

SÉMINAIRE D'ANALYSE FONCTIONNELLE ÉCOLE POLYTECHNIQUE

H. KÖNIG

Some estimates for type and cotype constants

Séminaire d'analyse fonctionnelle (Polytechnique) (1979-1980), exp. n° 28, p. 1-13

http://www.numdam.org/item?id=SAF_1979-1980__A25_0

© Séminaire analyse fonctionnelle (dit "Maurey-Schwartz")
(École Polytechnique), 1979-1980, tous droits réservés.

L'accès aux archives du séminaire d'analyse fonctionnelle implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. : (1) 941.82.00 - Poste N°

Télex : ECOLEX 691596 F

S E M I N A I R E

D ' A N A L Y S E F O N C T I O N N E L L E

1979-1980

SOME ESTIMATES FOR TYPE AND COTYPE CONSTANTS
=====

H. KÖNIG

(Universität de Bonn)

This is a presentation of some results on type and cotype constants which were obtained in joint work with L. Tzafriri.

To fix notations, let X be a Banach space, $r_n(t)$ be the sequence of Rademacher functions, $g_n(t)$ be a sequence of independent standard Gaussian variables on a probability space (Ω, P) . Given $1 \leq p \leq 2 \leq q \leq \infty$ and $n \in \mathbb{N}$, we define $a_{p,n}(X)$, $b_{q,n}(X)$, $\alpha_{p,n}(X)$ and $\beta_{q,n}(X)$ to be smallest constants such that for arbitrary $x_1 \dots x_n \in X$ the following inequalities hold

$$b_{q,n}(X)^{-1} \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q} \leq \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^2 dt \right)^{1/2} \leq a_{p,n}(X) \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p}$$

$$\beta_{q,n}(X)^{-1} \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q} \leq \left(\int_{\Omega} \left\| \sum_{j=1}^n g_j(\omega) x_j \right\|^2 dP(\omega) \right)^{1/2} \leq \alpha_{p,n}(X) \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p}$$

If $a_p(X) = \sup_n a_{p,n}(X) < \infty$ (resp. $b_q(X) = \sup_n b_{q,n}(X) < \infty$), X is of (Rademacher) type p (resp. (Rademacher) cotype q). Similarly, define the Gaussian type p and cotype q constants by $\alpha_p(X) = \sup_n \alpha_{p,n}(X)$ and $\beta_q(X) = \sup_n \beta_{q,n}(X)$. These quantities were investigated by Maurey and Pisier [7].

We have for some c and c_p independent of n and X

$$c_p^{-1} \alpha_{p,n}(X) \leq a_{p,n}(X) \leq c \alpha_{p,n}(X) \tag{1}$$

$$\beta_{q,n}(X) \leq c b_{q,n}(X) .$$

The last two inequalities result immediately from

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^2 dt \leq c^2 \int_{\Omega} \left\| \sum_{j=1}^n g_j(\omega) x_j \right\|^2 dP(\omega) ,$$

cf. Pisier [9]. To prove the first inequality, we have by the symmetry of the g_j 's

$$\begin{aligned} \int_{\Omega} \left\| \sum_{j=1}^n g_j(\omega) x_j \right\|^p dP(\omega) &= \int_0^1 \int_{\Omega} \left\| \sum_{j=1}^n r_j(t) g_j(\omega) x_j \right\|^p dP(\omega) dt \\ &\leq \int_{\Omega} \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) g_j(\omega) x_j \right\|^2 dt \right)^{p/2} dP(\omega) \\ &\leq a_{p,n}(X)^p \int_{\Omega} \sum_{j=1}^n |g_j(\omega)|^p \|x_j\|^p dP(\omega) \\ &= \tilde{c}_p^p a_{p,n}(X)^p \sum_{j=1}^n \|x_j\|^p \end{aligned}$$

The equivalence of the Gaussian p - and 2-moments yields the desired inequality $\alpha_{p,n}(X) \leq c_p a_{p,n}(X)$.

If X does not have some finite cotype, i.e. $b_q(X) < \infty$ for some $q < \infty$, the sequences $\beta_{q,n}(X)$ and $b_{q,n}(X)$ may be inequivalent: for $X = \ell_\infty^n$ one gets $b_{2,n}(X) \sim n^{1/2}$ but $\beta_{2,n}(X) \sim (n/\log n)^{1/2}$, cf. Figiel-Lindenstrauss-Milman [2].

