H. KÖNIG

Some estimates for type and cotype constants

Séminaire d'analyse fonctionnelle (Polytechnique) (1979-1980), exp. nº 28, p. 1-13
<http://www.numdam.org/item?id=SAF_1979-1980____A25_0>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. : (1) 941.82.00 - Poste N°

Télex : ECOLEX 691596 F

S E M I N A I R E

D ' A N A L Y S E F O N C T I O N N E L L E

1979-1980

SOME ESTIMATES FOR TYPE AND COTYPE CONSTANTS

H. KÖNIG

(Université de Bonn)

Exposé No XXVIII 13 Juin 1980

This is a presentation of some results on type and cotype constants which were obtained in joint work with L. Tzafriri.

To fix notations, let X be a Banach space, $r_n(t)$ be the sequence of Rademacher functions, $g_n(t)$ be a sequence of independent standard Gaussian variables on a probability space (Ω,P). Given $1 \le p \le 2 \le q \le \infty$ and $n \in \mathbb{N}$, we define $a_{p,n}(X)$, $b_{q,n}(X)$, $\alpha_{p,n}(X)$ and $\beta_{q,n}(X)$ to be smallest constants such that for arbitrary $x_1 \dots x_n \in X$ the following inequalities hold

$$b_{q,n}(X)^{-1}\Big(\sum_{j=1}^{n} \|x_j\|^q\Big)^{1/q} \le \Big(\int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)\, x_j\Big\|^2 dt\Big)^{1/2} \le a_{p,n}(X)\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p}$$

$$\beta_{q,n}(X)^{-1}\Big(\sum_{j=1}^{n} \|x_j\|^q\Big)^{1/q} \le \Big(\int_\Omega \Big\|\sum_{j=1}^{n} g_j(\omega)\, x_j\Big\|^2 dP(\omega)\Big)^{1/2} \le \alpha_{p,n}(X)\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p}$$

If $a_p(X) = \sup_n a_{p,n}(X) < \infty$ (resp. $b_q(X) = \sup_n b_{q,n}(X) < \infty$), X is of (Rademacher) type p (resp. (Rademacher) cotype q). Similarly, define the Gaussian type p and cotype q constants by $\alpha_p(X) = \sup_n \alpha_{p,n}(X)$ and $\beta_q(X) = \sup_n \beta_{q,n}(X)$. These quantities were investigated by Maurey and Pisier [7].

We have for some c and c_p independent of n and X

$$c_p^{-1}\, \alpha_{p,n}(X) \le a_{p,n}(X) \le c\, \alpha_{p,n}(X)$$
$$\beta_{q,n}(X) \le c\, b_{q,n}(X) \quad . \tag{1}$$

The last two inequalities result immediately from

$$\int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)\, x_j\Big\|^2 dt \le c^2 \int_\Omega \Big\|\sum_{j=1}^{n} g_j(\omega)\, x_j\Big\|^2 dP(\omega) \quad ,$$

cf. Pisier [9]. To prove the first inequality, we have by the symmetry of the g_j's

$$\begin{aligned}\int_\Omega \Big\|\sum_{j=1}^{n} g_j(\omega) x_j\Big\|^p dP(\omega) &= \int_0^1 \int_\Omega \Big\|\sum_{j=1}^{n} r_j(t)\, g_j(\omega) x_j\Big\|^p dP(\omega)\, dt \\ &\le \int_\Omega \Big(\int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)\, g_j(\omega)\, x_j\Big\|^2 dt\Big)^{p/2} dP(\omega) \\ &\le a_{p,n}(X)^p \int_\Omega \sum_{j=1}^{n} |g_j(\omega)|^p\, \|x_j\|^p\, dP(\omega) \\ &= \tilde{c}_p^{\,p}\, a_{p,n}(X)^p \sum_{j=1}^{n} \|x_j\|^p\end{aligned}$$

The equivalence of the Gaussian p- and 2-moments yields the desired inequality $\alpha_{p,n}(X) \le c_p\ a_{p,n}(X)$.

If X does not have some finite cotype, i.e. $b_q(X) < \infty$ for some $q < \infty$, the sequences $\beta_{q,n}(X)$ and $b_{q,n}(X)$ may be inequivalent : for $X = \ell_\infty^n$ one gets $b_{2,n}(X) \sim n^{1/2}$ but $\beta_{2,n}(X) \sim (n/\log n)^{1/2}$, cf. Figiel-Lindenstrauss-Milman [2].

