

SÉMINAIRE N. BOURBAKI

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Séminaire N. Bourbaki, 1958, exp. n° 160, p. 319-326

http://www.numdam.org/item?id=SB_1956-1958__4__319_0

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SOME APPLICATIONS OF INVARIANT DIFFERENTIAL OPERATORS

ON A SEMISIMPLE LIE ALGEBRA

by HARISH-CHANDRA

Let R and C be the fields of real and complex numbers respectively and E_0 a vector space of finite dimension over R . We assume that there is given on E_0 a real, non-degenerate, symmetric bilinear form $\langle X, Y \rangle$ ($X, Y \in E_0$). Let E denote the complexification of E_0 and $S(E)$ the symmetric algebra over E . By means of the above bilinear form, we can identify E with its dual. In this way any element of $S(E)$ becomes a polynomial function on E . Now let $C^\infty(E_0)$ denote the space of all indefinitely differentiable functions (with complex values) on E_0 . For any $X \in E_0$, we define a differential operator $\partial(X)$ on E_0 as follows :

$$(\partial(X)f)(Y) = \left\{ \frac{d}{dt} f(Y + tX) \right\}_{t=0} \quad (f \in C^\infty(E_0), Y \in E_0, t \in R).$$

Let \mathcal{L} be the algebra of all differential operators on E_0 . The mapping $X \rightarrow \partial(X)$ can obviously be extended uniquely to a homomorphism ∂ of $S(E)$ into \mathcal{L} . Thus for every $p \in S(E)$, we get a differential operator on E_0 . Moreover p , being a polynomial function on E_0 , is also a differential operator of order zero. Thus $S(E)$ and $\partial(S(E))$ are both subalgebras of \mathcal{L} . We denote by $\mathcal{D}(E)$ the subalgebra of \mathcal{L} generated by $S(E) \cup \partial(S(E))$. $\mathcal{D}(E)$ will be called the algebra of polynomial differential operators on E .

For any two elements p, q in $S(E)$, let $\langle p, q \rangle$ denote the value of the polynomial function $\partial(p)q$ at zero. It is easy to see that in this way we get an extension of our original bilinear form to a non-degenerate bilinear form on $S(E)$.

We fix the following notation. For any open set U in E_0 , $C^\infty(U)$ denotes the space of all indefinitely differentiable functions on U and $C_c^\infty(U)$ the subspace of $C^\infty(U)$ consisting of those functions which vanish outside some compact subset of U . Moreover $\mathcal{C}(U)$ is the space of those $f \in C^\infty(U)$ such that

$$\nu_D(f) = \sup_{X \in U} |(Df)(X)| < \infty$$

for every $D \in \mathcal{D}(E)$. We topologise $\mathcal{C}(U)$ by means of the seminorms ν_D ($D \in \mathcal{D}(E)$).

Now let \mathfrak{g}_0 be a semisimple Lie algebra over R . Put $\langle X, Y \rangle = \text{tr}(\text{ad } X \text{ ad } Y)$

$(X, Y \in \mathfrak{g}_0)$, where $X \rightarrow \text{ad } X$ is the adjoint representation of \mathfrak{g}_0 . Then the above procedure is applicable to \mathfrak{g}_0 . Let G denote the connected component of 1 in the adjoint group of \mathfrak{g}_0 . Naturally G operates on the algebra \mathcal{E} of all differential operators on \mathfrak{g}_0 in the obvious way. Moreover since the fundamental bilinear form is invariant under G , p^x is the function $X \rightarrow p(x^{-1}X)$ ($X \in \mathfrak{g}_0$) and $\partial(p^x) = (\partial(p))^x$ ($p \in S(\mathfrak{g})$, $x \in G$). It is clear that $\mathcal{D}(\mathfrak{g})$ is stable under the operations of G . Let $\mathcal{F}'(\mathfrak{g})$ denote the set of those elements of $\mathcal{D}(\mathfrak{g})$ which are invariant under G . Also let $I^\infty(\mathfrak{g}_0)$ denote the set of invariant functions in $C^\infty(\mathfrak{g}_0)$ (i.e. those f for which $f(xX) = f(X)$ for all $x \in G$ and $X \in \mathfrak{g}_0$). Then $I^\infty(\mathfrak{g}_0)$ is stable under any operator in $\mathcal{F}'(\mathfrak{g})$.