We will study the question whether the type and cotype constants of n -dimensional spaces X_n can be calculated essentially by n vectors, that is whether e.g.

$$a_p(X_n) \leq c_p a_{p,n}(X_n) \quad (2)$$

holds, with c_p depending only on p . For $p = 2$ one has the

Theorem (Tomczak-Jaegermann [10]) : For any n -dimensional space X_n ,

$$\alpha_2(X_n) \leq 2 \alpha_{2,n}(X_n) \quad \text{and} \quad \beta_2(X_n) \leq 2 \beta_{2,n}(X_n) .$$

The proof rests upon a corresponding statement for 2-absolutely summing norms of rank n operators, to which the Gaussian constants relate. Given $T: X \rightarrow Y$ and $1 \leq s \leq r < \infty$ we denote by $\pi_{r,s}^{(n)}(T)$ the smallest constant c such that for all $x_1, \dots, x_n \in X$

$$\left(\sum_{i=1}^n \|Tx_i\|^r \right)^{1/r} \leq c \sup_{\|x^*\|_{X^*} \leq 1} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^s \right)^{1/s} .$$

Clearly T is absolutely (r,s) -summing, $T \in \pi_{r,s}(X,Y)$ iff

$\pi_{r,s}^{(n)}(T) = \sup_n \pi_{r,s}^{(n)}(T) < \infty$. For $T: \ell_2^n \rightarrow X$, let

$\ell(T) := \left(\int_{\Omega} \left\| \sum_{i=1}^n g_i(\omega) Te_i \right\|^2 dP(\omega) \right)^{1/2}$ where e_i are the unit vectors in

ℓ_2^n ; ℓ has ideal norm properties.

The following lemma relates the Gaussian constant and $(q,2)$ -absolutely summing norms. It is due to Tomczak-Jaegermann [10] (for $p = 2 = q$, the generalization of the argument to $p < 2 < q$ is easy), for more details cf. also Pełczyński [8].

Lemma 1 : Let X be a Banach space, $1 \leq p \leq 2 \leq q \leq \infty$, $p' = p/(p-1)$ and $n \in \mathbb{N}$. Then

$$\beta_{q,n}(X) = \sup\{\pi_{q,2}^{(n)}(T) \mid T: \ell_2^n \rightarrow X \text{ with } \ell(T) \leq 1\}$$

$$\alpha_{p,n}(X) = \sup\{\ell(S) \mid S: \ell_2^n \rightarrow X \text{ with } (\pi_{p',2}^{(n)})^*(S^*) \leq 1\} .$$

Here $(\pi_{p',2}^{(n)})^*$ denotes the adjoint ideal norm to $\pi_{p',2}^{(n)}$.

Concerning problem (2) for $p \neq 2 \neq q$ we have the following positive answer

Theorem 1 : Let $1 < p < 2 < q < \infty$. There is $c_q \leq c/(q-2)$ such that for any $n \in \mathbb{N}$ and any n -dimensional space X_n

$$\alpha_p(X_n) \leq c_{p'}, \alpha_{p,n}(X_n) \text{ and } \beta_q(X_n) \leq c_q \beta_{q,n}(X_n)$$

and by (1), $a_p(X_n) \leq d_{p'}, a_{p,n}(X_n)$.

Theorem 1 results immediately from lemma 1 and proposition 1 below which we want to derive :

Proposition 1 : For any $q > 2$, there is $c_q \leq c/(q-2)$ such that for any $n \in \mathbb{N}$ and any rank n operator $T: X \rightarrow Y$

$$\pi_{q,2}(T) \leq c_q \pi_{q,2}^{(n)}(T) .$$

Defining the approximation numbers of $T: X \rightarrow Y$ by

$$\alpha_j(T) := \inf\{\|T - T_j\| \mid T_j: X \rightarrow Y \text{ of rank } < j\} , \quad j \in \mathbb{N}$$

we let for $0 < r \leq \infty$

$$S_r(X,Y) = \{T: X \rightarrow Y \mid \sigma_r(T) := (\sum_{j=1}^{\infty} \alpha_j(T)^r)^{1/r} < \infty\}$$

$$S_{2,1}(X,Y) = \{T: X \rightarrow Y \mid \sigma_{2,1}(T) := \sum_{j=1}^{\infty} \alpha_j(T) j^{-1/2} < \infty\} .$$

Thus $\sigma_{2,1}(T)$ is the norm of $(\alpha_j(T))_{j \in \mathbb{N}}$ in the Lorentz sequence space $\ell_{2,1}$ which can be written as a real interpolation space between ℓ_q -spaces ; in particular $\ell_{2,1} = (\ell_1, \ell_{\infty})_{\frac{1}{2}, 1}$, cf. [1].