We will study the question whether the type and cotype constants of n-dimensional spaces X_n can be calculated essentially by n vectors, that is whether e.g.

$$a_p(X_n) \le c_p\ a_{p,n}(X_n) \qquad (2)$$

holds, with c_p depending only on p. For $p = 2$ one has the

<u>Theorem</u> (Tomczak-Jaegermann [10]) : For any n-dimensional space X_n ,

$$\alpha_2(X_n) \le 2\ \alpha_{2,n}(X_n) \quad \text{and} \quad \beta_2(X_n) \le 2\ \beta_{2,n}(X_n) \quad .$$

The proof rests upon a corresponding statement for 2-absolutely summing norms of rank n operators, to which the Gaussian constants relate. Given $T : X \to Y$ and $1 \le s \le r < \infty$ we denote by $\pi_{r,s}^{(n)}(T)$ the smallest constant c such that for all $x_1, \dots, x_n \in X$

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^r\Big)^{1/r} \le c \sup_{\|x^*\|_{X^*} \le 1} \Big(\sum_{i=1}^{n} |\langle x^*, x_i\rangle|^s\Big)^{1/s} \quad .$$

Clearly T is absolutely (r,s)-summing, $T \in \pi_{r,s}(X,Y)$ iff $\pi_{r,s}(T) = \sup_n \pi_{r,s}^{(n)}(T) < \infty$. For $T : \ell_2^n \to X$, let $\ell(T) := \Big(\int_\Omega \|\sum_{i=1}^{n} g_i(\omega)\ Te_i\|^2\ dP(\omega)\Big)^{1/2}$ where e_i are the unit vectors in ℓ_2^n; ℓ has ideal norm properties.

The following lemma relates the Gaussian constant and (q,2)-absolutely summing norms. It is due to Tomczak-Jaegermann [10] (for $p = 2 = q$, the generalization of the argument to $p < 2 < q$ is easy), for more details cf. also Pełczynski [8].

Lemma 1 : Let X be a Banach space, $1 \le p \le 2 \le q \le \infty$, $p' = p/(p-1)$ and $n \in \mathbb{N}$. Then

$$\beta_{q,n}(X) = \sup\{\pi_{q,2}^{(n)}(T) \mid T : \ell_2^n \to X \text{ with } \ell(T) \le 1\}$$

$$\alpha_{p,n}(X) = \sup\{\ell(S) \mid S : \ell_2^n \to X \text{ with } (\pi_{p',2}^{(n)})^*(S^*) \le 1\} .$$

Here $(\pi_{p',2}^{(n)})^*$ denotes the adjoint ideal norm to $\pi_{p',2}^{(n)}$.

Concerning problem (2) for $p \ne 2 \ne q$ we have the following positive answer

Theorem 1 : Let $1 < p < 2 < q < \infty$. There is $c_q \le c/(q-2)$ such that for any $n \in \mathbb{N}$ and any n-dimensional space X_n

$$\alpha_p(X_n) \le c_{p'} \; \alpha_{p,n}(X_n) \quad \text{and} \quad \beta_q(X_n) \le c_q \; \beta_{q,n}(X_n)$$

and by (1), $a_p(X_n) \le d_{p'} \; a_{p,n}(X_n)$.

Theorem 1 results immediately from lemma 1 and proposition 1 below which we want to derive :

Proposition 1 : For any $q > 2$, there is $c_q \le c/(q-2)$ such that for any $n \in \mathbb{N}$ and any rank n operator $T : X \to Y$

$$\pi_{q,2}(T) \le c_q \; \pi_{q,2}^{(n)}(T) .$$

Defining the approximation numbers of $T : X \to Y$ by

$$\alpha_j(T) := \inf\{\|T - T_j\| \mid T_j : X \to Y \text{ of rank } < j\} \quad , \quad j \in \mathbb{N}$$

we let for $0 < r \le \infty$

$$S_r(X,Y) = \{T : X \to Y \mid \sigma_r(T) := (\sum_{j=1}^{\infty} \alpha_j(T)^r)^{1/r} < \infty\}$$

$$S_{2,1}(X,Y) = \{T : X \to Y \mid \sigma_{2,1}(T) := \sum_{j=1}^{\infty} \alpha_j(T) \; j^{-1/2} < \infty\} .$$

Thus $\sigma_{2,1}(T)$ is the norm of $(\alpha_j(T))_{j \in \mathbb{N}}$ in the Lorentz sequence space $\ell_{2,1}$ which can be written as a real interpolation space between ℓ_q-spaces ; in particular $\ell_{2,1} = (\ell_1, \ell_\infty)_{\frac{1}{2},1}$, cf. [1].

