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A class of elliptic differential equations with discontinuous coefficients


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A CLASS OF ELLIPTIC DIFFERENTIAL EQUATIONS
WITH DISCONTINUOUS COEFFICIENTS

par Charles B. MORREY, Jr

1. Introduction.

We shall discuss equations of the form

\[(1.1) \int_G \left\{ \sum_{\alpha=1}^{\nu} a^{\alpha_\beta} u^{\alpha_\beta} + b^\alpha u^\alpha + e^\alpha + v(\sum_{\alpha=1}^{\nu} c^\alpha u^\alpha + du + f) \right\} dx = 0 \]

in which \( G \) is a bounded domain in \( \nu \) dimensional space, the coefficients \( a^{\alpha_\beta} \), \( b^\alpha \), \( c^\alpha \), and \( d \) are bounded and measurable, \( e \) and \( f \) are in \( L_2(G) \), \( u \in H^1_p(D) \) for each domain \( D \) for which \( \overline{D} \) (the closure of \( D \)) \( \subset G \), and the equation is supposed to hold for all Lipschitz functions \( v \) with compact support. A function \( u \in H^1_p(D) \) if and only if \( u \) and its "distribution derivatives", which we denote by \( u_{\alpha}, \alpha = 1, \ldots, \nu, \in L_p(D) \); that is to say \( u \in L_p(D) \) and there are functions \( p^\alpha, \alpha = 1, \ldots, \nu, \) also in \( L_p(D) \), such that

\[(1.2) \int_D u(x) g^\alpha(x) dx = -\int_D p^\alpha(x) g(x) dx \]

for every function \( g \) of class \( C^\infty \) on \( D \) and having compact support; here, of course, \( g^\alpha \) denotes the ordinary partial derivative. These spaces are well known but are discussed rather completely in [7] (see the bibliography at the end).

In case the function \( u \) is of class \( C^2(G) \), the coefficients \( a^{\alpha_\beta} \), \( b^\alpha \), and \( c^\alpha \) in \( C^1(G) \), and \( d \) and \( f \) in \( C^0(G) \), then one sees that \( u \) satisfies (1.1) for all the \( v \) mentioned above if and only if \( u \) satisfies the differential equation

\[(1.3) \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x^\alpha} \left( \sum_{\beta=1}^{\nu} a^{\alpha_\beta} u^{\alpha_\beta} + b^\alpha u^\alpha \right) - \left( \sum_{\alpha=1}^{\nu} c^\alpha u^\alpha + du + f \right) = 0 \]
However, if the coefficients are not smooth, examples show that there may not be any solution $u$ of the equation (1.1) which is in $C^1(G)$, let alone $C^2(G)$; in such cases, of course, it is not legitimate to write the differential equation (1.3).

Equations of the form (1.1) with "rough" coefficients arise in attempting to prove the differentiability of the solutions of variational problems. For example, suppose that a function $z$ minimizes an integral of the form

$$I(z, G) = \int_G f[x, z(x), \nabla z(x)] \, dx \quad (x = (x^1, \ldots, x^N))$$

(1.4)

$$\nabla z(x) = \text{grad } z(x) = \{z'_1(x), \ldots, z'_N(x)\}$$

among all admitted functions having the same boundary values (in a generalized sense). Then, if the function $f(x^1, \ldots, x^N, z, p_1, \ldots, p_N)$ is of class $C^2$ in its arguments and satisfies a set of inequalities, too long to write here (but see [7]), one can proceed as follows: first, if $\zeta$ is any Lipschitz function with compact support, then $z + \lambda \zeta$ has the same boundary values as $z$ for any $\lambda$ and so the function

$$\varphi(\lambda) = \int_G f[x, z(x) + \lambda \zeta(x), \nabla z(x) + \lambda \nabla \zeta(x)] \, dx$$

(1.5)

has a minimum for $\lambda = 0$ and the first step in the derivation of the Euler equation can be carried through to yield

$$\varphi'(0) = 0 = \int_G \left( \sum_{\alpha=1}^{N} \zeta_\alpha f_{p_\alpha} + \zeta f_p \right) \, dx$$

(1.6)

for any Lipschitz function $\zeta$ with compact support; here

$$f_{p_\alpha} = f_{p_\alpha}[x, z(x), \nabla z(x)], \text{ etc.} \quad \text{and} \quad f_p = \partial f / \partial p_\alpha, \text{ etc.}$$

At this point, all we know about $z$ from the existence theory (see the notes [7] referred to above) is that it belongs to some space $H^1_p(G)$. To obtain more information about $z$, we next apply a difference-quotient procedure to the equation (1.6) as follows: let $\zeta$ be any Lipschitz function with compact
support in \( G \). Then there is an \( h_0 > 0 \) and a \( D' \) with \( D' \subset G \) such that the support of the function \( \zeta(x - he_\gamma) \subset D' \) for all \( h \) with \( 0 < |h| < h_0 \) and each \( \gamma \), \( 1 \leq \gamma \leq \nu \), \( e_\gamma \) being the unit vector in the \( x_\gamma \) direction.

For a fixed \( \gamma \) and \( h \), \( 0 < |h| < h_0 \), let us define

\[
\zeta_h(x) = h^{-1} [\zeta(x - he_\gamma) - \zeta(x)], \quad z_h(x) = h^{-1} [z(x + he_\gamma) - z(x)].
\]

If \( \zeta_h \) is inserted in (1.6), if next the integral is written as a sum of two integrals one for each term in \( \zeta_h \), if then the obvious change of variables is made in the integral involving \( \zeta(x - he_\gamma) \), and if finally the integrals are recombined, one obtains

\[
\int_G \left[ \sum_{\alpha=1}^\nu \zeta_\alpha(x) A^{\alpha}_h(x) + \zeta(x) B_h(x) \right] dx = 0,
\]

where

\[
(1.7) \quad A^{\alpha}_h(x) = \int_{p_{\alpha}} [x + he_\gamma, z(x + he_\gamma), p(x + he_\gamma)] - \int_{p_{\alpha}} [x, z(x), p(x)],
\]

\[
p(x) = \{p_1(x), \cdots, p_\nu(x)\}, \quad p_{\alpha}(x) = z_\alpha(x),
\]

