

# SÉMINAIRE D'ANALYSE FONCTIONNELLE ÉCOLE POLYTECHNIQUE

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**On operators fixing copies of  $c_0$  and  $\ell_\infty$**

*Séminaire d'analyse fonctionnelle (Polytechnique)* (1980-1981), exp. n° 12, p. 1-11

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1980-1981

ON OPERATORS FIXING COPIES OF  $c_0$  AND  $l_\infty$

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In this seminar, we report on a part of a joint work with W.B. Johnson and T. Figiel [1] concerning the structure of non-weakly compact operators on Banach lattices. First, we recall the following two fundamental theorems.

Theorem (A) : (A. Pełczyński [4]) . A non-weakly compact operator from a  $C(K)$ -space into any Banach space must preserve a copy of  $c_0$  ; that is there exists a subspace of  $C(K)$ , isomorphic to  $c_0$ , on which  $T$  acts as an isomorphism.

Theorem (B) : (H. Rosenthal [5]). If  $K$  is a  $\sigma$ -Stonian compact space, then every non-weakly compact operator from  $C(K)$  into any Banach space must preserve a copy of  $\ell_\infty$  .

Our goal is to see to which extent, one can replace  $C(K)$  in theorems (A) and (B) by a larger class of Banach spaces.

### § I. NON WEAKLY COMPACT OPERATORS :

The existence of the James space [2] eliminates the possibility of replacing  $C(K)$  in theorem (A) by any Banach space not containing a subspace isomorphic to  $\ell_1$ , since  $c_0$  and  $\ell_1$  do not embed in this space and yet it is not reflexive. However, the result does hold for the identity operator acting on a Banach lattice since if the latter is not reflexive, then it must contain a sublattice isomorphic either to  $\ell_1$  or  $c_0$  [3]. A natural problem is then to check if the result holds for any operator or equivalently if whether in theorem (A),  $C(K)$  can be replaced by any Banach lattice not containing  $\ell_1$ .

Surprisingly, Pełczyński's theorem does not extend even to this case as we show in the following counterexample.

Example (1) : For every  $p$ ,  $1 \leq p < \infty$ , there exists a Banach lattice

$X_p$  and a lattice homomorphism  $T_p$  from  $X_p$  onto  $c_0$  so that

- (i)  $T_p$  is strictly singular for each  $p$ ,  $1 \leq p < \infty$
- (ii)  $X_p$  contains no subspace isomorphic to  $\ell_1$  for  $p$ ,  $1 < p < \infty$ .

We first give the idea. Let  $c$  be the space of converging sequences and set  $X = \ell_1(c)$ ; that is the space of doubly-indexed sequences  $a = (a_{i,j})$ , where  $i = 1, 2, \dots$ ;  $j = 1, 2, \dots, \omega$  such that

$$\lim_{j \rightarrow \infty} a_{i,j} = a_{i,\omega} \text{ for } i = 1, 2, \dots$$

and 
$$\|a\|_X = \sum_{i=1}^{\infty} \sup_j |a_{i,j}| < \infty$$

Define the norm one operator  $T : X \longrightarrow c_0$  by

$$Ta = (a_{i,\omega})_{i=1}^{\infty}.$$

Clearly,  $T$  is weakly compact and  $X$  contains lots of sublattices isomorphic to  $\ell_1$ . However, we can turn  $T$  into a non-weakly compact operator by adding to the unit ball of  $X$  vectors  $(f_n)$  for which  $(Tf_n)$  is not weakly compact in  $c_0$  and taking for the new unit ball in  $X$  the absolute convex solid hull of the old unit ball and the  $f_n$ 's, in order to get a normed lattice. The completion of the resulting space probably still contains  $\ell_1$  complementably, but we can kill them by taking the  $p$ -convexification of the space for some  $1 < p < \infty$ .

Letting  $X$  and  $T$  be defined as above we define  $f_n \in X$  by

$$(f_n)_{i,j} = \begin{cases} 1 & , \text{ if } i \leq n \leq j \\ 0 & , \text{ otherwise} \end{cases}.$$

Clearly

$$Tf_n = \sum_{i=1}^n e_i$$

where  $(e_i)_{i=1}^{\infty}$  is the unit vector basis for  $c_0$ .