Proof of proposition 1 : Since $\pi_{q,2}(T) = \sup\{\pi_{q,2}(TA) : A : \ell_2 \rightarrow X, \|A\| \leq 1\}$, it suffices to prove the statement for maps $T : \ell_2^n \rightarrow Y$. We will show

$$\pi_{q,2}(T) \leq c_q \sigma_q(T) \leq c_q \pi_{q,2}^{(n)}(T) \quad . \quad (3)$$

Step 1 : We define inductively an orthonormal basis $(e_j)_{j=1}^n$ of ℓ_2^n with $\alpha_j(T) \leq \|Te_j\|$. For $j=1$, choose e_1 of norm one such that $\alpha_1(T) = \|T\| = \|Te_1\|$. If $j < n$ orthonormal vectors have been found, let $Y_j := [e_1, \dots, e_j]$ and $P_j : \ell_2^n \rightarrow Y_j \subseteq \ell_2^n$ be the orthogonal projection. Thus

$$\alpha_{j+1}(T) \leq \|T - TP_j\| = \|T|_{Y_j^\perp}\| \quad .$$

Hence there is $e_{j+1} \in Y_j^\perp$ of norm one such that $\alpha_{j+1}(T) \leq \|Te_{j+1}\|$. Since $\text{rank } T \leq n$, $\alpha_k(T) = 0$ for $k > n$. This yields

$$\sigma_q(T) = \left(\sum_{j=1}^n \alpha_j(T)^q \right)^{1/q} \leq \left(\sum_{j=1}^n \|Te_j\|^q \right)^{1/q} \leq \pi_{q,2}^{(n)}(T) \quad ,$$

the right side inequality in (3).

Step 2 : We show $S_{2,1}(X,Y) \leq \pi_2(X,Y)$ for any X and Y . Taking $S \in S_{2,1}(X,Y)$ choose $D_j : X \rightarrow Y$ of rank $D_j < 2^j$ with $\|S - D_j\| \leq 2\alpha_{2^j}(S)$,

$j = 0, 1, \dots$ ($D_0 = 0$). Let $S_j = D_{j+1} - D_j$. Then $S = \sum_{j=0}^{\infty} S_j$, $\|S_j\| \leq 4\alpha_{2^j}(S)$ and $\text{rank } S_j < 2^{j+2}$. Since the 2-absolutely summing norm of the identity on an n -dimensional space is $n^{1/2}$, we infer

$$\begin{aligned} \pi_2\left(\sum_{j=0}^N S_j\right) &\leq \sum_{j=0}^N \pi_2(S_j) \leq \sum_{j=0}^N \|S_j\| 2^{j/2+1} \\ &\leq 16 \sum_{j=0}^N 2^{j/2-1} \alpha_{2^j}(S) \\ &\leq 16 \sum_{k=1}^{\infty} k^{-1/2} \alpha_k(S) = 16 \sigma_{2,1}(S) \quad . \end{aligned}$$

Thus S is 2-summing with $\pi_2(S) \leq 16 \sigma_{2,1}(S)$.

Step 3 : The K -functional of the real interpolation theory [1] satisfies

$$K(t, T; S_1(X, Y), S_\infty(X, Y)) \sim K(t, (\alpha_j(T))_{j=1}^{\infty}; \ell_1, \ell_\infty) \quad . \quad (4)$$

Here $S_\infty = \mathcal{L} =$ all continuous linear maps. By definition of the K-functional,

$$\begin{aligned} K(t, T; S_1, S_\infty) &:= \inf \left\{ \sum_{j \in \mathbb{N}} \alpha_j(T_t) + t \|T - T_t\| \mid T_t : X \rightarrow Y \right\} \\ &\geq \inf \left\{ \sum_{j=1}^{[t]} (\alpha_j(T_t) + \|T - T_t\|) \right\} \geq \sum_{j=1}^{[t]} \alpha_j(T) \\ &\sim K(t, (\alpha_j(T))_{j=1}^\infty; \ell_1, \ell_\infty) \quad , \end{aligned}$$

for the last equivalence cf. [1]. For $t \geq 1$, choose $T_t : X \rightarrow Y$ with rank $T_t < [t]$ and $\|T - T_t\| \leq 2 \alpha_{[t]}(T)$. Then

$$\alpha_j(T_t) \leq \alpha_j(T) + \|T - T_t\| \leq 3 \alpha_j(T)$$

for all $j < [t]$ and $\alpha_j(T_t) = 0$ for $j \geq [t]$, hence

$$\begin{aligned} K(t, T; S_1, S_\infty) &\leq \sum_{j=1}^{[t]} \alpha_j(T_t) + t \|T - T_t\| \\ &\leq 3 \sum_{j=1}^{[t]} (\alpha_j(T) + \alpha_{[t]}(T)) \leq 6 \sum_{j=1}^{[t]} \alpha_j(T) \end{aligned}$$

which proves (4).