\( B_h \) being the corresponding difference of \( f_z \). Using the integral form of the theorem of the mean, we may write

\[
A^{\alpha}_h(x) = \sum_{\beta=1}^\nu a^{\alpha\beta}_h(x) z_{h,\beta}(x) + b^{\alpha}_h(x) z_h(x) + c^{\alpha}_h(x),
\]

\[
B_h(x) = \sum_{\alpha=1}^\nu c^{\alpha}_h(x) z_{h,\alpha}(x) + d_h(x) z_h(x) + e_h(x),
\]

\[
(1.8) \quad a^{\alpha\beta}_h(x) = \int_0^1 \int_{p_{\alpha}p_{\beta}} [x + he_\gamma, z(x) + t\Delta z, p(x) + t\Delta p] dt,
\]

\[
\Delta z = z(x + he_\gamma) - z(x), \quad \Delta p = p_\varepsilon(x + he_\gamma) - p_\varepsilon(x)
\]

for almost every \( x \); of course the other coefficients are given by corresponding formulas. Since we have the solution, we may regard the coefficients as known and we see that \( z_h \) satisfies an equation of the form (1.1); but of course the coefficients are known only to be measurable and, in the general
cases considered in the Notes [7], are not even known to be bounded. However, in case $f$ "has degree $2k$ at infinity", i.e., satisfies

$$mW^k - K \leq f(x, z, p) \leq M^k, \quad V = 1 + z^2 + \sum_{\alpha} p_{\alpha}^2 \quad 0 < m \leq M, \quad k \geq 1$$

and the other inequalities in equation (3.1) of the Notes [7], it is possible, by using interior boundedness properties something like those proved in § 3, to show that we may let $\varepsilon \to 0$ and conclude that the derivatives $p_{\gamma} \equiv z_{\gamma}$ and the function $U = V^{k/2}$ (see (1.9)) $\in H^1(D)$ for each $D$ with $\overline{D} \subset G$ and that the derivatives $p_{\gamma}$ satisfy the differentiated equations

$$\int_D V^{k-1} \left\{ \sum_{\alpha, \beta} \frac{\zeta_{\alpha \beta}}{\beta} p_{\beta, \gamma} + b_{\alpha} p_{\gamma} + V^{1/2} e^{\alpha \gamma} \right\} \, dx = 0, \quad \gamma = 1, \ldots, \nu,$$

(1.10)

$$V^{k-1} a_{\alpha \beta} = f_{p_{\beta, \gamma}}; \quad V^{k-1} b_{\alpha} = V^{k-1} c_{\alpha} = f_{p_{\alpha}}; \quad V^{k-1} d = f_{zz}; \quad V^{k-1/2} e^{\alpha \gamma} = f_{p_{\alpha} x^\gamma}; \quad V^{k-1/2} f_{\gamma} = f_{zz \gamma};$$

and the coefficients $a_{\alpha \beta}, b_{\alpha}, c_{\alpha}, d, e^{\alpha \gamma},$ and $f_{\gamma}$ are all bounded and measurable and satisfy

$$m \sum_{\alpha = 1}^{\nu} \lambda_\alpha^2 < \sum_{\alpha, \beta = 1}^{\nu} a_{\alpha \beta}(x) \lambda_\alpha \lambda_\beta, \quad m > 0,$$

(1.11)

$$\sum [(a_{\alpha \beta})^2 + (b_{\alpha})^2 + (c_{\alpha})^2 + d^2] \leq m^2.$$

By setting $\zeta = V^{-c} \psi_{p_{\gamma}}$ ($\psi$ Lipschitz with compact support in $D$) in equations (1.10) (this $\zeta$ is not Lipschitz but technical lemmas allow its use) and summing with respect to $\gamma$, it can be shown that the function $U = V^{k/2}$ mentioned above has the following property:

There is a number $\lambda, 1 < \lambda < 2$, such that the function $W = U^\lambda$ satisfies the differential inequality

$$\int_D \left[ \sum_{\alpha} \psi_{\alpha} (\sum_{\beta} a_{\alpha \beta} W_{\beta} + b_{\alpha} W) + \psi_{\alpha} (\sum_{\alpha} c_{\alpha} W_{\alpha} + dW) \right] \, dx \leq 0$$

(1.12)
for all $\psi \in \text{Lip}_c(G)$ with $\psi(x) \geq 0$, $\text{Lip}_c(G)$ denoting the set of Lipschitz functions having compact support in $G$.

This inequality is interesting, because if all functions are smooth, it is equivalent to the inequality

$$\int_G \psi \left( \sum \frac{\partial}{\partial x} (\Sigma a^{\alpha \beta} \omega^{\beta} + b^{\alpha} \omega) - (\Sigma c^{\alpha} \omega^{\alpha} + d\omega) \right) dx \geq 0$$

for all $\psi \geq 0$ from which one concludes that

$$(1.13) \quad \sum \frac{\partial}{\partial x} (\Sigma a^{\alpha \beta} \omega^{\beta} + b^{\alpha} \omega) - (\Sigma c^{\alpha} \omega^{\alpha} + d\omega) \geq 0$$

In case $a^{\alpha \beta} = \delta^{\alpha \beta}$ (the Kronecker delta) and $b^{\alpha} = c^{\alpha} = d = 0$, (1.13) implies that $W$ is sub-harmonic. In § 4, it is shown (not quite in all detail) using the method of MGSR [8] that $U$ is bounded on each domain $\Omega$ with $\Omega \subset G$. This implies that $z$ is Lipschitz and that all the $p_Y$ are bounded on such $\Omega$. Then since $V$ is bounded, we see that the equations (1.10) assume the form (1.1); in fact, we may absorb the terms $b^{\alpha} p_Y$ and $d p_Y$ into $e^{\alpha}$ and $f$, respectively. For such equations, we show in § 5 (not incomplete detail) that the solutions $p_Y$ are Hölder continuous on interior domains. Then, if the second derivatives of $f$ are Hölder continuous, the coefficients in (1.10) are Hölder continuous and this implies that the derivatives $p_Y^{\alpha \beta}$ are Hölder continuous, so that the second derivatives of $z$ are Hölder continuous (in the case $\nu = 2$, this result has been known a long time (see [1])). Higher differentiability can be deduced by repeating the difference-quotient procedure and using the other theorems.

Some of the techniques used in studying the equations (1.1) are useful even the coefficients are smooth. For example one of the simplest ways of proving the existence of the solutions of the equation (1.13) is to show first the existence of the solutions of the corresponding equations (1.1), using the result of § 2, and then applying the difference-quotient procedure illustrated above together with the interior boundedness theorem of § 3 to show that the second derivatives of $u$ are in $L_2(\Omega)$. Then the Hölder continuity theorems of § 5 show that the first derivatives of $u$ are Hölder continuous and the classical results show that the second derivatives are.

We are presenting these results before this seminar with the hope that the
solutions will satisfy some or all of the axioms of abstract potential theory. We believe, however, that the theorems which are necessary for this purpose have not all been proved.