Let  $X_0$  be the dense sublattice of  $X$  consisting of those vectors  $a = (a_{i,j})$  whose rows are eventually zero; i.e., for some  $n$ ,  $a_{i,j} = 0$  for all  $i \geq n$  and all  $j = 1, 2, \dots, \omega$ .

### XII.3

Let  $\|\cdot\|_1$  be the greatest lattice norm on  $X_0$  such that

$$\|f_n\|_1 \leq 1 \quad , \quad \|x\|_1 \leq \|x\|$$

for  $n = 1, 2, \dots$  and all  $x \in X_0$ . That is,  $\|\cdot\|_1$  is the gauge of the closed absolutely convex solid hull of the unit ball of  $X_0$  and the sequence  $(f_n)$ . Thus  $\|x\|_1 < 1$  if and only if there are  $g \in X_0^+$  and eventually zero sequence  $s_1, s_2, \dots$  in  $\mathbb{R}^+$  so that

$$|x| \leq g + \sum_{i=1}^{\infty} s_i f_i \quad \text{and}$$

$$\|g\|_X + \sum_{i=1}^{\infty} s_i < 1.$$

Let  $(X_1, \|\cdot\|_1)$  be the completion of  $(X_0, \|\cdot\|_1)$  and for  $1 < p < \infty$ , let  $(X_p, \|\cdot\|_p)$  be the completion of the  $p$ -convexification of  $(X_0, \|\cdot\|_1)$ ; that is, for  $x \in X_0$ ,

$$\|x\|_p = \| |x|^p \|_1^{1/p}.$$

(See chapter 1.e in [3] for a discussion of  $p$ -convexity.)

We claim that  $\|T\|_p = 1$  for every  $1 \leq p < \infty$ ; i.e.,  $T$  has norm one as an operator from  $(X_0, \|\cdot\|_p)$  into  $c_0$ . This claim is a consequence of the observation that for each  $i$  and  $j$ , the coordinatewise evaluation functional on  $X_0$  defined by  $a \longrightarrow a_{i,j}$  has  $\|\cdot\|_p$ -norm one. (For  $p=1$  this is clear, because  $|f_n| \leq 1$  for each  $n$ , the general case then follows from the definition of  $\|\cdot\|_p$ .)

Since  $X_0$  is dense in  $X_p$ ,  $T$  extends to a norm one operator,  $T_p$ , from  $X_p$  into  $c_0$ . Note also that  $T_p$  is a lattice homomorphism and for every choice of signs  $\pm$  and  $n = 1, 2, \dots$ ; there is  $g \in X_0$ ,  $|g| \leq f_n$ , so that  $Tg = \sum_{i=1}^{\infty} \pm e_i$  which shows that  $T_p$  is a quotient map.

In the sequel, we shall say that a sequence  $(x_n)_{n=1}^{\infty}$  in  $X_p$  is a special  $c_0$ -sequence if there exist  $K < \infty$  and integers  $i_1 < i_2 < \dots$  such that for every  $n = 1, 2, \dots$ ,

$$x_n \geq 0 \quad , \quad \|x_n\|_p = 1$$

$$(x_n)_{i,j} = 0 \quad \text{if } i \neq i_n$$

XII.4

$$\left\| \sum_{k=1}^n x_k \right\|_p < K.$$

Note that if  $1 \leq i < \infty$  and  $x \in X_0$  with

$$x_{\ell, j} = 0 \text{ for } \ell \neq i,$$

then

$$\|x\|_X = \sup_j |x_{i, j}| ;$$

consequently,

$$\|x\|_p = \sup_j |x_{i, j}|$$

for  $p = 1$  and hence for all  $1 \leq p < \infty$ . In particular, all the terms of a special  $c_0$ -sequence lie in  $X_0$ .

We now show that  $X_1$  contains no special  $c_0$ -sequence.

If such a sequence  $(x_n)_{n=1}^\infty$  exists in  $X_1$ , pick for each  $n$  an index  $j_n < \omega$  so that

$$(x_n)_{i_n, j_n} \geq 1/2 \sup_j (x_n)_{i_n, j} = 1/2 \|x_n\|_1 = 1/2 .$$

By passing to a subsequence, we may assume that  $i_{n+1} > j_n$  for each  $n$ .

