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S E M I N A I R E

D ' A N A L Y S E F O N C T I O N N E L L E

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SOME GEOMETRIC PROPERTIES OF FINITE DIMENSIONAL, SYMMETRIC BANACH SPACES

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We consider here two problems, the relationship of the projection constant $\lambda(E)$ and the Banach-Mazur distance $d(E,\ell_n^\infty)$ and the uniqueness of symmetric bases.

In [2] it was shown that for spaces E with unconditional bases, $ubc(E) = 1$, one has

$$\lambda(E) \le d(E,\ell_n^\infty) \le K_G^2 \lambda(E)^2 \quad .$$

On the other hand, we know [1], [3] that for spaces E with an unconditional basis $\{e_i\}_{i=1}^n$, $ubc(\{e_i\}_{i=1}^n) = 1$, and $\|\sum_{i=1}^n a_i e_i\| \le (\sum_{i=1}^n |a_i|^2)^{1/2}$ for all $a_i \in \mathbb{R}$, $i = 1,\ldots,n$ we have

$$(1) \qquad \lambda(E) \le d(E,\ell_n^\infty) \le \|\sum_{i=1}^n e_i\| \le \sqrt{2}\,\lambda(E) \quad .$$

Thus we ask whether in general the projection constant $\lambda(E)$ and Banach-Mazur distance $d(E,\ell_n^\infty)$ are up to a constant the same ? We get for symmetric spaces, $sbc(E) = 1$, a result that is very close to it :

$$\lambda(E) \le d(E,\ell_n^\infty) \le C\,(\log 2\lambda(E))^{7/2}\,\lambda(E) \quad .$$

We also consider uniqueness of symmetric bases. Suppose E and F have each a symmetric basis $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$, $sbc(\{e_i\}_{i=1}^n) = sbc(\{f_i\}_{i=1}^n) = 1$. Then, we ask whether there is a constant $C = C(d(E,F))$ depending only on the Banach-Mazur distance $d(E,F)$ such that

$$(2) \qquad C^{-1}\|\sum_{i=1}^n a_i e_i\| \le \|\sum_{i=1}^n a_i f_i\| \le C\,\|\sum_{i=1}^n a_i e_i\|$$

for all $a_i \in \mathbb{R}$, $i = 1,\ldots,n$?
In [1], this was verified for certain classes of spaces, namely the classes of q-concave spaces, $1 \le q < 2$, with q-concavity constant K_q. Thus the constant C appearing in (2) depended also on the q-concavity constant.

This result is generalized in [5]. Instead of considering the classes of q-concave spaces we considered the classes of spaces E with $d(E,\ell_n^2) \ge n^r$, $r > 0$. For these classes one also has uniqueness.

0. NOTATIONS.

The symmetric basis constant $sbc(\{e_i\}_{i=1}^n)$ of a basis $\{e_i\}_{i=1}^n$ of a Banach E is the infimum of all numbers $C > 0$ such that

$$\|\sum_{i=1}^n a_i e_i\| \le C \|\sum_{i=1}^n \varepsilon_i a_i e_{\pi(i)}\|$$

for all signs $\varepsilon_i = \pm 1$, all $a_i \in \mathbb{R}$, $i = 1,\dots,n$ and all permutations π of $\{1,\dots,n\}$. The unconditional basis constant $ubc(\{e_i\}_{i=1}^n)$ is the infimum of all numbers $C > 0$ such that

$$\|\sum_{i=1}^n a_i e_i\| \le C \|\sum_{i=1}^n \varepsilon_i a_i e_i\|$$

for all signs $\varepsilon_i = \pm 1$, and all $a_i \in \mathbb{R}$, $i = 1,\dots,n$.

We put $sbc(E) = \inf\{sbc(\{e_i\}_{i=1}^n) \mid \{e_i\}_{i=1}^n \text{ is a basis}\}$

$ubc(E)$ is defined analogously.

We say that E has a symmetric (unconditional) basis $\{e_i\}_{i=1}^n$ if it is normalized and $sbc(\{e_i\}_{i=1}^n) = 1$ ($ubc(\{e_i\}_{i=1}^n) = 1$). The Banach-Mazur distance of two Banach spaces E and F is given by

$$d(E,F) = \inf\{\|I\| \, \|I^{-1}\| \mid I \in L(E,F)\} \quad .$$

The dual basis of $\{e_i\}_{i=1}^n$ is denoted by $\{e_i^*\}_{i=1}^n$.

1. PROJECTION CONSTANT AND BANACH-MAZUR DISTANCE.

In the following we give an estimate of $d(E,\ell_n^\infty)$ by $\lambda(E)$. Moreover, we estimate $\lambda(E)$ and $d(E,\ell_n^\infty)$ by an expression that can easily be computed [4].

