mathfrak{h})$ denote the set of those elements in $\mathfrak{D}(\mathfrak{h})$ which are invariant under W. Also put $I(\mathfrak{h}) = S(\mathfrak{h}) \cap \mathfrak{I}'(\mathfrak{h})$. Then CHEVALLEY has proved the following result (see [1], p. 10).

LEMMA 2 (CHEVALLEY). - <u>The mapping</u> $p \to \bar{p}$ $(p \in I(\mathfrak{g}))$ <u>is an isomorphism of</u> $I(\mathfrak{g})$ <u>onto</u> $I(\mathfrak{h})$.

Now introduce some lexicographic order among the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be all the distinct positive roots under this order. Put $\pi = \alpha_1 \alpha_2 \dots \alpha_r$. Then π is a polynomial function on $\mathfrak{h}$.

LEMMA 3. - <u>Let</u> p <u>be an element in</u> $I(\mathfrak{g})$. <u>Then</u> $\delta'(\partial(p)) = \pi^{-1}\ \partial(\bar{p}) \circ \pi$ (<u>where</u> $\circ$ <u>denotes the product of two differential operators</u>). See [4], p. 98, for the proof.

Let $\mathfrak{I}(\mathfrak{h})$ denote the subalgebra of $\mathfrak{I}'(\mathfrak{h})$ generated by $I(\mathfrak{h}) \cup \partial(I(\mathfrak{h}))$. Then it is easy to obtain the following theorem from lemmas 1, 2 and 3.

THEOREM 1. - <u>There exists a unique homomorphism</u> δ <u>of</u> $\mathfrak{I}(\mathfrak{g})$ <u>onto</u> $\mathfrak{I}(\mathfrak{h})$ <u>such that</u>

(i) $$\delta(p) = \bar{p} \quad \text{and} \quad \delta(\partial(p)) = \partial(\bar{p}) \qquad (p \in I(\mathfrak{g}))$$

(ii) $$\delta'(D) = \pi^{-1}\ \delta(D) \circ \pi \qquad (D \in \mathfrak{I}(\mathfrak{g})).$$

We shall now derive some consequences of this theorem. First consider the case when $\mathfrak{g}_0$ is compact (i.e. the quadratic form $\langle X, X \rangle$ is negative definite on $\mathfrak{g}_0$). For any $f \in C^\infty(\mathfrak{g}_0)$, put

$$\Phi_f(H) = \pi(H) \int_G f(xH)\, dx \qquad (H \in \mathfrak{h}_0)$$

where dx is the normalized Haar measure on G. If follows from theorem 1 that $\Phi_{Df} = \delta(D)\Phi_f$ for $D \in \mathfrak{I}(\mathfrak{g})$. Hence in particular $\Phi_{\partial(p)f} = \partial(\bar{p})\Phi_f$ $(p \in I(\mathfrak{g}))$. Apply this in particular to the function $f = e^{H_0}$ where H_0 is a fixed element in $\mathfrak{h}$. (We recall that H_0 is a linear function on $\mathfrak{g}_0$ and (therefore $f(X) = e^{\langle H_0, X \rangle}$ for $X \in \mathfrak{g}_0$). Obviously $\partial(p)f = p(H_0)f$ for any $p \in S(\mathfrak{g})$. Hence

$$\partial(\bar{p})\Phi_f = \Phi_{\partial(p)f} = p(H_0)\Phi_f \qquad (p \in I(\mathfrak{g})).$$