Let \mathfrak{h}_0 be a Cartan subalgebra of \mathfrak{g}_0 . For any $f \in I^\infty(\mathfrak{g}_0)$, let \bar{f} denote the restriction of f on \mathfrak{h}_0 . Then for a fixed $D \in \mathcal{F}'(\mathfrak{g})$, we seek the relation between the two functions \bar{f} and \overline{Df} ($f \in I^\infty(\mathfrak{g}_0)$).

Let ℓ be the rank of \mathfrak{g} . An element $X \in \mathfrak{g}$ is called regular if $\text{ad } X$ takes the eigenvalue zero exactly with the multiplicity ℓ . Let \mathfrak{g}'_0 denote the set of all regular elements in \mathfrak{g}_0 and put $\mathfrak{h}'_0 = \mathfrak{g}'_0 \cap \mathfrak{h}_0$. Then \mathfrak{g}'_0 and \mathfrak{h}'_0 are both open and dense subsets of \mathfrak{g}_0 and \mathfrak{h}_0 respectively.

LEMMA 1. - For each $D \in \mathcal{F}'(\mathfrak{g})$ there exists a unique differential operator $\delta'(D)$ on \mathfrak{h}'_0 such that

$$\overline{Df} = \delta'(D)\bar{f} \quad \text{on } \mathfrak{h}_0$$

for every $f \in I^\infty(\mathfrak{g}_0)$. Moreover $D \rightarrow \delta'(D)$ is a homomorphism of $\mathcal{F}'(\mathfrak{g})$ into the algebra of all differential operators on \mathfrak{h}'_0

So now we have to determine the operator $\delta'(D)$. Let $I(\mathfrak{g})$ denote the algebra of invariant elements in $S(\mathfrak{g})$ so that $I(\mathfrak{g}) = S(\mathfrak{g}) \cap \mathcal{F}'(\mathfrak{g})$. Then $I(\mathfrak{g})$ and $\partial(I(\mathfrak{g}))$ are both subalgebras of $\mathcal{F}'(\mathfrak{g})$. Denote by $\mathcal{F}(\mathfrak{g})$ the subalgebra of $\mathcal{F}'(\mathfrak{g})$ generated by $I(\mathfrak{g}) \cup \partial(I(\mathfrak{g}))$. We intend to give an explicit formula for $\delta'(D)$ in case $D \in \mathcal{F}'(\mathfrak{g})$. First of all notice that if $p \in I(\mathfrak{g})$, then $\overline{pf} = \bar{p}\bar{f}$. Hence $\delta'(p) = \bar{p}$. In view of the fact that δ' is a homomorphism and $\mathcal{F}(\mathfrak{g})$ is generated by $I(\mathfrak{g})$ and $\partial(I(\mathfrak{g}))$, it is sufficient to determine $\delta'(\partial(p))$ for $p \in I(\mathfrak{g})$.

The restriction of our fundamental bilinear form on \mathfrak{h}_0 is also non-degenerate. Hence we can take $E_0 = \mathfrak{h}_0$ in our earlier set up. Then for any $q \in S(\mathfrak{h})$, $\partial(q)$ is a differential operator on \mathfrak{h}_0 . Also $\mathcal{D}(\mathfrak{h})$ is the algebra of all polynomial differential operators on \mathfrak{h}_0 . Let W denote the Weyl group

of \mathfrak{g} with respect to \mathfrak{h} . Then W operates on \mathfrak{h} and therefore also on $S(\mathfrak{h})$ and $\mathfrak{D}(\mathfrak{h})$. Moreover our bilinear form on \mathfrak{h} is invariant under W . Let $\mathfrak{I}'(\mathfrak{h})$ denote the set of those elements in $\mathfrak{D}(\mathfrak{h})$ which are invariant under W . Also put $I(\mathfrak{h}) = S(\mathfrak{h}) \cap \mathfrak{I}'(\mathfrak{h})$. Then CHEVALLEY has proved the following result (see [1], p. 10).