2. The existence theory.

We shall consider the equations (1.1) in which the coefficients $a^\alpha $, $b^\alpha $, $c^\alpha $, and $d$ satisfy (1.11) but we shall allow $e^\alpha $ and $f$ to be in $L_2$; the domain $G$ will always be bounded. We shall not assume that $c^\alpha = b^\alpha $ or that $a^\beta = a^\beta $; this makes no difference in the proofs and is useful in studying certain non-linear equations of the form

$$\sum_{\alpha =1}^\nu \frac{\partial}{\partial x^\alpha } a^\alpha (x, z, \nabla z) = B(x, z, \nabla z),$$

which have the same form as the Euler equations for the integral (1.4) except that we do not assume that $A^\alpha = f^\alpha $ and hence don't assume that $\partial A^\alpha /\partial p^\alpha = \partial A^\beta /\partial p^\alpha $.

We are looking for a solution $u$ of (1.1) which "has given boundary values". This has, originally, to be interpreted (since we have abandoned continuity in using the spaces $H^1_2$) to mean that $u - u^* \in H^1_2(G)$, $u^*$ being a given function $H^1_2(G)$ and $H^1_2(G)$ denoting the closure in $H^1_2(G)$ of the set Lipschitz functions with compact support in $G$. This implies that $u \in H^1_2(G)$ and so has finite Dirichlet integral. Smoothness on the interior and at the boundary is considered later, but we shall not present any such results here. But solutions of (1.1) will be shown in § 3 and § 4 to have boundedness properties on interior domains even if they are not in $H^1_2$ over the whole domain $G$.

It is clear that our problem may be reduced, by setting $U = u - u^*$ ($u^*$ given), to that where our desired solution $H^1_2(G)$; the resulting terms involving $u^*$ can be absorbed into the non-homogeneous terms $e^\alpha $ and $f$. Moreover, for well-known reasons, we shall modify (1.1) by allowing the functions $u$, $v$, $e^\alpha $, and $f$ to be complex-valued (keeping the others real), replacing $v$ by its conjugate $\overline{v}$, and adding the term $\lambda \overline{u} v$ in the integral ($\lambda$ complex). This last has the effect, in the smooth case, of replacing the equation (1.3) by $Lu - \lambda u = \varphi$, where $Lu$ and $\varphi$ denote the left and right sides of that equation.
We now define

\[ B(u, v) = B_1(u, v) + B_2(u, v); \quad C(u, v) = \int_G uv \, dx \]
\[ B_1(u, v) = \int_G \sum_{\alpha, \beta} \bar{v} \alpha a_{\alpha \beta}^u \bar{u} \beta \, dx \]
\[ B_2(u, v) = \int_G [\sum_{\alpha} (b^\alpha \bar{v}, \alpha u - c^\alpha u, \alpha \bar{v}) + duv] \, dx \]
\[ L(v) = -\int_G (\sum_{\alpha} \bar{e}^\alpha \bar{v} \alpha - f \bar{v}) \, dx. \]

We shall assume that \( u \) and \( v \in H^{\frac{1}{20}}(G) \) and shall use as inner product, the expression

\[ ((u, v)) = \int_G \sum_{\alpha} \bar{u} \alpha \bar{v} \alpha \, dx. \]

This is legitimate since we have (see the Notes [7]).

**Lemma 2.1 (Poincaré's inequality).** - If \( G \subset B(x_0, R) \), then

\[ \int_G |u|^2 \, dx \leq (\frac{R^2}{2}) \int_G \sum_{\alpha} |u, \alpha|^2 \, dx, \quad u \in H^{\frac{1}{20}}(G) \]

here \( B(x_0, R) \) denotes the ball with center \( x_0 \) and radius \( R \).

In terms of this notation our altered equations (1.1) become

\[ B(u, v) + \lambda C(u, v) = L(v), \quad v \in H^{\frac{1}{20}}(G) \]

that (2.4) holds for all \( v \) in \( H^{\frac{1}{20}}(G) \) if it does for all \( v \in Lip_c(G) \) is evident from the fact that \( Lip_c(G) \) is dense in \( H^{\frac{1}{20}}(G) \).

We first prove:

**Theorem 2.1.** - There is a real number \( \lambda_0 \) such that
where \(|u_1|\) and \(|u_0|\) denote the norms in \(H^1_{20}(G)\) and \(L_2(G)\), respectively; we may take \(\lambda_0 = M(1 + 2M/m)\).

**Proof.** - If we write \(u = u_1 + iu_2\), the first inequality follows from (1.11), since

\[
\text{Re} B_1(u, u) = \int_G \sum_{\alpha, \beta} a_{\alpha, \beta} (u_1, \alpha u_1, \beta + u_2, \alpha u_2, \beta) \, dx
\]

The second and third inequalities are immediate consequences of (1.11) and the Schwarz inequality (and the fact that \(\sum_{\alpha} b_{\alpha} v_{\alpha} = (\sum_{\alpha} |b_{\alpha}|^2)^{1/2} (\sum_{\alpha} |v_{\alpha}|^2)^{1/2}\), etc.). The fourth follows by setting \(v = u\) in the third inequality and using the Cauchy inequality

\[
2|u_1| \cdot |u_0| \leq \varepsilon |u_1|^2 + \varepsilon^{-1} \cdot |u_0|^2, \quad \varepsilon = M/2m
\]

**Theorem 2.2 (Lema of Lax and Milgram) [2].** - Suppose, in a Hilbert space \(H\), \(B_0(u, v)\) is linear in \(u\) for each \(v\) and conjugate linear in \(v\) for each \(u\) and suppose

\[
|B_0(u, v)| \leq M_1 |u| \cdot |v|
\]

\[
|B_0(u, u)| \geq m_1 |u|^2, \quad m_1 > 0
\]

Suppose the transformation \(T_0\) is defined by the condition

\[
B_0(u, v) = (T_0 u, v)
\]
Then $T_0$ and $T_0^{-1}$ are operators with bounds $M_1$ and $m_1^{-1}$, respectively.

Proof. It is clear that $T_0$ is a linear operator with bound $M_1$. From (2.6) (ii) and (2.7), we see that

$$m_1\|u\|^2 \leq |B_0(u, u)| = |(T_0 u, u)| \leq \|u\| \cdot \|T_0 u\|$$

so that

$$\|T_0 u\| \geq m_1 \|u\|$$

It follows easily that the range of $T_0$ is closed. If the range were not the whole space, there would be a $v$ such that $B_0(u, v) = (T_0 u, v) = 0$ for every $u$. But, by setting $u = v$, it follows from (ii) that $v = 0$. Thus $T_0^{-1}$ is a bounded operator with norm $\leq m_1^{-1}$.