Step 4 : Since $\ell_{2,1} = (\ell_1, \ell_\infty)_{\frac{1}{2}, 1}$, the equivalence (4) yields

$S_{2,1}(X, Y) = (S_1(X, Y), S_\infty(X, Y))_{\frac{1}{2}, 1}$. Let $q > 2$, $\frac{1}{q} = \frac{1-\theta}{2}$ and $\eta = \frac{1}{2}(1+\theta)$. Then

by the reiteration theorem [1]

$$(S_{2,1}(X, Y), S_\infty(X, Y))_{\theta, q} = (S_1(X, Y), S_\infty(X, Y))_{\eta, q} = S_q(X, Y) \quad (5)$$

where the last equality follows from (4) and $(\ell_1, \ell_\infty)_{\eta, q} = \ell_q$. It is an easy consequence from

$$\ell_q(X) = (\ell_2(X), \ell_\infty(X))_{\theta, q} \quad , \quad \frac{1}{q} = \frac{1-\theta}{2} \quad , \quad 0 < \theta < 1$$

that

$$(\pi_2(X, Y), \mathcal{L}(X, Y))_{\theta, q} \leq \pi_{q, 2}(X, Y) \quad .$$

This, (5) and step 2 show $S_q(X, Y) \leq \pi_{q, 2}(X, Y)$ for any $q > 2$ and thus

$\pi_{q, 2}(T) \leq c_q \sigma_q(T)$ for any $T \in S_q(X, Y)$, where c_q depends only on $q > 2$;

the bound $c_q \leq c/(q-2)$ can be derived by checking the constants occurring in the reiteration theorem. This proves the left side in (3) and thus proposition 1.

As a corollary to the proof we note a fact which is false for $q = 2$:

Corollary 1 : For any $2 < q < \infty$ and any Banach space Y

$$\pi_{q,2}(\ell_2, Y) = S_q(\ell_2, Y) \quad .$$

Corollary 2 : For any $2 < q < \infty$, there is c_q such that for any $n \in \mathbb{N}$ and any rank n operator $T : X \rightarrow Y$

$$\pi_{q,1}(T) \leq c_q \pi_{q,1}^{(n)}(T) \quad .$$

Proof : It is well-known that

$$\begin{aligned} \pi_{q,1}(T) &= \sup\{\pi_{q,1}(TA) \mid \|A : \ell_\infty \rightarrow X\| \leq 1\} \\ &\leq \sup\{\pi_{q,2}(TA) \mid \|A : \ell_\infty \rightarrow X\| \leq 1\} \\ &\leq c_q \sup\{\pi_{q,2}^{(n)}(TA) \mid \|A : \ell_\infty \rightarrow X\| \leq 1\} \quad . \end{aligned}$$

By Maurey [6], the $(q,1)$ - and $(q,2)$ -absolutely summing norms are equivalent on $\pi_{q,1}(\ell_\infty, Y)$; the argument does not depend on the number of vectors considered. Thus

$$\pi_{q,1}(T) \leq d_q \sup\{\pi_{q,1}^{(n)}(TA) \mid \|A : \ell_\infty \rightarrow X\| \leq 1\} \leq d_q \pi_{q,1}^{(n)}(T) \quad .$$

Theorem 1 gives no answer to the question whether the (Rademacher) cotype constants on n -dimensional spaces X_n can be calculated by n vectors, i.e. whether

$$b_q(X_n) \leq c_q b_{q,n}(X_n)$$

does hold or not. We only have a partial answer :

Proposition 2 : Let $q > 2$. There is $c_q > 0$ such that for any $n \in \mathbb{N}$ and any n -dimensional space

$$b_q(X_n) \leq c_q \beta_{q,n}(X_n) (\log b_{q,n}(X_n))^{1/2} \quad (6)$$

and thus by (1)

$$b_q(X_n) \leq c_q b_{q,n}(X_n) (\log b_{q,n}(X_n))^{1/2} \quad (7)$$

Remark : c_q can be bounded by $c \sqrt{q}$ as q tends to ∞ .

We postpone the proof of proposition 2. It is well-known, Maurey-Pisier [7], that a Banach space X has some finite cotype $q < \infty$ if it has type p for some $p > 1$. More quantitative information is given in

Theorem 2 : Let $p > 1$, $p' = p/(p-1)$ and X be a Banach space.