LEMMA 2 (CHEVALLEY). - The mapping $p \rightarrow \bar{p}$ ($p \in I(\mathfrak{g})$) is an isomorphism of $I(\mathfrak{g})$ onto $I(\mathfrak{h})$.

Now introduce some lexicographic order among the roots of \mathfrak{g} with respect to \mathfrak{h} and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be all the distinct positive roots under this order. Put $\pi = \alpha_1 \alpha_2 \dots \alpha_r$. Then π is a polynomial function on \mathfrak{h} .

LEMMA 3. - Let p be an element in $I(\mathfrak{g})$. Then $\delta'(\partial(p)) = \pi^{-1} \partial(\bar{p}) \circ \pi$ (where \circ denotes the product of two differential operators). See [4], p. 98, for the proof.

Let $\mathfrak{I}(\mathfrak{h})$ denote the subalgebra of $\mathfrak{I}'(\mathfrak{h})$ generated by $I(\mathfrak{h}) \cup \partial(I(\mathfrak{h}))$. Then it is easy to obtain the following theorem from lemmas 1, 2 and 3.

THEOREM 1. - There exists a unique homomorphism δ of $\mathfrak{I}(\mathfrak{g})$ onto $\mathfrak{I}(\mathfrak{h})$ such that

- (i) $\delta(p) = \bar{p}$ and $\delta(\partial(p)) = \partial(\bar{p})$ ($p \in I(\mathfrak{g})$)
- (ii) $\delta'(D) = \pi^{-1} \delta(D) \circ \pi$ ($D \in \mathfrak{I}(\mathfrak{g})$).

We shall now derive some consequences of this theorem. First consider the case when \mathfrak{g}_0 is compact (i.e. the quadratic form $\langle X, X \rangle$ is negative definite on \mathfrak{g}_0). For any $f \in C^\infty(\mathfrak{g}_0)$, put

$$\Phi_f(H) = \pi(H) \int_G f(xH) dx \quad (H \in \mathfrak{h}_0)$$

where dx is the normalized Haar measure on G . It follows from theorem 1 that $\Phi_{Df} = \delta(D)\Phi_f$ for $D \in \mathfrak{I}(\mathfrak{g})$. Hence in particular $\Phi_{\partial(p)f} = \partial(\bar{p})\Phi_f$

($p \in I(\mathfrak{g})$). Apply this in particular to the function $f = e^{H_0}$ where H_0 is a fixed element in \mathfrak{h} . (We recall that H_0 is a linear function on \mathfrak{g}_0 and

(therefore $f(X) = e^{\langle H_0, X \rangle}$ for $X \in \mathfrak{g}_0$). Obviously $\partial(p)f = p(H_0)f$ for any $p \in S(\mathfrak{g})$. Hence

$$\partial(\bar{p})\Phi_f = \Phi_{\partial(p)f} = p(H_0)\Phi_f \quad (p \in I(\mathfrak{g})).$$

Hence by Chevalley's theorem (Lemma 2), $\partial(q)\Phi_f = q(H_0)\Phi_f$ for every $q \in I(\mathfrak{h})$. Let χ be any homomorphism of $I(\mathfrak{h})$ into \mathbb{C} . We consider the system of differential equations $\partial(q)\Phi = \chi(q)\Phi$ ($q \in I(\mathfrak{h})$) on a non-empty connected open set U of \mathfrak{h}_0 . First of all, one sees easily that this system always contains equations of the elliptic type. Hence every solution Φ of this system is analytic. Let w be the order of the group W . It follows from a result of CHEVALLEY [2] that $S(\mathfrak{h})$ is a free abelian module over $I(\mathfrak{h})$ of rank w . Hence we can select $u_1, \dots, u_w \in S(\mathfrak{h})$ such that $\sum_{1 \leq i \leq w} I(\mathfrak{h}) u_i = S(\mathfrak{h})$. Therefore it is clear that if the derivatives $\partial(u_i)\Phi$ vanish simultaneously at some point H of U for some solution Φ of our system, all derivatives $\partial(u)\Phi$ are zero at H and therefore Φ , being analytic, is identically zero on U . Hence our system can have at most w linearly independent solutions. Now assume that H_0 is regular. Then $sH_0 \neq H_0$ for $s \neq 1$ in W . Then e^{sH_0} ($s \in W$) are w linearly independent solutions of the system $\partial(q)\Phi = q(H_0)\Phi$ ($q \in I(\mathfrak{h})$) on \mathfrak{h}_0 . Therefore $\Phi_f = \sum_{s \in W} c_s e^{sH_0}$ where c_s are constants. On the other hand it is known that $\pi^2 \in I(\mathfrak{h})$ and therefore $\pi^s = \xi(s)\pi$ ($s \in W$) where $\xi(s) = \pm 1$. Moreover \mathfrak{g}_0 being compact, for every $s \in W$, we can choose $x \in G$ such that $sH = xH$ for all $H \in \mathfrak{h}$. Hence it is obvious from its definition that $\Phi_f(sH) = \xi(s)\Phi_f(H)$. Therefore