Proof of proposition 1 : Since $\pi_{q,2}(T) = \sup\{\pi_{q,2}(TA) : A : \ell_2 \to X, \|A\| \le 1\}$, it suffices to prove the statement for maps $T : \ell_2^n \to Y$. We will show

$$\pi_{q,2}(T) \le c_q \ \sigma_q(T) \le c_q \ \pi_{q,2}^{(n)}(T) \quad . \tag{3}$$

Step 1 : We define inductively an orthonormal basis $(e_j)_{j=1}^n$ of ℓ_2^n with $\alpha_j(T) \le \|Te_j\|$. For $j = 1$, choose e_1 of norm one such that $\alpha_1(T) = \|T\| = \|Te_1\|$. If $j < n$ orthonormal vectors have been found, let $Y_j := [e_1, \ldots, e_j]$ and $P_j : \ell_2^n \to Y_j \subseteq \ell_2^n$ be the orthogonal projection. Thus

$$\alpha_{j+1}(T) \le \|T - TP_j\| = \|T|_{Y_j^\perp}\| \quad .$$

Hence there is $e_{j+1} \in Y_j^\perp$ of norm one such that $\alpha_{j+1}(T) \le \|Te_{j+1}\|$. Since rank $T \le n$, $\alpha_k(T) = 0$ for $k > n$. This yields

$$\sigma_q(T) = (\sum_{j=1}^n \alpha_j(T)^q)^{1/q} \le (\sum_{j=1}^n \|Te_j\|^q)^{1/q} \le \pi_{q,2}^{(n)}(T) \quad ,$$

the right side inequality in (3).

Step 2 : We show $S_{2,1}(X,Y) \le \pi_2(X,Y)$ for any X and Y. Taking $S \in S_{2,1}(X,Y)$ choose $D_j : X \to Y$ of rank $D_j < 2^j$ with $\|S - D_j\| \le 2\alpha_{2^j}(S)$, $j = 0,1,\ldots$ $(D_o = 0)$. Let $S_j = D_{j+1} - D_j$. Then $S = \sum_{j=0}^\infty S_j$, $\|S_j\| \le 4\ \alpha_{2^j}(S)$ and rank $S_j < 2^{j+2}$. Since the 2-absolutely summing norm of the identity on an n-dimensional space is $n^{1/2}$, we infer

$$\begin{aligned}
\pi_2(\sum_{j=0}^N S_j) &\le \sum_{j=0}^N \pi_2(S_j) \le \sum_{j=0}^N \|S_j\|\ 2^{j/2+1} \\
&\le 16 \sum_{j=0}^N 2^{j/2-1} \alpha_{2^j}(S) \\
&\le 16 \sum_{k=1}^\infty k^{-1/2} \alpha_k(S) = 16\ \sigma_{2,1}(S) \quad .
\end{aligned}$$

Thus S is 2-summing with $\pi_2(S) \le 16\ \sigma_{2,1}(S)$.

Step 3 : The K-functional of the real interpolation theory [1] satisfies

$$K(t,T;S_1(X,Y),S_\infty(X,Y)) \sim K(t,(\alpha_j(T))_{j=1}^\infty ; \ell_1,\ell_\infty) \quad . \tag{4}$$

Here $S_\infty = \mathcal{L}$ = all continuous linear maps. By definition of the K-functional,

$$K(t,T;S_1,S_\infty) := \inf\{ \sum_{j\in\mathbb{N}} \alpha_j(T_t) + t\,\|T - T_t\| \mid T_t : X \to Y\}$$

$$\geq \inf\{ \sum_{j=1}^{[t]} (\alpha_j(T_t) + \|T - T_t\|)\} \geq \sum_{j=1}^{[t]} \alpha_j(T)$$

$$\sim K(t,(\alpha_j(T))_{j=1}^{\infty} ; \ell_1,\ell_\infty) \quad ,$$

for the last equivalence cf. [1]. For $t \geq 1$, choose $T_t : X \to Y$ with rank $T_t < [T]$ and $\|T - T_t\| \leq 2\,\alpha_{[t]}(T)$. Then

$$\alpha_j(T_t) \leq \alpha_j(T) + \|T - T_t\| \leq 3\,\alpha_j(T)$$

for all $j < [t]$ and $\alpha_j(T_t) = 0$ for $j \geq [t]$, hence

$$K(t,T;S_1,S_\infty) \leq \sum_{j=1}^{[t]} \alpha_j(T_t) + t\,\|T - T_t\|$$

$$\leq 3 \sum_{j=1}^{[t]} (\alpha_j(T) + \alpha_{[t]}(T)) \leq 6 \sum_{j=1}^{[t]} \alpha_j(T)$$

which proves (4).