THEOREM 2.3. Suppose the transformation $U$ is defined on $H^1_{20}(G)$ by the condition that

$$(2.8) \quad C(u, v) = ((Uu, v))_{20}^1, \quad v \in H^1_{20}(G).$$

Then $U$ is a completely continuous operator.

Proof. That $U$ is an operator follows from Poincaré's inequality (lemma 2.1) since

$$\|Uu\| = \sup (Uu, v) = \sup_G uv \, dx \leq 2^{-1} \|v\| \quad \text{if} \quad \|v\| = 1.$$

Next, suppose $u_n \to u$ (locale convergence) in $H^1_{20}(G)$. Then $u_n \to u$ (strongly) in $L_2$ ([7], theorem 1.10 (d)) and

$$\|U(u_n - u)\| = \sup \int_G (u_n - u) \, \overline{v} \, dx \leq 2^{-1/2} R \|v\| \cdot \|u_n - u\|_0 \to 0.$$

so that $U$ is compact.
THEOREM 2.4. - If \( \lambda \) is not in a set \( \mathbb{C} \), which has no limit points in the plane, the equation (2.4) has a unique solution \( u \) in \( H^1_0(G) \) for each given \( e \) and \( f \) in \( L^2(G) \). If \( \lambda \in \mathbb{C} \), there are solutions of (2.4) in which \( u \neq 0 \) and \( e = f = 0 \), but the manifold of these is finite dimensional. If \( \lambda_0 \) is defined as in theorem 2.1, then no real number \( \lambda^*_1 > \lambda_0 \) is in \( \mathbb{C} \).

Proof. - Let us define \( \lambda_0 \) as in theorem 2.1 and \( B_0 \) by

\[
B_0(u, v) = B(u, v) + \lambda_0 C(u, v)
\]

and define \( T_0 \) by (2.7). Then, equation (2.4) is equivalent to

\[
T_0 u + (\lambda - \lambda_0) Uu = w, \quad \text{where} \quad ((w, v)) = L(v)
\]

\( L \) being a linear functional. Moreover, from theorem 2.1, it follows that \( B_0 \) satisfies the conditions of the lemma of Lax and Milgram with \( m_1 = m/2 \). Accordingly \( T_0 \) has a bounded inverse so (2.10) is equivalent to

\[
\begin{align*}
\frac{1}{T_0^{-1} U} & \quad \frac{1}{T_0} u \quad \frac{1}{T_0^{-1} w} \quad \frac{1}{T_0^{-1} u} \\
& \quad \frac{1}{T_0^{-1} u} \quad \frac{1}{T_0^{-1} w} \quad \frac{1}{T_0^{-1} u} \\
& \quad \frac{1}{T_0^{-1} u} \quad \frac{1}{T_0^{-1} w} \quad \frac{1}{T_0^{-1} u} \\
\end{align*}
\]

Since \( T_0^{-1} U \) is compact, the theorem follows from the Riesz theory of linear operators.

As an immediate consequence of the theorems of this section and Poincaré's inequality we obtain the following theorem:

THEOREM 2.5. (Local existence and uniqueness theorem). - There is an \( R_0 > 0 \), depending only on \( m \) and \( M \) such that if \( 0 < R < R_0 \) and \( G \subset B(x_0, R) \), then \( \lambda = 0 \) is not in the set \( \mathbb{C} \) of theorem 2.4 and, in fact if \( u \) is the solution of (2.4), then

\[
\|u\|_1 \leq 4m_1^{-1}(\|e\|_0 + R\|f\|_0)
\]

3. First interior boundedness and approximation theorems.

In this section, we continue to assume that the coefficients \( a, b, c, \) and \( d \) satisfy (1.11) and that \( G \) is bounded and \( G \subset B(x_0, R) \).
THEOREM 3.1 (Interior boundedness in $H^1_2$). - Suppose that $u$, $e$ and $f \in L^2(G)$ and that $u \in H^1_2(D)$ for each $D$ with $D \subset G$ and satisfies (1.1) for each $\nu \in H^1_2(G)$ which vanishes in $G - D$ for some $D \subset G$. Then there is a constant $C$ depending only on $m$ and $M$ (and $R$) such that

$$\|\nu_u\|_{0,D} \leq C[\delta^{-1} \|u\|_0 + \|e\|_0 + \delta \|f\|_0], \quad \delta \leq 1,$$

being the distance from $D$ to $\partial G$ (the boundary of $G$).

Notations. - $|\nu_u| \geq 0$, $|\nu_u|^2 = \sum |\nu_u, \alpha|^2$; $\|\nu_u\|_{0,D}$ denotes the $L^2$ norm of $|\nu_u|$ over $D$.

Proof. - Let us define $\eta(x) = 1$ on $D$, $\eta(x) = 1 - 2\delta^{-1} d(x, D)$ for $0 \leq d(x, D)$ (the distance of $x$ from $D$) $\leq \delta/2$, and $\eta(x) = 0$ otherwise; and let us define

$$v = \eta \nu, \quad U = \nu u$$

and substitute that $v$ in our equation (1.1) as altered to (2.4) with $\lambda = 0$ and take the real part. Then $U \in H^1_2(G)$ and we have

$$\overline{\nu_u, \alpha} = \eta(\overline{u, \alpha} + \eta, \alpha \overline{u}) \quad \eta u, \beta = U, \beta - \eta, \beta u$$

and our equation becomes

$$(3.2) \quad 0 = \text{Re}B(U, U) + \text{Re} \int_G \sum [a^\alpha \eta, \alpha \overline{\nu_u, \beta} - \overline{U, \alpha} \eta, \beta u] - a^\alpha \eta, \alpha \eta, \beta uu + \eta u, \alpha U, \alpha + \eta, \alpha \eta \overline{u} \alpha U + \eta, \alpha \eta \overline{u} \alpha - c^\alpha \eta, \alpha \overline{U} \alpha + \overline{U} \alpha]$$

Using theorem 2.1 for $U$ and relations like

$$|\sum a^\alpha \overline{\varphi, \alpha} \varphi, \beta| \leq M(\overline{\varphi, \alpha}^2)^{1/2} (\overline{\varphi, \alpha}^2)^{1/2}$$

and then the Cauchy inequality to eliminate the terms involving the $U, \alpha$ in the remaining integral, we see that
from which the theorem follows easily.