$$\Phi_f = c \sum_{s \in W} \xi(s) e^{sH_0}$$

where c is a constant. On the other hand, it is obvious that

$$(\partial(\pi)\Phi_f)_{H=0} = \langle \pi, \pi \rangle f(0) = \langle \pi, \pi \rangle .$$

But $\partial(\pi)\Phi_f = \xi(s)\pi(\pi_0)e^{sH_0}$. Hence

$$\langle \pi, \pi \rangle = c \pi(H_0)_w$$

and so we get the following result.

THEOREM 2. - Suppose G is compact. Then

$$\pi(H_0)\pi(H) \int_G e^{\langle H_0, xH \rangle} dx = w^{-1} \sum_{s \in W} \xi(s) e^{\langle H_0, sH \rangle}$$

for $H_0, H \in \mathfrak{h}$. (Here dx is the normalized Haar measure on G).

We actually proved this result for $H_0 \in \mathfrak{h}'_0$ and $H \in \mathfrak{h}_0$. But since both sides are holomorphic in H_0, H the more general case follows immediately.

For later use we also note the formula

$$(1) \quad (\partial(\pi)\Phi_f)_{H=0} = \langle \pi, \bar{\pi} \rangle f(0) \quad (f \in C^\infty(\mathfrak{g}_0)) .$$

The proof is trivial.

Now we take up the more difficult case when \mathfrak{g}_0 is not compact so that the quadratic form $\langle X, X \rangle$ is indefinite on \mathfrak{g}_0 . Let A be the Cartan subgroup of G corresponding to \mathfrak{h}_0 . (By definition, A is the centralizer of \mathfrak{h}_0 in G). We denote by $x \rightarrow x^*$ the natural mapping of G on $G^* = G/A$. Put $x^*H = xH$ ($x \in G, H \in \mathfrak{h}_0$) and let dx^* denote the invariant measure on G^* (normalized in some fixed but arbitrary way). For any $f \in C(\mathfrak{g}_0)$, put

$$\Phi_f(H) = \int_{G^*} f(x^*H) dx^* \quad (H \in \mathfrak{h}'_0)$$

Then it can be shown without difficulty that the integral converges for $H \in \mathfrak{h}'_0$ and that Φ_f is of class C^∞ on \mathfrak{h}_0 . Again, we can conclude from theorem 1 that $\Phi_{Df} = \delta(D)\Phi_f$ for all $D \in \mathcal{I}(\mathfrak{g})$ and so in particular $\Phi_{\partial(p)f} = \partial(\bar{p})\Phi_f$ for $p \in I(\mathfrak{g})$. Now an important consequence of this relation is the following result (see [5], theorem 3, p. 225).

LEMMA 4. - For any $f \in C(\mathfrak{g}_0)$, Φ_f lies in $C(\mathfrak{h}'_0)$. Moreover $f \rightarrow \Phi_f$ is a continuous mapping of $C(\mathfrak{g}_0)$ into $C(\mathfrak{h}'_0)$.