<u>Step 4</u> : Since $\ell_{2,1} = (\ell_1,\ell_\infty)_{\frac{1}{2},1}$, the equivalence (4) yields $S_{2,1}(X,Y) = (S_1(X,Y),S_\infty(X,Y))_{\frac{1}{2},1}$. Let $q > 2$, $\frac{1}{q} = \frac{1-\theta}{2}$ and $\eta = \frac{1}{2}(1+\theta)$. Then by the reiteration theorem [1]

$$(S_{2,1}(X,Y)),S_\infty(X,Y))_{\theta,q} = (S_1(X,Y),S_\infty(X,Y))_{\eta,q} = S_q(X,Y) \quad (5)$$

where the last equality follows from (4) and $(\ell_1,\ell_\infty)_{\eta,q} = \ell_q$. It is an easy consequence from

$$\ell_q(X) = (\ell_2(X),\ell_\infty(X))_{\theta,q} \quad , \quad \frac{1}{q} = \frac{1-\theta}{2} \quad , \quad 0 < \theta < 1$$

that

$$(\pi_2(X,Y),\mathcal{L}(X,Y))_{\theta,q} \leq \pi_{q,2}(X,Y) \quad .$$

This, (5) and step 2 show $S_q(X,Y) \leq \pi_{q,2}(X,Y)$ for any $q > 2$ and thus $\pi_{q,2}(T) \leq c_q\ \sigma_q(T)$ for any $T \in S_q(X,Y)$, where c_q depends only on $q > 2$;

the bound $c_q \le c/(q-2)$ can be derived by checking the constants occuring in the reiteration theorem. This proves the left side in (3) and thus proposition 1.

As a corollary to the proof we note a fact which is false for $q = 2$:

Corollary 1 : For any $2 < q < \infty$ and any Banach space Y

$$\pi_{q,2}(\ell_2, Y) = S_q(\ell_2, Y) \quad .$$

Corollary 2 : For any $2 < q < \infty$, there is c_q such that for any $n \in \mathbb{N}$ and any rank n operator $T : X \to Y$

$$\pi_{q,1}(T) \le c_q \; \pi_{q,1}^{(n)}(T) \quad .$$

Proof : It is well-known that

$$\begin{aligned} \pi_{q,1}(T) &= \sup\{\pi_{q,1}(TA) \mid \|A : \ell_\infty \to X\| \le 1\} \\ &\le \sup\{\pi_{q,2}(TA) \mid \|A : \ell_\infty \to X\| \le 1\} \\ &\le c_q \; \sup\{\pi_{q,2}^{(n)}(TA) \mid \|A : \ell_\infty \to X\| \le 1\} \quad . \end{aligned}$$

By Maurey [6], the (q,1)- and (q,2)-absolutely summing norms are equivalent on $\pi_{q,1}(\ell_\infty, Y)$; the argument does not depend on the number of vectors considered. Thus

$$\pi_{q,1}(T) \le d_q \; \sup\{\pi_{q,1}^{(n)}(TA) \mid \|A : \ell_\infty \to X\| \le 1\} \le d_q \; \pi_{q,1}^{(n)}(T) \quad .$$

Theorem 1 gives no answer to the question whether the (Rademacher) cotype constants on n-dimensional spaces X_n can be calculated by n vectors, i.e. whether

$$b_q(X_n) \le c_q \; b_{q,n}(X_n)$$

does hold or not. We only have a partial answer :

Proposition 2 : Let $q > 2$. There is $c_q > 0$ such that for any $n \in \mathbb{N}$ and any n-dimensional space

$$b_q(X_n) \leq c_q\ \beta_{q,n}(X_n)(\log b_{q,n}(X_n))^{1/2} \tag{6}$$

and thus by (1)

$$b_q(X_n) \leq c_q\ b_{q,n}(X_n)(\log b_{q,n}(X_n))^{1/2}\ . \tag{7}$$

Remark : c_q can be bounded by $c\sqrt{q}$ as q tends to ∞.