**THEOREM 3.2** (Approximation theorem). - Suppose that the coefficients \( a_n, b_n, c_n, d_n \) satisfy (1.11) for each \( n \) on \( G \) and converge almost everywhere on \( G \) to \( a, b, c, d \), respectively, and suppose that \( e_n \to e \) and \( f_n \to f \) in \( L_2(G) \). Suppose that \( u_n \to u \) in \( H_2^1(G) \) and that \( u_n \) is a solution of (1.1) for each \( n \). Then \( u \) is a solution of (1.1).

**Notation.** - \( \to \) denotes weak convergence.

**Proof.** - For each \( v \in H_2^0(G) \), we see that

\[
\begin{align*}
& a_n^{\alpha \beta} v^\alpha \to a^{\alpha \beta} v^\alpha, \\
& b_n^\alpha v^\alpha \to b^\alpha v^\alpha, \\
& c_n^\alpha v^\alpha \to c^\alpha v^\alpha, \\
& d_n v^\alpha \to d v^\alpha, \\
& \text{etc.}
\end{align*}
\]

in \( L_2(G) \) (\( \to \) denotes strong or ordinary convergence) so that

\[
B(u_n, v) \to B(u, v), \quad C(u_n, v) \to C(u, v), \quad \text{and} \quad L_n(v) \to L(v).
\]

4. Interior boundedness.

Suppose that a function \( u \in H_2^1(B(x_0, R)) \). Then there is a lemma of SOBOLEV ([9], [7]) which states that \( u \in L_2(B(x_0, R)) \) and that

\[
(4.1) \quad \int_{B(x_0, R)} |u(x)|^{2s} \, dx \frac{1}{s} \leq C_0 \int_{B(x_0, R)} [||u||^2 - R^{-2} |u|^2] \, dx,
\]

where \( 1 \leq s \leq \nu/(\nu - 2) \) if \( \nu > 2 \), \( s \geq 1 \) if \( \nu = 2 \).

In the case \( \nu = 2 \), \( u \) still need not be bounded. The function \( U \), mentioned in § 1, \( \in L_2(G) \) and to \( H_2^1(D) \) for each \( D \) with \( \overline{D} \subset G \). If \( U \) also satisfies the conditions near (1.12), then \( U^s \in L_2(D) \) and it turns out that we can conclude that \( U^s \in H_2^1(\Delta) \) for each \( \Delta \) with \( \Delta \subset D \). Indeed, it is possible to prove the following lemma:
LEMMA 4.1. - Suppose that \( U \) is real, \( U(x) > 1 \), and satisfies the underlined
conditions near and including \((1.12)\) and, in addition that

\[
w = U^\tau \in L_2[B(x_0, R + a)]
\]

for some \( \tau > 1 \) where we assume that \( B(x_0, R + a) \subset G \) and \( 0 < a < R \). Then

\[
\int_{B(x_0, R)} |\nabla w|^2 \, dx \leq c_1 \tau^2 a^{-2} \int_{B(x_0, R-a)} w^2 \, dx, \quad c_1 > 1
\]

where \( c_1 \) depends only on \( \nu, m, M \), and \( \lambda \).

Proof. - A technical lemma allows us to substitute

\[
\psi = \eta^2 U^{2-\lambda} U_L^{2\tau-2}
\]

in \((1.12)\), \( U_L \) being the truncated function, defined by

\[
U_L(x) = U(x) \text{ if } U(x) < L, \quad U_L(x) = L \text{ if } U(x) \geq L
\]

and \( \eta \) being defined by \( \eta(x) = 1 \) on \( B(x_0, R) \), equal to \( a^{-1}(|x - x_0| - R) \)
for \( R \leq |x - x_0| < R + a \) and 0 otherwise. Since \( U_L, \alpha = 0 \) almost everywhere
on \( E_L \), the set where \( U(x) \geq L \), we see that

\[
\psi, \alpha = \eta^2 U^{1-\lambda} U_L^{2\tau-2} \left[ (2 - \lambda) U_L + (2\tau - 2) U_L, \alpha \right] + 2\eta \alpha U^{2-\lambda} U_L^{2\tau-2}
\]

the inequality \((1.12)\) becomes (again using \( \nabla U_L = 0 \) on \( E_L \)):

\[
\int_G \left[ \eta^2 U_L^{2\tau-2} \left[ (\lambda(2 - \lambda) \nabla U_L a \cdot \nabla U + (2 - \lambda) \nabla b \cdot \nabla U - \lambda U_L b \cdot \nabla U_L \right] + dU^2 \\
+ (2\tau - 2)(\lambda \nabla U_L a \cdot \nabla U_L + U_L b \cdot \nabla U_L) + 2\eta U_L^{2\tau-2} \nabla \eta(\lambda a \cdot \nabla U + b) \right] \, dx = 0
\]

where we have abbreviated \( \sum U_L a^\alpha b^\beta U_L, \alpha \) to \( \nabla U_L a \cdot \nabla U, b \cdot \nabla U_L, \) etc.
Using the bounds for the coefficients and the inequalities of Cauchy and Schwarz as usual, we conclude that

\[
(4.5) \quad \int_G \eta^2 u_L^{2\tau-2} \left[ |\nabla u|^2 + (\tau - 1) |u_L|^2 \right] \, dx \\
\leq C_1 \int_G \left[ \eta^2 u_L^{2\tau-2} u^2 + |\nabla \eta|^2 u^2 u_L^{2\tau-2} \right] \, dx.
\]

If we now set \( u_L = \eta u_L^{p-1} \), we find (again using \( \nabla u_L = 0 \) on \( E_L \)) that

\[
(4.6) \quad \nabla u_L = u u_L^{p-1} \nabla \eta + \eta u_L^{p-1}[\nabla u + (\tau - 1) \nabla u_L].
\]

It follows from (4.5) and (4.6) that

\[
(4.7) \quad \int_{B(x_0, R+a)} |\nabla u_L|^2 \, dx \leq C_2 \tau^2 \int_{B(x_0, R+a)} (\eta^2 + |\nabla \eta|^2) u^2 u_L^{2\tau-2} \, dx.
\]

Since \( u \in L^2(B(x_0, R+a)) \), we may let \( L \to \infty \) to obtain our result.

**Theorem 4.1.** Suppose \( U \) satisfies the hypotheses of lemma 4.1 with \( \tau = 1 \). Then \( U \) is bounded on each domain \( \overline{D} \subset G \) and

\[
(4.8) \quad |U(x)|^2 \leq C a^{-\nu} \int_{B(x_0, R+a)} |U(y)|^2 \, dy, \quad x \in B(x_0, R),
\]

\[
0 < a \leq R, \; B(x_0, R+a) \subset G, \; \nu > 2
\]

where \( C \) depends only on \( \nu, m, M, \) and \( \lambda \).