The main point of interest here is the fact that $\partial(q)\Phi_f$ remains bounded on \mathfrak{h}'_0 for every $q \in \delta(\mathfrak{h})$. The proof of this fact in the general case is rather complicated. So, as an illustration, let us consider the following example. Take \mathfrak{g}_0 to be the Lie algebra of all 2×2 real matrices with trace zero and \mathfrak{h}_0 the Cartan subalgebra of \mathfrak{g}_0 spanned over \mathbb{R} by the matrix $H_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then A is compact and zero is the only singular point in \mathfrak{h}_0 . Hence we can write

$$\Phi_f(\theta H_0) = 2i\theta \int_G f(\theta x H_0) dx \quad (f \in C_c^\infty(\mathfrak{g}_0), \theta \in \mathbb{R}, \theta \neq 0)$$

because $\pi(H_0) = 2i$. Put

$$F_f(\theta) = \theta \int_G f(\theta x H_0) dx \quad (\theta \neq 0) .$$

We have to show that $\frac{d^k}{d\theta^k} F_f$ remains bounded around $\theta = 0$ for every $k \geq 0$.

This is done as follows. Consider the polynomial ω on \mathfrak{g} given by $\omega(X) = \text{tr}(X^2)$ ($X \in \mathfrak{g}$). Then $\omega \in I(\mathfrak{g})$ and $\omega(\theta H_0) = -2\theta^2$. Therefore since $\Phi_{\partial(\omega)f} = \partial(\bar{\omega})\Phi_f$, we conclude that

$$\frac{d^2}{d\theta^2} F_f = -2 F_{\partial(\omega)f} \cdot$$

Now first one proves by a crude estimate that there exists an integer $n \geq 0$ with the property that $c(f) = \sup_{\theta} |\theta^n F_f(\theta)| < \infty$ for every $f \in C_c^\infty(\mathcal{Q}_0)$. Assume now that n is the least possible such integer. We claim $n = 0$. For otherwise suppose $n \geq 1$. Then

$$\left| \frac{d^2}{d\theta^2} F_f \right| \leq 2 |F_{\partial(\omega)f}| \leq 2 |\theta|^{-n} c(\partial(\omega)f).$$

Hence if $n \geq 2$, it follows easily by integration that

$$|F_f| \leq |\theta|^{2-n} c'(f)$$

where $c'(f)$ is a positive constant depending on f . As this contradicts the choice of n , we must have $n = 1$. But since $\log |\theta|$ is locally summable around $\theta = 0$, it follows by the same argument that $|F_f| \leq c_1(f)$ where $c_1(f)$ is another constant depending on f . Thus we again get a contradiction. Hence $|F_f|$ remains bounded and therefore

$$\frac{d^{2k}}{d\theta^{2k}} F_f = (-2)^k F_{\partial(\omega)^k f}$$

also remains bounded for every $k \geq 0$. But then by integration we can conclude the same for

$$\frac{d^{2k-1}}{d\theta^{2k-1}} F_f \quad (k \geq 1)$$

The reasoning in the general case, although more complicated, is essentially the same.

Let dX and dH denote the Euclidean measures on \mathcal{Q}_0 and \mathcal{H}_0 respectively. For any $f \in C(\mathcal{Q}_0)$ and $g \in C(\mathcal{H}_0)$, put

$$\tilde{f}(Y) = \int_{\mathcal{Q}_0} e^{i \langle Y, X \rangle} f(X) dX \quad (Y \in \mathcal{Q}_0)$$

$$\tilde{g}(H') = \int_{\mathcal{H}_0} e^{i \langle H', H \rangle} g(H) dH \quad (H' \in \mathcal{H}_0).$$

Then, in the compact case, theorem 2 can be interpreted to mean that $\tilde{\Phi}_f$ and $\tilde{\Phi}_f$ are the same except for a constant factor which is independent of f . Similar but more complicated results hold when \mathcal{Q}_0 is not compact (see [5], lemma 24). We give only one such result here (see [5] theorem 4, p. 247). Let K be a maximal compact subgroup of G and let dk denote the normalized Haar measure of K .