We postpone the proof of proposition 2. It is well-known, Maurey-Pisier [7], that a Banach space X has some finite cotype $q < \infty$ if it has type p for some $p > 1$. More quantitative information is given in

Theorem 2 : Let $p > 1$, $p' = p/(p-1)$ and X be a Banach space.
(a) If $\dim X = 2^n$ and $q = 2 + 2^{p'}\ a_{p,n}(X)^{p'}$, then $b_q(X) \leq 2$.
(b) If X has type $p > 1$, it has cotype q for $q = 2 + 2^{p'}\ a_p(X)^{p'}$.

Proof : Clearly it suffices to prove part (a).
Step 1 : Let $\dim X = 2^n$ and $\{z_i\}_{i\in\mathbb{N}}$ be dense in X. Denote the span of $\{r_i(t)z_i\}_{i\in\mathbb{N}}$ in $L_p(X)$ by Z. Then $\{r_i(t)z_i\}_{i\in\mathbb{N}}$ is a 1-unconditional basis of Z. We now use a Shimogaki-Pisier-type argument : define γ_j to be the smallest constant such that for any sequence $(u_i)_{i=1}^{j} \subseteq Z$ of pairwise disjoint elements (relative to the lattice structure inherited from the 1-unconditional basis $(r_i(t)z_i)_{i\in\mathbb{N}}$ of Z) one has

$$\inf_{1\leq i\leq j} \|u_i\|_Z \leq \gamma_j \left\| \sum_{i=1}^{j} u_i \right\|_Z\ .$$

Then $1 = \gamma_1 \geq \gamma_2 \geq \ldots$ and $\gamma_{jk} \leq \gamma_j\ \gamma_k$ for all j and k in $\mathbb{N}$, cf. Lindenstrauss-Tzafriri [5], p. 91. Given $m \in \mathbb{N}$, choose $\{w_i\}_{i=1}^{2^m}$ such that

$$\inf_{1\leq i\leq 2^m} \|w_i\|_Z = 1\quad ,\quad \left\| \sum_{i=1}^{2^m} w_i \right\|_Z = \frac{1}{\gamma_{2^m}}\ .$$

It is a consequence of the 1-unconditionality of $\{r_i(t)z_i\}$ in Z that we may assume w.l.o.g. $\|w_i\|_Z = 1$ for all $i = 1,\ldots,2^m$ and moreover

$$\sup_{1\leq i\leq 2^m} |c_i| \leq \left\| \sum_{i=1}^{2^m} c_i\ w_i \right\|_Z \leq \frac{1}{\gamma_{2^m}} \sup_{1\leq i\leq 2^m} |c_i|$$

for all scalar sequences $(c_i) \in \mathbb{R}^{2^m}$. Thus $\mathrm{span}[w_i]$ is $\frac{1}{\gamma_{2^m}}$-isomorphic to $\ell_\infty^{2^m}$ which contains (in the real case) isometrically ℓ_1^m. Thus there are $v_1,\ldots,v_m \in Z$ (which can be realized as the Rademacher elements over the

$\{w_i\}$) such that

$$\sum_{i=1}^{m} |c_i| \le \| \sum_{i=1}^{m} c_i v_i \|_Z \le \frac{1}{\gamma_{2^m}} \sum_{i=1}^{m} |c_i| \qquad (8)$$

for all scalar sequences (c_i). It is well-known and easily checked by integral inequalities that $a_{p,m}(X) = a_{p,m}(L_p(X)) \ge a_{p,m}(Z)$. Integrating (8) with $c_i = r_i(t)$, we get

$$m \le \left(\int_0^1 \| \sum_{i=1}^{m} r_i(t) v_i \|_Z^p \, dt\right)^{1/p} \le a_{p,m}(X) \frac{1}{\gamma_{2^m}} m^{1/p} \quad ,$$

$$\gamma_{2^m} \le a_{p,m}(X) \; m^{-1/p'} \quad . \qquad (9)$$

Step 2 : We first consider the case $a_{p,n}(X) n^{-1/p'} \ge 1/2$. Then $q := 2 + 2^{p'} \; a_{p,n}(X)^{p'} \ge 2 + n > n$. If follows from

$$b_2(\ell_2^{2^n}) = 1 \quad \text{and} \quad d(X, \ell_2^{2^n}) \le 2^{n/2}$$

that $b_2(X) \le 2^{n/2}$. An interpolation argument shows

$$b_q(X) \le b_2(X)^{2/q} \le 2^{n/q} < 2 \quad .$$

Step 3 : We now derive the same conclusion in the other case $a_{p,n}(X) \; n^{-1/p'} < 1/2$, for the same value of q. Let $(y_i)_{i=1}^{\ell} \subseteq X$ be an arbitrary finite sequence. We may assume that $\|y_i\|$ is non-increasing. Since the $(z_i)_{i=1}^{\infty}$ were dense in X,