**Remark.** If \( \nu = 2 \), \( U \) is still bounded on interior domains but the inequality in (4.8) must be replaced by

\[
(4.8') \quad |U(x)|^2 \leq C(\epsilon) a^{-\nu-\epsilon} \int_{B(x_0, R+a)} |U(y)|^2 \, dy, \quad \nu = 2.
\]

This result is not good enough to obtain the results in the next section. However, the writer proved the results in the next section in the case \( \nu = 2 \) for
more general systems of equations many years ago [3], [4]. A simplified version of this old work appears in [6], chapter 4. So in the next section, we assume \( \nu > 2 \).

Proof. - Let us define

\[
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s = \sqrt{\nu - 2}, \quad n = 0, \quad v_n = u^{2^n}, \quad B_n = B(x_0, R + 2^{-n} a), \quad w_n = \int_{B_n} w_n^2 \, dx
\]

Using the lemma we conclude in turn that \( w_1 = w_0 \in H^1(B_1), \quad v_1 \in H^1_2(B_2), \)

\( w_2 = w_1 \in L^2(B_2), \quad w_2 \in H^1_2(B_3), \) etc. Then, using the inequalities (4.1) and (4.2) with \( s = \frac{n}{2} \) and \( a \) replaced by \( 2^{-n} a \), we obtain the recurrence relation

\[
w_n^1 \leq C_0 \int_{B_n} \left( |\nabla w_n| + R^{-2} w_n^2 \right) \, dx
\]

\[
\leq 2C_0 C_1 s^{2^2 - 2} a^{2} \int_{B_n} w_{n-1}^2 \, dx = K_0 K_1 w_{n-1}
\]

\[
K_0 = 2C_0 C_1 s^2 a^2, \quad K_1 = 4s^2
\]

From this recurrence relation for each \( n \), we conclude that

\[
w_n^1 \leq K_0 K_1 w_0 = C a^{\nu} w_0, \quad \alpha = (1 - s^{-1})^{-1} = \nu / 2, \quad \beta = \alpha^2
\]

The theorem follows by letting \( n \to \infty \).

5. Hölder continuity of the solutions.

In this section we shall assume that \( \nu > 2 \) (see the remark after theorem 4.1) and shall restrict ourselves to the special equations

\[
(5.1) \int_G \sum(\zeta, \alpha \beta u, - \zeta c \alpha u, \alpha) \, dx = \int_G (\nabla \zeta \cdot a - \nabla u \cdot c - \nabla u) \, dx = 0
\]

(5.2)

\[
\int_G [\nabla \zeta (a \cdot \nabla u + e) + \zeta (c \cdot \nabla u + f)] \, dx = 0
\]
It was pointed out in the introduction that the type (5.2) with \( e \) and \( f \) bounded is sufficient for the application to the calculus of variations. The general equations (1.1) have been treated in [5] by a somewhat longer method.

We need the following two generalizations of Poincaré's inequality:

**Lemma 5.1.** There are constants \( C_1(\nu) \) and \( C_2(\nu, c) \) such that

\[
\int_{B(x_0, R)} |u|^2 \, dx \leq C_1 R^2 \int_{B(x_0, R)} |\nabla u|^2 \, dx \quad \text{if} \quad \int_{B(x_0, R)} u \, dx = 0
\]

\[
\int_{B(x_0, R)} |u|^2 \, dx \leq C_2 R^2 \int_{B(x_0, R)} |\nabla u|^2 \, dx \quad \text{if} \quad |S| > c|B(x_0, R)|, \quad c > 0,
\]

for all \( u \in H^1_2[B(x_0, R)] \); here \( S \) is the set of \( x \) where \( u(x) = 0 \) and \( |S| \) is its measure.

**Proof.** It is sufficient to prove these for \( R = 1 \) and \( x_0 = 0 \). We prove the second, the first is proved similarly. Suppose the second is false. Then there exists a sequence \( \{u_n\} \) with \( \|u_n\|_1 \) (the full norm in \( H^1_2 \)) = 1 such that

\[
|S_n| > c|B(0, 1)|
\]

and

\[
(5.3) \quad \int_{B(0, 1)} |u_n^2| \, dx > n \int_{B(0, 1)} |\nabla u_n|^2 \, dx.
\]

We may assume that \( u_n \to u \) in \( H^1_2 \) ([7], theorem 1.10 b) so that \( u_n \to u \) in \( L^2 \) ([7], theorem 1.10 d). From (5.3) we conclude that \( \nabla u_n \to 0 \) in \( L^2 \), so that \( u_n \to u \) in \( H^1_2 \). Then \( u \) must be a constant \( d \) ([7], theorem 1.1) \( \neq 0 \) since \( \|u\|_1 = 1 \). But then

\[
0 = \lim_{n \to \infty} \int_{B(0, 1)} |u_n - u|^2 \, dx \geq \lim_{n \to \infty} \int_{S_n} |u_n - u|^2 \, dx \geq \lim d^2 |S_n|
\]

which is a contradiction.

**Definition.** A function \( v \in H^1_2(D) \) for each \( D \) with \( \overline{D} \subset G \) is a subsolution of (5.1) if and only if
\[ \int_{G} (\nabla \zeta \cdot a \nabla v + \zeta c \cdot \nabla v) \, dx < 0 \text{ for each } \zeta \in \text{Lip}_c(G), \ \zeta(x) > 0 \]

Remarks. - This condition is formally equivalent to the condition

\[
\frac{d}{dx^\alpha} e^{\alpha \beta} v, \beta - c^\alpha v, \alpha > 0
\]

**Lemma 5.2.** - Suppose that

(i) \( F \) is non-negative and convex on the interval \((0, \infty)\),
(ii) \( H = -e^{-F} \) is convex on that interval,
(iii) \( u \) is a non-negative solution of \((5.1)\) on \( G \),
(iv) \( v(x) = F[u(x)] \), and
(v) \( v \in L_2(G) \).

Then \( v \) is a sub-solution of \((5.1)\) on \( G \) and

\[
\int_{D} |\nabla v|^2 \, dx \leq C \alpha^{-2} |G| \text{ if } D \subset G_a, \ G \subset B(x_1, R)
\]

where \( C \) depends only on \( \nu, m, M, \) and \( R \) and \( G_a \) is the set of \( x \) in \( G \) such that \( B(x, a) \subset G \).