Hence by Chevalleys' theorem (Lemma 2), $\partial(q)\Phi_f = q(H_0)\Phi_f$ for every $q \in I(\mathfrak{h})$. Let χ be any homomorphism of $I(\mathfrak{h})$ into C, We consider the system of differential equations $\partial(q)\Phi = \chi(q)\Phi$ $(q \in I(\mathfrak{h}))$ on a non-empty connected open set U of $\mathfrak{h}_0$. First of all, one sees easily that this system always contains equations of the elliptic type. Hence every solution Φ of this system is analytic. Let w be the order of the group W. It follows from a result of CHEVALLEY [2] that $S(\mathfrak{h})$ is a free abelian module over $I(\mathfrak{h})$ of rank w. Hence we can select $u_1, \dots, u_w \in S(\mathfrak{h})$ such that $\sum_{1 \leq i \leq w} I(\mathfrak{h})u_i = S(\mathfrak{h})$. Therefore it is clear that if the derivatives $\partial(u_i)\Phi$ vanish simultaneously at some point H of U for some solution Φ of our system, all derivatives $\partial(u)\Phi$ are zero at H and therefore Φ, being analytic, is identically zero on U. Hence our system can have at most w linearly independent solutions. Now assume that H_0 is regular. Then $sH_0 \neq H_0$ for $s \neq 1$ in W. Then e^{sH_0} $(s \in W)$ are w linearly independent solutions of the system $\partial(q)\Phi = q(H_0)\Phi$ $(q \in I(\mathfrak{h}))$ on $\mathfrak{h}_0$. Therefore $\Phi_f = \sum_{s \in W} c_s e^{sH_0}$ where c_s are constants. On the other hand it is known that $\pi^2 \in I(\mathfrak{h})$ and therefore $\pi^s = \epsilon(s)\pi$ $(s \in W)$ where $\epsilon(s) = \pm 1$. Moreover $\mathfrak{g}_0$ being compact, for every $s \in W$, we can choose $x \in G$ such that $sH = xH$ for all $H \in \mathfrak{h}$. Hence it is obvious from its definition that $\Phi_f(sH) = \epsilon(s)\Phi_f(H)$. Therefore

$$\Phi_f = c \sum_{s \in W} \epsilon(s)\, e^{sH_0}$$

where c is a constant. On the other hand, it is obvious that

$$(\partial(\pi)\Phi_f)_{H=0} = \langle \pi, \pi \rangle f(0) = \langle \pi, \pi \rangle .$$

But $\partial(\pi)e^{sH_0} = \epsilon(s)\pi(H_0)e^{sH_0}$. Hence

$$\langle \pi, \pi \rangle = c\, \pi(H_0) w$$

and so we get the following result.

THEOREM 2. - Suppose G is compact. Then

$$\pi(H_0)\, \pi(H) \int_G e^{\langle H_0, xH \rangle}\, dx = w^{-1} \sum_{s \in W} \epsilon(s)\, e^{\langle H_0, sH \rangle}$$

for $H_0, H \in \mathfrak{h}$. (Here dx is the normalized Haar measure on G).

We actually proved this result for $H_0 \in \mathfrak{h}_0'$ and $H \in \mathfrak{h}_0$. But since both sides are holomorphic in H_0, H the more general case follows immediately.

For later use we also note the formula

$$(1) \qquad (\partial(\pi)\Phi_f)_{H=0} = \langle \pi, \pi \rangle \; f(0) \qquad (f \in C^{\infty}(\mathfrak{g}_0)) .$$

The proof is trivial.

Now we take up the more difficult case when $\mathfrak{g}_0$ is not compact so that the quadratic form $\langle X , X \rangle$ is indefinite on $\mathfrak{g}_0$. Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}_0$. (By definition, A is the centralizer of $\mathfrak{h}_0$ in G). We denote by $x \to x^*$ the natural mapping of G on $G^* = G/A$. Put $x^*H = xH$ $(x \in G , \; H \in \mathfrak{h}_0)$ and let dx^* denote the invariant measure on G^* (normalized in some fixed but arbitrary way). For any $f \in \mathcal{C}(\mathfrak{g}_0)$, put

$$\Phi_f(H) = \pi(H) \int_{G^*} f(x^*H) \, dx^* \qquad (H \in \mathfrak{h}_0')$$

Then it can be shown without difficulty that the integral converges for $H \in \mathfrak{h}_0'$ and that Φ_f is of class C^{∞} on $\mathfrak{h}_0$. Again, we can conclude from theorem 1 that $\Phi_{Df} = \delta(D)\Phi_f$ for all $D \in \mathfrak{I}(\mathfrak{g})$ and so in particular $\Phi_{\partial(p)f} = \partial(\bar{p})\Phi_f$ for $p \in I(\mathfrak{g})$. Now an important consequence of this relation is the following result (see [5], theorem 3, p. 225).

LEMMA 4. - <u>For any</u> $f \in \mathcal{C}(\mathfrak{g}_0)$, Φ_f <u>lies in</u> $\mathcal{C}(\mathfrak{h}_0')$. <u>Moreover</u> $f \to \Phi_f$ <u>is a continuous mapping of</u> $\mathcal{C}(\mathfrak{g}_0)$ <u>into</u> $\mathcal{C}(\mathfrak{h}_0')$.