Given an integer  $N$ , find  $g \in X_0^+$  and  $(s_i)_{i=1}^\infty \subseteq \mathbb{R}^+$  so that

$$\sum_{n=1}^N x_n \leq g + \sum_{i=1}^\infty s_i f_i ,$$

$$\|g\|_X + \sum_{i=1}^\infty s_i < \left\| \sum_{n=1}^N x_n \right\|_1 + 1.$$

Evaluating both sides of the first inequality at  $(i_n, j_n)$ , we get

$$1/2 \leq (g)_{i_n, j_n} + \sum_{i=i_n}^{j_n} s_i \text{ for } n = 1, 2, \dots, N.$$

It follows that

$$N/2 \leq \sum_{n=1}^N (g)_{i_n, j_n} + \sum_{n=1}^N \sum_{i=i_n}^{j_n} s_i \leq$$

$$\leq \|g\|_X + \sum_{i=1}^{\infty} s_i < \left\| \sum_{n=1}^N x_n \right\|_1 + 1$$

which for large  $N$  contradicts the inequality

$$\left\| \sum_{n=1}^N x_n \right\|_1 < K.$$

To prove (i), suppose that  $T_p : X_p \longrightarrow c_0$  is an isomorphism on an infinite dimensional subspace  $E$  of  $X_p$  which we may assume is isomorphic to  $c_0$ . Let  $(z_n)_{n=1}^{\infty}$  be a normalized basis for  $E$  which is  $K$ -equivalent to the unit vector basis of  $c_0$ ; since  $X_0$  is dense in  $X_p$ , we can assume that each  $z_n$  lies in  $X_0$ .

Since

$$\|T_p z_n\| = \max_i |(z_n)_{i,\omega}| \text{ and}$$

$$\lim_{n \rightarrow \infty} (z_n)_{i,\omega} = 0 \text{ for each } i \in \mathbb{N},$$

we can find a sequence  $i_1 < i_2 < \dots$  and  $\delta > 0$  such that for all  $n$ ,

$$|(z_n)_{i_n,\omega}| > \delta.$$

Define the band projection  $P_n : X_p \longrightarrow X_p$  by

$$(P_n x)_{i,j} = \begin{cases} x_{i,j}, & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

By the diagonal principle (cf. p.20 in [2]) it follows that the disjoint sequence  $(P_{i_n} z_n)_{n=1}^{\infty}$  is  $K/\delta$ -equivalent to the unit vector basis of  $c_0$ .

Consequently,

$$y_n = \|P_{i_n} z_n\|_p^{-1} |P_{i_n} z_n|$$

is a special  $c_0$ -sequence in  $X_p$  and hence the sequence  $x_n = y_n^p$  is a special  $c_0$ -sequence in  $X_1$ , which is a contradiction.

To prove (ii), note that if  $E$  is a subspace of  $X_p$  isomor-



phic to  $\ell_1$ , and if  $S_m X_p = \sum_{i=1}^m P_i X_p$  determines the natural Schauder decomposition of  $X_p$ , then  $S_m \Big|_E$  cannot be an isomorphism for any  $m$  because  $S_m X_p$  is isomorphic to  $c_0$ . Thus there exists a normalized sequence  $(x_n)_{n=1}^\infty$  in  $E$  which is equivalent to the unit vector basis for  $\ell_1$  and a disjoint sequence  $(y_n)_{n=1}^\infty$  in  $X_0$  so that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\|_p = 0.$$

It follows that the sublattice of  $X_p$  generated by  $(y_n)_{n=1}^\infty$  is isomorphic to  $\ell_1$ , which is impossible for  $p > 1$  because  $X_p$  is  $p$ -convex.  $\square$

§ II. OPERATORS WHOSE ADJOINT ARE NOT WEAK\*-SEQUENTIALLY COMPACT :

To study the extensions of theorem (B), we note first that if  $K$  is  $\sigma$ -Stonian, then  $C(K)$  is a Grothendieck space, that is the weak-star sequential convergence in its dual coincide with the weak convergence. The problem then reduces to the study of the structure of operators whose adjoints are not weak-star sequentially compact and whose domain is a Banach lattice which contains no complemented copy of  $\ell_1$ . The first theorem reduces the problem to  $C(K)$ -spaces, where much is known.