$$\|y_j\| = \inf_{1 \le i \le j} \|y_i\| \le \gamma_j \left(\int_0^1 \| \sum_{i=1}^{j} r_i(t) \; y_i \|^p \, dt\right)^{1/p}$$

holds for all j. Hence

$$\left(\sum_j \|y_j\|^q\right)^{1/q} \le \left(\sum_j \gamma_j^q\right)^{1/q} \left(\int_0^1 \|\sum_i r_i(t) \; y_i\|^p \, dt\right)^{1/p}$$

$$\le \left(\sum_j \gamma_j^q\right)^{1/q} \left(\int_0^1 \|\sum_i r_i(t) \; y_i\|^2 \, dt\right)^{1/2}$$

and thus
$$b_q(X) \le \left(\sum_j \gamma_j^q\right)^{1/q} \quad .$$

Let k be the first integer $\le n$ such that

$$\gamma_{2^k} \le a_{p,n}(X) \; k^{-1/p'} \le 1/2 \quad .$$

Then

$$\left(\sum_{j=1}^{\infty} \gamma_j^q\right)^{1/q} \le 2^{k/q} \left(\sum_{i=0}^{\infty} 2^{ik}\, \gamma_{2^{ik}}^q\right)^{1/q}$$

$$\le 2^{k/q} \left(\sum_{i=0}^{\infty} 2^{ik}\, \gamma_{2^k}^{iq}\right)^{1/q}$$

$$\le 2^{k/q} \left(\sum_{i=0}^{\infty} \left(\frac{1}{2^{q-k}}\right)^i\right)^{1/q} \le 2^{k/q}\, 2^{1/q} \le 2 \quad .$$

Note here $k - 1 \le 2^{p'}\, a_{p,n}(X)^{p'}$ and thus $k + 1 \le q$ as well as $q - k \ge 1$. Hence $b_q(X) \le 2$ in the second case, too.

Example : The value of $q = 2 + 2^{p'}\, a_p(X)^{p'}$ may be slightly improved : choosing $c = e^{1/p'}$ instead of $c = 2$ in $a_{p,n}(X)\, n^{-1/p'} \gtrless c^{-1}$, one gets that X is of type q for any $q > p' \ell n\, 2 + (e\, p' \ell n\, 2)\, a_p(X)^{p'}$. However, in general this value of q has to be at least as large as $q \ge e\, a_p(X)^{p'}$ as shown by $X = L_r(\mu)$, $r > 2$: clearly X is not of cotype q for $q < r$. On the other hand, $a_p(X) \le a_2(X)^{2/p'} \le B_r^{1/p'}$ where B_r is the Khintchin constant for Rademacher r-averages. By the estimates of Haagerup [3], $B_r \sim r/e$ as $r \to \infty$.

If $T : X \to Y$ factors through a Hilbert space, let $\gamma_2(T)$ denote the factorization norm.

Corollary 3 : Assume X is a Banach space of type 2. Let $\varepsilon = (6\, a_2(X)^2)^{-1}$. Then onto any n-dimensional subspace X_n of X there is a projection P_n with

$$\gamma_2(P_n) \le 8\, a_2(X)\, n^{1/2-\varepsilon} \quad .$$

In particular, one has for the relative projection constant of X_n in X and the Banach-Mazur distance to Hilbert space

$$\lambda_X(X_n) \le 8\, a_2(X)\, n^{1/2-\varepsilon}$$

$$d(X_n, X) \le 8\, a_2(X)\, n^{1/2-\varepsilon} \quad .$$

Proof : It is known (cf. [4]) that there is a projection $P_n : X \to X_n$ with

$$\gamma_2(P_n) \le 4\, a_2(X)\, \beta_2(X_n) \quad .$$

Since $\beta_2(X_n) \le b_2(X_n) \le n^{1/2 - 1/q}\, b_q(X_n)$, theorem 2 implies for $q = 6$ $a_2(X)^2 \ge 2 + 2^2\, a_2(X)^2$ that $\beta_2(X_n) \le 2\, n^{1/2 - \varepsilon}$.