Proof. - First, we assume that

\[ H \in C^2(0, \infty), \ -1 \leq H(u) \leq -\epsilon \quad (\epsilon > 0) \]

and that \( H'' \) is bounded on \((0, \infty)\). Then \( F \in C^2(0, \infty) \), and \( F, F', F'' \) are bounded there with \( F''(u) > [F'(u)]^2 \). Let us set \( \zeta = \eta^2 F'(u) \) in equation \((5.1)\), where \( \eta \) is defined as usual. It follows that

\[
0 = \int_{G} [2\eta \nabla \eta \cdot a \nabla v + \eta^2 F''(u) \nabla u \cdot a \cdot \nabla u + \eta^2 c \cdot \nabla v] \, dx
\]

\[
> \int_{G} [\eta^2 (\nabla v \cdot a \cdot \nabla v + c \cdot \nabla v) + 2\eta \nabla \eta \cdot a \cdot \nabla v] \, dx
\]

since \( F'' > (F')^2 \). Finally
from which the inequality follows easily.

In the general case, $H$ is convex with $-\leq H(u) < 0$ on $(0, \infty)$. It is easy to see that $H$ can be approximated from below by functions $H_n$ having the properties in the preceding paragraph. It follows that the functions $v_n(x) \to v(x)$ from below and hence strongly in $L^2(G)$. Clearly, also, $V_n \to v$ in $H^1(D)$ for each $D$ with $D \subset G$, on account of the inequality (5.4) which holds for each $n$. The inequality holds in the limit by lower-semicontinuity.

**Theorem 5.1 (Harnack type).** Suppose that

(i) $u$ is a non-negative solution of (5.1) on $B_{2R} = B(x_0, 2R)$ and
(ii) the set $S$ where $u(x) > 1$ has measure $\geq c_1 |B_{2R}|$, $c_1 > 0$. Then

$$u(x) \geq c_2 > 0 \text{ for } x \in B_R$$

where $c_2$ depends only on $n$, $m$, $N$, and $c_1$.

**Proof.** There is a $k$, $1 < k < 2$, such that $|B_{2R} - B_{kR}| = (1/2)c_1 |B_{2R}|$. Then $|S \cap B_{kR}| \geq (1/2)c_1 |B_{kR}|$. Let us define $F(u) = \max[- \log(u + \epsilon), 0]$, where $0 < \epsilon < 1$. It is easy to see that $F$ satisfies the hypotheses of lemma 5.1. Consequently

$$\int_{B_{kR}} |\nabla v|^2 \, dx \leq C_1 R^{n-2} \text{ where } v(x) = F[u(x)]$$

Since $v(x) = 0$ on $S$ and $|S \cap B_{kR}| \geq (c_1/2)|B_{kR}|$, it follows from lemma 5.1 that

$$\int_{B_{kR}} v^2 \, dx \leq C_2 R^b$$

The theorem follows from this and theorem 4.1.

**Notation.** $u \in C^0_\mu(G)$ if and only if $u$ satisfies a uniform Hölder condition with exponent $\mu$ on $G$; $u \in C^0_\mu(D)$ if and only if $u \in C^0_\mu(D)$ for each $D$ with $\overline{D} \subset G$. 

\[(5.4) \quad \int_G \eta^2 |\nabla v|^2 \, dx \leq C \int_G (\eta^2 \sigma^2 + |\nabla \eta|^2) \, dx\]
THEOREM 5.2. - Suppose $u$ is a solution of (5.1) on $G$. Then $u \in C^{0}_{\mu_0}(G)$ where $0 < \mu_0 < 1$ and $\mu_0$ depends only on $\nu$, $m$, and $M$.

More precisely

$$|u(x) - u(x_0)| \leq C \delta^{-\tau} \left( |x - x_0|/R \right)^{\mu_0}, \quad x \in B(x_0, R),$$

where

$$L = \|u\|^{0}_{2,R+\delta}, \quad B(x_0, R + \delta) \subset G, \quad \tau = \nu/2, \quad \delta \leq R,$$

and $C$ depends only on $\nu$, $m$, and $M$.

Proof. - It is sufficient to prove the inequality. It follows from theorem 4.1 that

$$|u(x)| \leq C_0 L^{\delta^{-\tau}}, \quad x \in B_R = B(x_0, R).$$

Let us define $m^*$ and $M^*$ as the essential inf and sup of $u(x)$ on $B_R$ and let us choose $\bar{m}$ (unique) so that $|S^+| \leq |B_R|/2$, $S^+$ and $S^-$ being the sets of points $x \in B_R$ for which $u(x) > \bar{m}$ and $u(x) < \bar{m}$, respectively.

If $m^* < \bar{m} < M^*$, the functions $[m^* - u(x)]/(M^* - m^*)$ and $[u(x) - m^*]/(\bar{m} - m^*)$ satisfy the hypotheses of theorem 5.1 on $B_R$ with $c_1 = 1/2$. It follows that $m_1 \leq u(x) \leq M_1$ for $x \in B_{r/2}$, where

$$m_1 = \bar{m} - h(\bar{m} - m^*), \quad M_1 = \bar{m} + h(M^* - \bar{m}), \quad h = 1 - c_2 < 1,$$

c_2 being the constant of theorem 5.1 with $c_1 = 1/2$. The same results hold if $\bar{m} = m^*$ or $\bar{m} = M^*$ or both.

Now, let us define

$$\varphi(r) = [\text{ess sup } u(x)] - [\text{ess inf } u(x)] \quad \text{for } x \in B_r, \quad r \leq R.$$

We conclude from the preceding paragraph that
Thus

\[ \log \varphi(r) \leq \log S - \log h + (n + 1) \log h - \frac{\log h}{\log 2} \log \left( \frac{R}{r} \right) , \]

if \( n \log 2 < \log \left( \frac{R}{r} \right) < (n + 1) \log 2 \).