Given any  $u$  in the positive cone  $L^+$  of a Banach lattice  $L$ , denote by  $L_u$  the (not necessarily closed) ideal generated by  $u$ . The canonical injection from  $L_u$  into  $L$  is denoted by  $j_u$  or just  $j$  if there is no ambiguity. If we put the natural norm on  $L_u$ , defined by

$$\|x\|_u = \inf \{ \lambda > 0 : |x| \leq \lambda u \}$$

then  $(L_u, \|\cdot\|_u)$  is an abstract  $M$ -space with unit  $u$  and hence is isometrically isomorphic to a  $C(K)$  space by Kakutani's Theorem. The operator  $j_u : (L_u, \|\cdot\|_u) \longrightarrow L$  obviously has norm  $\|u\|$ .

**Theorem 2** : Let  $L$  be a Banach lattice which does not contain a copy of  $\ell_1$  as a sublattice and let  $T$  be an operator from  $L$  into a Banach space  $X$  such that  $T^* \text{Ball}(X^*)$  is not weak\* sequentially compact. Then there exists  $u \in L^+$  so that  $(Tj_u)^* \text{Ball}(X^*)$  is not weak\* sequen-

initially compact.

To prove the theorem we will need a few lemmas. Given an infinite subset of  $\mathbb{N}$ , denote by  $[M]$  the set of all infinite subsets of  $M$ . Given a Banach space  $L$  and a bounded sequence  $(f_n)$  in  $L^*$ , we define for  $x \in L$  and  $M \in [\mathbb{N}]$

$$\alpha_M(x) = \limsup_{m \in M} f_m(x) - \liminf_{m \in M} f_m(x) .$$

Note that

$$\alpha_M(x) \leq 2 \sup_{m \in M} \|f_m\| \|x\|$$

and there exists  $P \in [M]$  so that

$$|\lim_{p \in P} f_p(x)| \geq 1/2 \alpha_M(x) .$$

Given  $A \subseteq L$ , define

$$\alpha_M(A) = \sup\{\alpha_M(x) : x \geq 0, \|x\| \leq 1, x \in A\}$$

$$\beta_M(A) = \inf\{\alpha_P(A) : P \in [M]\} .$$

**Lemma (3)** : Let  $L$  be a Banach space and let  $(f_n)$  be a bounded sequence in  $L^*$ . If  $A \subseteq \text{Ball}(L)$  and  $M \in [\mathbb{N}]$ , then either  $\beta_P(A) > 0$  for some  $P \in [M]$  or there exists  $P \in [M]$  such that  $(f_p)_{p \in P}$  converges pointwise on  $A$ .

**Proof** : If  $\beta_P(A) = 0$  for all  $P \in [M]$ , we can recursively define infinite sets  $M \supseteq P_1 \supseteq P_2 \supseteq \dots$  so that  $\alpha_{P_n}(A) < \frac{1}{n}$ . If  $P$  is a diagonal sequence with respect to the  $P_n$ 's, then  $\alpha_P(A) = 0$ ; i.e.,  $(f_p)_{p \in P}$  converges on  $A$ .

From lemma (3) it follows that if  $L$  is a Banach lattice and  $(f_n) \subseteq \text{Ball}(L^*)$  has no weak\* convergent subsequence, then we may assume, by passing to a subsequence of  $(f_n)$  that  $\beta_{\mathbb{N}}(L^+) > 0$ .