Remarks :

(1) The existence of an $\varepsilon > 0$ as in corollary 3 is an open problem in spaces of type $p > 1$. Theorem 2 yields a positive answer for small type p constants and values p near 2 : if

$$a_p(X) < \left(\frac{2(p-1) - (2-p)\,\ell n\, 2}{e(2-p)\,\ell n\, 2} \right)^{1/p'} \tag{10}$$

corollary 3 holds for X and $\varepsilon = \frac{1}{2} - (\frac{1}{p} - \frac{1}{r})$ (> 0) where $r > p'\,\ell n\, 2 + (e\, p'\,\ell n\, 2)\, a_p(X)^{p'}$. This condition makes sense only if the right side of (9) is > 1 which means $p > 1.56$.

(2) In general estimates of the form $b_2(X_n) \le c\, f(a_2(X_n))\, n^{1/2 - \varepsilon}$, $\varepsilon > 0$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, f cannot be of polynomial growth because necessarily $n^{\varepsilon} \le d\, f(\sqrt{\ell n\, n})$ for some $d > 0$: take $X_n = \ell_r^n$, $r = \ell n\, n > 2$ to see this, $a_2(X_n) \le \sqrt{r}$, $n^{1/2 - 1/r} \le b_2(X_n)$.

Proof of proposition 2 : The proof is a combination of ideas of Maurey-Pisier [7], p. 68 and the proof of theorem 2. We will show for $\dim X_n = n$, $q > 2$, $n > 2^q\, b_{q,n}(X_n)^q$ and any finite sequence $(z_i)_{i=1}^m \subseteq X$:

$$(\int_\Omega \|\sum_{i=1}^m g_i(\omega) z_i\|^2\, dP(\omega) \le (c_q^2 \log(b_{q,n}(X_n) + 1))(\int_\Omega \|\sum_{i=1}^m r_i(t)\, z_i\|^2\, dt) \tag{11}$$

Theorem 1 and inequalities (1) and (11) yield

$$\begin{aligned} b_q(X_n) &\le c_q\, \{\log(b_{q,n}(X_n) + 1)\}^{1/2}\, \beta_q(X_n) \\ &\le c'_q\, \{\log(b_{q,n}(X_n) + 1)\}^{1/2}\, \beta_{q,n}(X_n) \\ &\le c''_q\, \{\log(b_{q,n}(X_n) + 1)\}^{1/2}\, b_{q,n}(X_n) \quad . \end{aligned}$$

In the case $n \le 2^q\, b_{q,n}(X_n)^q$ the same estimate is trivial, since necessarily $b_q(X_n) \le n^{1/q}$ and thus

$$b_q(X_n) \le 2\, b_{q,n}(X_n) \quad .$$

Thus it suffices to prove (11). Let $(z_i)_{i=1}^m \subseteq X$ be given,

$Z = \text{span}\{r_i(t)z_i\}_{i=1}^m \subseteq L_2(X_n)$. Define $(\gamma_j)_{j=1}^m$ as in the proof of theorem 2. Let $k = [2^q\ b_{q,n}(X_n)^q] + 1$. Then for any set $(v_i)_{i=1}^k \subseteq Z$ of pairwise disjoint vectors

$$(\inf_{1\le i\le k} \|v_i\|)k^{1/q} \le (\sum_{i=1}^k \|v_i\|_Z^q)^{1/q} \le b_{q,n}(X_n)\ \|\sum_{i=1}^k v_i\|_Z$$

since $b_{q,n}(Z) \le b_{q,n}(X_n)$. Thus

$$\gamma_k \le b_{q,n}(X_n)\ k^{-1/q} \le 1/2 \quad .$$

Let $r = 2\log_2 k$. Then, with $\gamma_j = 0$ for $j > m$

$$\begin{aligned}(\sum_{j=1}^m \gamma_j^r)^{1/r} &\le k^{1/r}\ (\sum_{j=0}^\infty k^j\ \gamma_k^{rj})^{1/r} \\ &\le k^{1/r}\ (\sum_{j=0}^\infty [k/2^r]^j)^{1/r} \le 2 \quad .\end{aligned}$$

This implies for arbitrary pairwise disjoint vectors $(v_j)_{j=1}^m$ in Z,

$$(\sum_{j=1}^m \|v_j\|_Z^r)^{1/r} \le 2\ \|\sum_{j=1}^m v_i\|_Z \quad . \tag{12}$$