The main point of interest here is the fact that $\partial(q)\Phi_f$ remains bounded on $\mathfrak{h}_0'$ for every $q \in S(\mathfrak{h})$. The proof of this fact in the general case is rather complicated. So, as an illustration, let us consider the following example. Take $\mathfrak{g}_0$ to be the Lie algebra of all 2×2 real matrices with trace zero and $\mathfrak{h}_0$ the Cartan subalgebra of $\mathfrak{g}_0$ spanned over R by the matrix $H_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then A is compact and zero is the only singular point in $\mathfrak{h}_0$. Hence we can write

$$\Phi_f(\theta H_0) = 2i\theta \int_G f(\theta x H_0) \, dx \qquad (f \in C_c^{\infty}(\mathfrak{g}_0) , \; \theta \in R , \; \theta \neq 0)$$

because $\pi(H_0) = 2i$. Put

$$F_f(\theta) = \theta \int_G f(\theta x H_0) \, dx \qquad (\theta \neq 0) .$$

We have to show that $\frac{d^k}{d\theta^k} F_f$ remains bounded around $\theta = 0$ for every $k \geq 0$. This is done as follows. Consider the polynomial ω on $\mathfrak{g}$ given by $\omega(X) = \mathrm{tr}(X^2)$ $(X \in \mathfrak{g})$. Then $\omega \in I(\mathfrak{g})$ and $\omega(\theta H_0) = -2\theta^2$. Therefore since $\Phi_{\partial(\omega)f} = \partial(\bar{\omega})\Phi_f$, we conclude that

$$\frac{d^2}{d\theta^2} F_f = -2 F_{\partial(\omega)f} .$$

Now first one proves by a crude estimate that there exists an integer $n \geqslant 0$ with the property that $c(f) = \sup_\theta |\theta^n F_f(\theta)| < \infty$ for every $f \in C_c^\infty(\mathfrak{g}_0)$. Assume now that n is the least possible such integer. We claim $n = 0$. For otherwise suppose $n \geqslant 1$. Then

$$\left|\frac{d^2}{d\theta^2} F_f\right| \leqslant 2 \left|F_{\partial(\omega)f}\right| \leqslant 2 |\theta|^{-n} c(\partial(\omega)f) .$$

Hence if $n \geqslant 2$, it follows easily by integration that

$$|F_f| \leqslant |\theta|^{2-n} c'(f)$$

where $c'(f)$ is a positive constant depending on f. As this contradicts the choice of n, we must have $n = 1$. But since $\log |\theta|$ is locally summable around $\theta = 0$, it follows by the same argument that $|F_f| \leqslant c_1(f)$ where $c_1(f)$ is another constant depending on f. Thus we again get a contradiction. Hence $|F_f|$ remains bounded and therefore

$$\frac{d^{2k}}{d\theta^{2k}} F_f = (-2)^k F_{\partial(\omega^k)f}$$

also remains bounded for every $k \geqslant 0$. But then by integration we can conclude the same for

$$\frac{d^{2k-1}}{d\theta^{2k-1}} F_f \qquad (k \geqslant 1)$$

The reasoning in the general case, althrough more complicated, is essentially the same.

Let dX and dH denote the Euclidean measures on $\mathfrak{g}_0$ and $\mathfrak{h}_0$ respectively. For any $f \in \mathcal{C}(\mathfrak{g}_0)$ and $g \in \mathcal{C}(\mathfrak{h}_0')$, put

$$\tilde{f}(Y) = \int_{\mathfrak{g}_0} e^{i\langle Y,X\rangle} f(X)\, dX \qquad (Y \in \mathfrak{g}_0)$$

$$\tilde{g}(H') = \int_{\mathfrak{h}_0} e^{i\langle H',H\rangle} g(H)\, dH \qquad (H' \in \mathfrak{h}_0) .$$

Then, in the compact case, theorem 2 can be interpreted to mean that $\Phi_{\tilde{f}}$ and $\tilde{\Phi}_f$ are the same except for a constant factor which is independent of f. Similar but more complicated results hold when $\mathfrak{g}_0$ is not compact (see [5], lemma 24). We give only one such result here (see [5] theorem 4, p. 247). Let K be a maximal compact subgroup of G and let dk denote the normalized Haar measure of K.