<u>Proposition 1.1</u> : Let $\{e_i\}_{i=1}^n$ be a symmetric basis of a Banach space E. Then there is a constant $C > 0$ such that

$$C^{-1}(\log 2\|\sum_{i=1}^n e_i\|)^{-2} \min_{1\le k\le n} \max_{n/k\le j\le n} \sqrt{\frac{n}{kj}} \, \|\sum_{i=1}^j e_i\| \, \|\sum_{i=1}^k e_i^*\| \le \lambda(E) \quad .$$

<u>Proposition 1.2</u> : Let $\{e_i\}_{i=1}^n$ be a symmetric basis of a Banach space E. Then there is a constant $C > 0$ such that

$$d(E,\ell_n^\infty) \le C(\log 2 \|\sum_{i=1}^n e_i\|)^{3/2} \min_{1\le k\le n} \max_{n/k\le j\le n} \sqrt{\frac{n}{kj}} \|\sum_{i=1}^j e_i\| \|\sum_{i=1}^k e_i^*\| .$$

Theorem 1.3 : Suppose E is a symmetric, n-dimensional Banach space. Then there is a constant $C > 0$ such that

$$\lambda(E) \le d(E,\ell_n^\infty) \le C(\log 2\, \lambda(E))^{7/2} \lambda(E) .$$

Obviously, the theorem follows from the two preceding propositions. The only observation we need is that $\log 2\, \lambda(E)$ and $\log 2 \|\sum_{i=1}^n e_i\|$ are up to a constant the same. This follows from the next proposition [3].

Proposition 1.4 : Suppose $\{e_i\}_{i=1}^n$ is an unconditional basis of E. Then

$$\|\sum_{i=1}^n e_i\| \left(\max_{(a_i)} \frac{\|\sum_{i=1}^n a_i e_i\|}{(\sum_{i=1}^n |a_i|^2)^{1/2}} \right)^{-1} \le \sqrt{2}\, \lambda(E) .$$

In particular,

$$\|\sum_{i=1}^n e_i\| \le 2\, \lambda(E)^2 .$$

2. UNIQUENESS OF SYMMETRIC BASES.

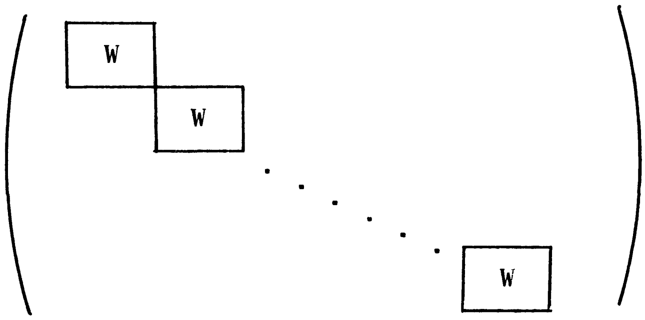
Theorem 2.1 [5] : Let E and F be Banach spaces with symmetric bases $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$. Suppose that $d(E,F) \le C$ and $d(E,\ell_n^2) \ge n^r$ for some $C > 0$ and $r > 0$. Then there is a constant $C_r = C_r(C) > 0$ such that for all $a_i \in \mathbb{R}$, $i = 1,\dots,n$ we have

$$C_r^{-1} \|\sum_{i=1}^n a_i e_i\| \le \|\sum_{i=1}^n a_i f_i\| \le C_r \|\sum_{i=1}^n a_i e_i\| .$$

The strange fact that the estimates in this theorem get worse if E "approaches" ℓ_n^2 is due to the method. In fact, we prove something stronger : consider the matrix of the map $A \in L(E,F)$ with $\|A\| \|A^{-1}\| = d(E,F)$ with respect to the bases $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$. Then, roughly speaking, it is shown that the matrix has a "big" diagonal. Of course, this cannot be shown in a Hilbert space since in a Hilbert space all orthonormal bases are symmetric.

Finally we want to mention some relationships. In [1] the uniqueness of symmetric bases was proved for classes of q-concave spaces. The starting point of the proof was formula (1) ensuring that $\|\sum_{i=1}^{n} e_i\|$ and $\|\sum_{i=1}^{n} f_i\|$ are proportional. Dropping the condition of q-concavity one cannot apply (1) anymore.

The proof of Theorem 2.1 had its starting point actually in the proof of Theorem 1.3. There it was shown that up to some logarithmic factor the distance $d(E,\ell_n^\infty)$ is attained for a matrix



where W are Walsh matrices of rank k, $1 \leq k \leq n$. The question was (and is) whether this is still true if we substitute ℓ_n^∞ by some other symmetric space F and consider d(E,F). We gave a "weak" answer to this which lead to the proof of Theorem 2.1

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