To prove Theorem 2, we fix a sequence  $(f_n) \subseteq T^*\text{Ball}(X^*)$  with  $\sup_n \|f_n\| \leq 1$  so that  $\beta_{\mathbb{N}}(L^+) > 0$ . We assume that  $\beta_M(L_x) = 0$  for

all  $x \in L^+$  and  $M \in [\mathbb{N}]$  since this is the case if  $(j_x^* r_m)_{m \in M}$  has a subsequence which converges weak\* in  $L_x^*$ . The conclusion that this set-up implies that  $L$  must contain a disjoint positive sequence equivalent to the unit vector basis of  $\ell_1$  is an immediate consequence of the next two lemmas. Lemma (4), produces an "almost disjoint" sequence in  $\text{Ball}(L^+)$  which, by Lemma (5), has a subsequence which is a small perturbation of a disjoint  $\ell_1$  sequence.

**Lemma (4)** : Suppose that  $L$  is a Banach lattice,  $(r_n) \subseteq \text{Ball}(L^*)$ ,  $\alpha_{\mathbb{N}}(L^+) > \delta > 0$ ,  $\beta_M(L_x) = 0$  for all  $M \in [\mathbb{N}]$  and  $x \in L^+$ , and  $\varepsilon_n \downarrow 0$ . Then there exists  $f \in \text{weak}^* \text{ closure } (f_n)$  and  $(y_n) \subseteq \text{Ball}(L^+)$  so that for each  $n = 1, 2, \dots$ ,

$$(i) \quad \left\| \left( \sum_{i=1}^{n-1} y_i \right) \wedge y_n \right\| < \varepsilon_n$$

$$(ii) \quad |f(y_n)| \geq \delta/2 .$$

**Proof** : By induction we construct a sequence  $(y_n) \subseteq \text{Ball}(L^+)$  and  $(M_n) \subseteq [\mathbb{N}]$  to satisfy for each  $n = 1, 2, \dots$  condition (i) and

$$(iii) \quad M_{n+1} \subset M_n$$

$$(iv) \quad |f_m(y_n)| > \delta/2 \text{ for all } m \in M_n .$$

Having done this, we simply let  $f$  be any element of  $\text{Ball}(L^*)$  which is a weak\* cluster point of  $(r_k)_{k \in M_n}$  for each  $n = 1, 2, \dots$ .

Choosing  $y_1 \in \text{Ball}(L^+)$  so that  $\alpha_{\mathbb{N}}(y_1) > \delta$ , we have that

$$\limsup_{m \in \mathbb{N}} |f_m(y_1)| > \delta/2$$

so that we can choose  $M_1 \in [\mathbb{N}]$  to satisfy (iv) for  $n = 1$ .

Having defined  $(M_n)_{n=1}^N$  and  $(y_n)_{n=1}^N$  to satisfy (i), (iii), and (iv) for  $n \leq N$ , we pick  $M \in [M_N]$  so that

$$\alpha_M \left( \left[ 0, \sum_{i=1}^N y_i \right] \right) = 0$$

and choose  $z \in \text{Ball}(L^+)$  so that  $\alpha_M(z) > \delta$ . Define

$$y_{N+1} = z - z \wedge \left( \varepsilon_{N+1}^{-1} \sum_{i=1}^N y_i \right).$$

Since

$$\alpha_M \left( z \wedge \varepsilon_{N+1}^{-1} \sum_{i=1}^N y_i \right) = 0.$$

we have that

$$\alpha_M(y_{N+1}) = \alpha_M(z) > \delta.$$

Thus we can choose  $M_{N+1} \in [M]$  so that for all  $m \in M_{N+1}$ ,

$$|f_m(y_{N+1})| > \delta/2.$$

To check (i), just note that if  $z, x \in L^+$  and  $\lambda \in \mathbb{R}^+$ , then

$$(z - z \wedge \lambda x) \wedge x = (z - \lambda x)^+ \wedge x \leq \lambda^{-1} z. \quad \square$$

**Lemma (5)** : Suppose that  $L$  is a Banach lattice,  $f \in \text{Ball}(L^*)$ ,  $(y_n) \subseteq \text{Ball}(L^+)$ , and  $0 < \delta < \delta + \varepsilon$ . Suppose that for each  $n = 1, 2, \dots$ ,  $f(y_n) \geq \delta + \varepsilon$  and  $\lim_{k \rightarrow \infty} \left\| \left( \sum_{i=1}^n y_i \right) \wedge y_k \right\| = 0$ . Then there is a subsequence  $(y_{n(i)})$  of  $(y_n)$  and a disjoint sequence  $(x_i)$  in  $L^+$  with  $x_i \leq y_{n(i)}$  so that for each  $i = 1, 2, \dots$

$$\|y_{n(i)} - x_i\| < 4^{-i+1} \varepsilon.$$

Consequently,  $|f(x_i)| > \delta$  for each  $i = 1, 2, \dots$ , and hence  $(x_i)$  is  $1/\delta$ -equivalent to the unit vector basis for  $\ell_1$  and  $[x_i]$  is  $1/\delta$ -complemented in  $L$ .