Define a map $T : \ell_\infty^m \to Z$ by $(a_i)_{i=1}^m \mapsto \sum_{i=1}^m a_i\ r_i(t)\ z_i$. Then T is a positive map relative to the lattice structure on Z inherited from the 1-unconditional basis $(r_i(t)z_i)_{i=1}^m$ of Z. By Lindenstrauss-Tzafriri [5], p. 84 and 55, (12) implies for $a^{(j)} \in \ell_\infty^m$

$$\begin{aligned}(\sum_{j=1}^m \|Ta^{(j)}\|_Z^r)^{1/r} &\le 2\ \|\sum_{j=1}^m |Ta^{(j)}|\ \|_Z \\ &\le 2\ \|T\|\ \|\sum_{j=1}^m |a^{(j)}|\ \|_\infty\end{aligned}$$

and thus $\pi_{r,1}(T : \ell_\infty^m \to Z) \le 2\ \|T\|$. Since T is defined on ℓ_∞^m, one has by Maurey [6]

$$\pi_{2r}(T) \le c/2\ \pi_{r,1}(T)$$

for some absolute constant c (even independent of r). Thus $\pi_{2r}(T) \le c\ \|T\|$.

The Pietsch factorization theorem yields the existence of a sequence $\delta_i \ge 0$ with $\sum_{i=1}^m \delta_i = 1$ and

$$\|T(a_i)_{i=1}^m\| \le \pi_{2r}(T)\ (\sum_{i=1}^m |a_i|^{2r}\ \delta_i)^{1/2r}$$
$$\le c\ \|T\|\ (\sum_{i=1}^m |a_i|^{2r}\ \delta_i)^{1/2r} \quad .$$

Integrating this inequality with $a_i = g_i(\omega)$ gives

$$(\int_\Omega \int_0^1 \|\sum_{i=1}^\infty g_i(\omega)\ r_i(t)\ z_i\|^2\ dt\ dP(\omega))^{1/2}$$
$$\le c\ \|T\|\ (\int_\Omega (\sum_{i=1}^m |g_i(\omega)|^{2r}\ \delta_i)^{2/2r}\ dP(\omega))^{1/2}$$
$$\le c\ \|T\|\ (\int_\Omega \sum_{i=1}^m |g_i(\omega)|^{2r}\ \delta_i\ dP(\omega))^{1/2r}$$
$$\le c\ \|T\|\ \|g_1\|_{2r} \quad . \tag{13}$$

Now $\|g_1\|_{2r} \le c'\ \sqrt{r}$ and $\|T\| = (\int_0^1 \|\sum_{i=1}^m r_i(t)\ x_i\|^2\ dt)^{1/2}$. The left side of (13) is nothing but the Gaussian average $(\int_\Omega \|\sum_{i=1}^m g_i(\omega)\ x_i\|^2\ dP(\omega))^{1/2}$. This proves (11) since $r \sim q \log b_{q,n}(X_n)$.

<u>Remark</u> : Inequality (6) of proposition 2 is asymptotically optimal, in general : let $X_n = \ell_\infty^n$ and $q = 2$. Then

$$b_q(X_n) \sim b_{q,n}(X_n) \sim n^{1/2} \quad , \quad \beta_{q,n}(X_n) \sim (n/\log n)^{1/2} \quad ,$$

cf. [2]. It seems unknown, however, whether (7) could be improved to $b_q(X_n) \le c_q\ b_{q,n}(X_n)$.

REFERENCES

[1] J. Bergh, J. Löfström : Interpolation spaces, Springer, 1976.

[2] T. Figiel, J. Lindenstrauss , V. Milman : The dimension of almost spherical sections of convex bodies, Acta Math., 139 (1977), 53-94.

[3] U. Haagerup : The best constants in the Khintchine inequality, Odense Universitet, preprint 5, 1979.

[4] H. König, J.R. Retherford, N. Tomczak-Jaegermann : Eigenvalues of (p,2)-summing operators and constants associated with normed spaces, to appear in J. Funct. Anal.

[5] J. Lindenstrauss, L. Tzafriri : Classical Banach spaces, II, Springer, 1978.

[6] B. Maurey : Une nouvelle caractérisation des applications (p,q)-sommantes, Séminaire Maurey-Schwartz 1973-74, Exposé XII.

[7] B. Maurey, G. Pisier : Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, Studia Math. 58 (1976), 45-90.

[8] A. Pełczynski : Geometry of finite dimensional Banach spaces and operator ideals, in : Notes in Banach spaces, Univ. of Texas Press (1980).

[9] G. Pisier : Types des espaces normés, Note aux C. R. Acad. Sc. Paris, t. 276 (1973), 1673-1674.

[10] N. Tomczak-Jaegermann : Computing 2-summing norm with few vectors, Arkiv für Math. 17 (1979), 273-277.