THEOREM 3. – Suppose $\mathfrak{h}_0$ is contained in the Lie algebra of K. Then it follows easily that $s\mathfrak{h}_0 = \mathfrak{h}_0$ for every $s \in W$. For any $f \in \mathcal{C}(\mathfrak{h}_0)$, put

$$\hat{f}(X) = \int_{K\times\mathfrak{h}_0} e^{i\langle X,kH\rangle}\,\pi(H)^2 \sum_{s\in W} f(sH)\,dk\,dH \qquad (x \in \mathfrak{g}_0).$$

Then the integral

$$\Phi_{\hat{f}}(H) = \pi(H)\int_{G^*} \hat{f}(x^*H)\,dx^*$$

converges for $H \in \mathfrak{h}_0$. Moreover there exists a complex number $c \neq 0$ such that

$$\sum_{s\in W} \varepsilon(s)\Phi_{\hat{f}}(sH') = c\int_{\mathfrak{h}_0} \sum_{s\in W} \varepsilon(s)\,e^{i\langle H',sH\rangle}\,\pi(H)\,f(H)\,dH$$

for all $H' \in \mathfrak{h}_0'$ and $f \in \mathcal{C}(\mathfrak{h}_0)$.

The main object of this theory is to obtain the analogue of (1) in the non-compact case. Fix a connected component $\mathfrak{h}_1$ of $\mathfrak{h}_0'$ and put

$$T(f) = \lim_{H\to 0} \partial(\pi)\Phi_f \qquad (H \in \mathfrak{h}_1)$$

for $f \in \mathcal{C}(\mathfrak{g}_0)$. It follows from lemma 4 that this limit exists and that T is a distribution on $\mathfrak{g}_0$. The main task now is to show that T is a constant multiple of the δ-distribution corresponding to the unit mass at the origin. Let $\tilde{T}$ denote the Fourier transform of T. Then we have to prove that $\tilde{T}$ is a constant. As before, let $\mathfrak{g}_0'$ be the set of all regular elements of $\mathfrak{g}_0$ and $\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_N$ all the distinct connected components of $\mathfrak{g}_0'$. It follows without much difficulty (again by using theorem 1) that on each $\mathfrak{g}_i$ $\tilde{T}$ coincides with a constant c_i. The main remaining difficulty is to show that $c_1, \dots, c_N$ are all equal (see [5], paragraphe 7). This however requires considerable work and a rather detailed investigation [6]. The final result can be stated as follows.

THEOREM 4. – There exists a real number c such that

$$\lim_{H\to 0} \partial(\pi)\Phi_f = c\,f(0) \qquad (H \in \mathfrak{h}_0')$$

for every $f \in \mathcal{C}(\mathfrak{g}_0)$.

Actually it turns out that $c = 0$ most of the time. Put $\omega(X) = \langle X, X\rangle$. Then ω is a quadratic form on $\mathfrak{g}_0$ and its restriction $\bar{\omega}$ on $\mathfrak{h}_0$ is a quadratic form on $\mathfrak{h}_0$. Let ℓ_- denote the number of negative eigen-values of $\bar{\omega}$ (taking into account their multiplicity). Then we say that $\mathfrak{h}_0$ is a fundamental

Cartan subalgebra of $\mathfrak{g}_0$ if ℓ_- has the maximum possible value. Any two fundamental Cartan subalgebras are conjugate under G. Moreover, the constant c of theorem 4 is different from zero, if and only if, $\mathfrak{h}_0$ is fundamental. In view of the arbitrary normalization of the measure dx^* on G^*, it is only the sign of c which is of interest (in case $\mathfrak{h}_0$ is fundamental). Let K be a maximal compact subgroup of G. Then c has the sign $(-1)^q$ where

$$q = \frac{1}{2}(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)$$

(see the remark at the end of [7]).

Theorem 4 had been announced by GELFAND and GRAEV [3] in the case of the Lie algebra $\mathfrak{g}_0$ of all $n \times n$ real matrices with trace zero. However the reasoning sketched by them appears to me to be incorrect because they seem to assume (or to assert) that Φ_f (or rather $\frac{|\pi|}{\pi}\Phi_f$) can always be extended to a function of class C^∞ on $\mathfrak{h}_0$ (see the lines between equations (4) and (5) on p. 462 of [3]). This is false even in the case of the algebra of all 2×2 real matrices with trace zero.

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