**Proof** : Assume, by passing to a subsequence of  $(y_n)$ , that for  $n = 1, 2, \dots$

$$(a) \left\| y_{n+1} \wedge \sum_{i=1}^n y_i \right\| < 4^{-n} \varepsilon$$

We define by recursion a double sequence  $(y_{n,k})_{n=1, k=n}^{\infty, \infty} \subseteq \text{Ball}(L^+)$  to satisfy

$$(b) (y_{n,k})_{n=1}^k \text{ is disjoint for } k = 1, 2, \dots$$

$$(c) y_{n,k+1} \leq y_{n,k} \leq y_n \text{ for } 1 \leq n \leq k.$$

$$(d) \|y_n - y_{n,n}\| < 4^{-n}\varepsilon \text{ for } n = 1, 2, \dots .$$

$$(e) \|y_{n,k} - y_{n,k+1}\| < 4^{-k}\varepsilon \text{ for } 1 \leq n \leq k.$$

Once this is done, we can in view of (e) set

$$x_n = \lim_{k \rightarrow \infty} y_{n,k} ;$$

from (b) and (c) we have that  $(x_n)_{n=1}^{\infty}$  is disjoint and  $0 \leq x_n \leq y_n$  for each  $n = 1, 2, \dots$ . From (d) and (e) we infer that

$$\|y_n - x_n\| < 4^{-n+1}\varepsilon.$$

We turn now to the construction of the  $y_{n,k}$ 's. Set  $y_{1,1} = y_1$ .

Suppose that  $(y_{n,k})_{n=1}^N_{k=n}$  has been defined. Let

$$y_{N+1,N+1} = y_{N+1} - y_{N+1} \wedge \left( \sum_{k=1}^N y_{N,k} \right)$$

and, for  $1 \leq n \leq N + 1$ , set

$$y_{n,N+1} = y_{n,N} - y_{n,N} \wedge y_{N+1} .$$

We leave the verification of (b) - (e) to the reader.  $\square$

By applying Theorem (B) we obtain the following two corollaries of Theorem (2).

Corollary (6) : Let  $L$  be a  $\sigma$ -complete Banach lattice which does not contain a copy of  $\ell_1$  as a sublattice. If  $T$  is an operator from  $X$  into some Banach space  $Y$  and  $T^* \text{Ball}(Y^*)$  is not weak\* sequentially compact, then  $T$  preserves a copy of  $\ell_\infty$ .

Proof : By Theorem (2) there is  $u \in L^+$  so that  $(Tj_u)^* \text{Ball}(Y^*)$  is not weak\* sequentially compact and hence not weakly compact. When  $L_u$  is represented as  $C(K)$  space,  $K$  is  $\sigma$ -Stonian because  $L$  is  $\sigma$ -complete. Therefore, by Theorem (B)  $Tj_u$ , hence also  $T$ , preserves a copy of  $\ell_\infty$ .  $\square$

Corollary (7) : If  $L$  is a  $\sigma$ -complete Grothendieck Banach lattice, then every non-weakly compact operator from  $L$  into any Banach space preserves a copy of  $\ell_\infty$ .

Proof : A Grothendieck space cannot contain  $\ell_1$  (or any other non-reflexive separable space) as a complemented subspace, and non-weakly compact operators from a Grothendieck space have adjoints which are not weak\* sequentially compact, and hence Corollary (6) can be applied to any non-weakly compact operator from a  $\sigma$ -complete Grothendieck Banach lattice. □

Problem : It is still unknown whether every non-weakly compact operator from a Grothendieck space into any Banach space preserves a copy of  $\ell_\infty$ .

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