From this it follows that

\[ \varphi(r) \leq h^{-1} S \left( \frac{r}{R} \right)^{\mu_0} , \quad \mu_0 = - \frac{\log h}{\log 2} . \]

**THEOREM 5.3.** - There are constants \( R_1 > 0 \) and \( C \) which depend only on \( \nu, m, \) and \( N, \) such that

\[ \| \nabla u \|_{L^2}^{\nu} \leq C \left( \frac{r}{R} \right)^{\tau + 1 - \mu_0} , \quad 0 < r \leq R , \quad L = \| u \|_{L^2}^{\nu} , \quad \tau = \nu/2 . \]

for each \( R, 0 < R \leq R_1 , \) and each solution of (5.1) with \( \| \nabla u \|_{L^2}^{\nu} < + \infty . \)

**Proof.** - Evidently we may suppose that the average value of \( u = 0 \). From lemma 5.1, we conclude that

\[ \| u \|_{L^2}^{\nu} \leq CIR . \]

From theorem 5.2, we then obtain

\[ |u(x) - u(x_0)| \leq z_1 \| u \|_{L^2}^{\nu} \left( \frac{R}{2} \right)^{-\tau} \left| \frac{x - x_0}{R} \right|^{\mu_0} \]

\[ \leq z_2 \left( \frac{R}{2} \right)^{\mu_0 - 1 - \tau} |x - x_0|^{\mu_0} , \quad |x - x_0| \leq R/2 . \]

We define \( \eta \) as usual with \( a, G, \) and \( D \) replaced by \( r, B(x_0, 2r) \) and \( B(x_0, r) \), respectively, and put

\[ \zeta(x) = \eta^2 [u(x) - u(x_0)] , \quad x \in B(x_0, 2r) , \quad 0 < r \leq R/4 . \]
in (5.1). We obtain

$$0 = \int_{B_{2r}} \eta^2 [\nabla u \cdot \nabla u + c(u - u_0) \cdot \nabla u + 2\eta(u - u_0) \nabla \eta \cdot \nabla u] \, dx$$

The theorem follows easily by using (5.5) and the inequalities of Cauchy and Schwarz.

**Theorem 5.4.** Suppose that \( u \in H^1(G) \) and is a solution of (5.2) there, where \( f \) is bounded and \( e \in L^2(G) \) and satisfies

$$\int_{B(x_0, r)} |e|^2 \, dx \leq L^2(r/R)^{\nu-2+2\mu},$$

$$0 < \mu < \mu_0, \quad 0 \leq r \leq R \leq R_0 \quad \text{for every} \quad B(x_0, R) \subset G$$

\( R_0 \) being the number in theorem 2.5. Then \( u \in C^0_\mu(G) \) and, in fact, satisfies a condition of the form

$$\int_{B(x_0, r)} |\nabla u|^2 \, dx \leq K^2(r/R)^{\nu-2+2\mu}$$

for all \( x_0, r \) and \( R \) as above.

**Proof.** Let \( V \) be the potential of \( f \). It is well known that \( V \) is of class \( C^1 \) with \( |\nabla V| \leq C_p \max |f(x)| \) with \( |B(0, p)| = |G| \). Also

$$\int vf \, dx = \sum \int V \cdot \partial V \, dx$$

so equation (5.2) is equivalent to another such with \( f \equiv 0 \) and \( e \) replaced by \( \eta \nabla V \), which satisfies a condition (5.7) with a different \( L \). Moreover, by virtue of an old theorem of the writer ([7], theorem 1.12) it is sufficient to prove (5.8) for some \( K \).

Since we have assumed that \( R \leq R_0 \), we conclude from theorem 2.5 that \( u = U + H \) on \( B_R \equiv B(x_0, R) \), where \( U \) is the solution of (5.2) which \( e \in H^1_0(B_R) \) and \( H \) is the solution of (5.1) such that \( H - u \in H^1_0(B_R) \), and we also conclude that
(5.9) $||\nabla u||_R \leq z_1 ||e||_R \leq z_1 ||e||_G$, $||\nabla H||_R \leq z_2 ||\nabla u||_R \leq z_2 ||\nabla u||_G$,

where we have chosen a fixed ball $B(x_0, R) \subset G$ and will denote the $L_2$ norm of $\psi$ on $B_r = B(x_0, r)$ by $||\psi||_r$. Then it follows from theorem 5.3 that

$$||\nabla H||_r \leq C_3 ||\nabla u||_G (r/R)^{\tau-1+\mu_0}.$$ 

Now, let us define $\varphi(s) = L^{-1} \sup ||\nabla u||_{\mathbb{S}_s}$ for all $s$ which satisfy (5.7) with $L_1$ replaced by $L$, $R$ replaced by $S \leq R$, $U$ being the solution of (5.2) $\in H^1_{20}(B_S)$. Next, choose an arbitrary $s$ which satisfies (5.7) ($L_1$ replaced by $L$). We may write $U = U_S + H_S$ on $B_S$ where $U_S$ is the solution of (5.2) $\in H^1_{20}(B_S)$. Obviously $s$ satisfies

$$\int_B |\varphi|^2 \, dx \leq [L^2(S/R)^{\nu-2+2\mu}] (r/S)^{\nu-2+2\mu}, \quad 0 \leq r \leq S.$$ 

Thus, using the ideas of (5.9) and the definition of $\varphi$, we conclude that

$$||\nabla u||_S \leq Z_1 L(S/R)^{\tau-1+\mu}, \quad ||\nabla H||_S \leq Z_2 ||\nabla u||_S \leq Z_2 L\varphi(S/R).$$

Now, suppose that $0 < r < S < R$. Then

$$||\nabla u||_r \leq ||\nabla u||_r + ||\nabla H||_r \leq L(S/R)^{\tau-1+\mu} \varphi(r/S) + Z_3 L\varphi(S/R) (r/S)^{\tau-1+\mu_0}.$$ 

Since $e$ is arbitrary, we conclude (setting $s = r/R$, $t = S/R$) that

$$\varphi(s) \leq t^{\tau-1+\mu} \varphi(s/t) + Z_3 \varphi(t)(s/t)^{\tau-1+\mu_0}.$$ 

Obviously $\varphi$ is monotone and $\varphi(1) \leq Z_1$. So let us choose $\delta$, $0 < \delta < 1$. Then, obviously

$$\varphi(s) \leq S_0 s^{\tau-1+\mu}, \quad 0 \leq s \leq 1, \quad s_0 \leq Z_1 \delta^{\tau-1+\mu}.$$ 

Using (5.10) with $\delta^2 \leq s \leq \delta$ and $t = \delta^{-1} s$, we obtain
\( \varphi(s) \leq S_1 s^{\tau-1+\mu} \), where \( S_1 = S_0(1 + z_3 w) \), \( w = o^{\nu \mu} \).

Since \( S_1 > S_0 \), (5.11) holds for \( o^2 < s < 1 \). Using (5.10) with \( o^4 < s < o^2 \) and \( t = o^{-2} s \), we conclude that

\[ \varphi(s) \leq S_2 s^{\tau-1+\mu}, \quad o^4 < s < 1, \quad S_2 = S_0(1 + z_3 w)(1 + z_3 w^2) \]

By repeating the argument, we obtain

\[ \varphi(s) \leq S s^{\tau-1+\mu}, \quad 0 < s < 1, \quad S = S_0(1 + z_3 w)(1 + z_3 w^2)(1 + z_3 w^4) \ldots \]

from which the theorem follows immediately.

**BIBLIOGRAPHY**