THEOREM 3. - Suppose \mathfrak{h}_0 is contained in the Lie algebra of K . Then it follows easily that $s\mathfrak{h}_0 = \mathfrak{h}_0$ for every $s \in W$. For any $f \in \mathcal{C}(\mathfrak{h}_0)$, put

$$\hat{f}(X) = \int_{K \times \mathfrak{h}_0} e^{i \langle X, kH \rangle} \frac{1}{\pi(H)^2} \sum_{s \in W} f(sH) dk dH \quad (x \in \mathfrak{g}_0).$$

Then the integral

$$\Phi_f^\wedge(H) = \pi(H) \int_{\mathfrak{G}^*} \hat{f}(x^*H) dx^*$$

converges for $H \in \mathfrak{h}_0$. Moreover there exists a complex number $c \neq 0$ such that

$$\sum_{s \in W} \varepsilon(s) \Phi_f^\wedge(sH') = c \int_{\mathfrak{h}_0} \sum_{s \in W} \varepsilon(s) e^{i \langle H', sH \rangle} \pi(H) f(H) dH$$

for all $H' \in \mathfrak{h}'_0$ and $f \in \mathcal{C}(\mathfrak{h}_0)$.

The main object of this theory is to obtain the analogue of (1) in the non-compact case. Fix a connected component \mathfrak{h}'_1 of \mathfrak{h}'_0 and put

$$T(f) = \lim_{H \rightarrow 0} \partial(\pi) \Phi_f \quad (H \in \mathfrak{h}'_1)$$

for $f \in \mathcal{C}(\mathfrak{g}_0)$. It follows from lemma 4 that this limit exists and that T is a distribution on \mathfrak{g}_0 . The main task now is to show that T is a constant multiple of the δ -distribution corresponding to the unit mass at the origin. Let \tilde{T} denote the Fourier transform of T . Then we have to prove that \tilde{T} is a constant. As before, let \mathfrak{g}'_0 be the set of all regular elements of \mathfrak{g}_0 and $\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_N$ all the distinct connected components of \mathfrak{g}'_0 . It follows without much difficulty (again by using theorem 1) that on each \mathfrak{g}_i \tilde{T} coincides with a constant c_i . The main remaining difficulty is to show that c_1, \dots, c_N are all equal (see [5], paragraphe 7). This however requires considerable work and a rather detailed investigation [6]. The final result can be stated as follows.

THEOREM 4. - There exists a real number c such that

$$\lim_{H \rightarrow 0} \partial(\pi) \Phi_f = c f(0) \quad (H \in \mathfrak{h}'_0)$$

for every $f \in \mathcal{C}(\mathfrak{g}_0)$.

Actually it turns out that $c = 0$ most of the time. Put $\omega(X) = \langle X, X \rangle$. Then ω is a quadratic form on \mathfrak{g}_0 and its restriction $\bar{\omega}$ on \mathfrak{h}_0 is a quadratic form on \mathfrak{h}_0 . Let l_- denote the number of negative eigen-values of $\bar{\omega}$ (taking into account their multiplicity). Then we say that \mathfrak{h}_0 is a fundamental

Cartan subalgebra of \mathfrak{g}_0 if \mathfrak{l}_- has the maximum possible value. Any two fundamental Cartan subalgebras are conjugate under G . Moreover, the constant c of theorem 4 is different from zero, if and only if, \mathfrak{h}_0 is fundamental. In view of the arbitrary normalization of the measure dx^* on G^* , it is only the sign of c which is of interest (in case \mathfrak{h}_0 is fundamental). Let K be a maximal compact subgroup of G . Then c has the sign $(-1)^q$ where

$$q = \frac{1}{2}(\dim G/K - \text{rank } G + \text{rank } K)$$

(see the remark at the end of [7]).

Theorem 4 had been announced by GELFAND and GRAEV [3] in the case of the Lie algebra \mathfrak{g}_0 of all $n \times n$ real matrices with trace zero. However the reasoning sketched by them appears to me to be incorrect because they seem to assume (or to assert) that Φ_f (or rather $\frac{|\pi|}{\pi} \Phi_f$) can always be extended to a function of class C^∞ on \mathfrak{h}_0 (see the lines between equations (4) and (5) on p. 462 of [3]). This is false even in the case of the algebra of all 2×2 real matrices with trace zero.

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