(a) If $\dim X = 2^n$ and $q = 2 + 2^{p'} a_{p,n}(X)^{p'}$, then $b_q(X) \leq 2$.

(b) If X has type $p > 1$, it has cotype q for $q = 2 + 2^{p'} a_p(X)^{p'}$.

Proof : Clearly it suffices to prove part (a).

Step 1 : Let $\dim X = 2^n$ and $\{z_i\}_{i \in \mathbb{N}}$ be dense in X . Denote the span of $\{r_i(t)z_i\}_{i \in \mathbb{N}}$ in $L_p(X)$ by Z . Then $\{r_i(t)z_i\}_{i \in \mathbb{N}}$ is a 1-unconditional basis of Z . We now use a Shimogaki-Pisier-type argument : define γ_j to be the smallest constant such that for any sequence $(u_i)_{i=1}^j \subseteq Z$ of pairwise disjoint elements (relative to the lattice structure inherited from the 1-unconditional basis $(r_i(t)z_i)_{i \in \mathbb{N}}$ of Z) one has

$$\inf_{1 \leq i \leq j} \|u_i\|_Z \leq \gamma_j \left\| \sum_{i=1}^j u_i \right\|_Z \quad .$$

Then $1 = \gamma_1 \geq \gamma_2 \geq \dots$ and $\gamma_{jk} \leq \gamma_j \gamma_k$ for all j and k in \mathbb{N} , cf. Lindenstrauss-Tzafriri [5], p. 91. Given $m \in \mathbb{N}$, choose $\{w_i\}_{i=1}^{2^m}$ such that

$$\inf_{1 \leq i \leq 2^m} \|w_i\|_Z = 1 \quad , \quad \left\| \sum_{i=1}^{2^m} w_i \right\|_Z = \frac{1}{\gamma_{2^m}} \quad .$$

It is a consequence of the 1-unconditionality of $\{r_i(t)z_i\}$ in Z that we may assume w.l.o.g. $\|w_i\|_Z = 1$ for all $i = 1, \dots, 2^m$ and moreover

$$\sup_{1 \leq i \leq 2^m} |c_i| \leq \left\| \sum_{i=1}^{2^m} c_i w_i \right\|_Z \leq \frac{1}{\gamma_{2^m}} \sup_{1 \leq i \leq 2^m} |c_i|$$

for all scalar sequences $(c_i) \in \mathbb{R}^{2^m}$. Thus $\text{span}\{w_i\}$ is $\frac{1}{\gamma_{2^m}}$ -isomorphic to

$\ell_\infty^{2^m}$ which contains (in the real case) isometrically ℓ_1^m . Thus there are $v_1, \dots, v_m \in Z$ (which can be realized as the Rademacher elements over the

$\{w_i\}$) such that

$$\sum_{i=1}^m |c_i| \leq \left\| \sum_{i=1}^m c_i v_i \right\|_Z \leq \frac{1}{\gamma_{2^m}} \sum_{i=1}^m |c_i| \quad (8)$$

for all scalar sequences (c_i) . It is well-known and easily checked by integral inequalities that $a_{p,m}(X) = a_{p,m}(L_p(X)) \geq a_{p,m}(Z)$. Integrating (8) with $c_i = r_i(t)$, we get

$$m \leq \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t) v_i \right\|_Z^p dt \right)^{1/p} \leq a_{p,m}(X) \frac{1}{\gamma_{2^m}} m^{1/p} \quad ,$$

$$\gamma_{2^m} \leq a_{p,m}(X) m^{-1/p'} \quad . \quad (9)$$

Step 2 : We first consider the case $a_{p,n}(X) n^{-1/p'} \geq 1/2$. Then $q := 2 + 2^{p'}$ $a_{p,n}(X)^{p'} \geq 2 + n > n$. It follows from

$$b_2(\ell_2^{2^n}) = 1 \quad \text{and} \quad d(X, \ell_2^{2^n}) \leq 2^{n/2}$$

that $b_2(X) \leq 2^{n/2}$. An interpolation argument shows

$$b_q(X) \leq b_2(X)^{2/q} \leq 2^{n/q} < 2 \quad .$$

Step 3 : We now derive the same conclusion in the other case $a_{p,n}(X) n^{-1/p'} < 1/2$, for the same value of q . Let $(y_i)_{i=1}^{\ell} \subseteq X$ be an arbitrary finite sequence. We may assume that $\|y_i\|$ is non-increasing. Since the $(z_i)_{i=1}^{\infty}$ were dense in X ,

$$\|y_j\| = \inf_{1 \leq i \leq j} \|y_i\| \leq \gamma_j \left(\int_0^1 \left\| \sum_{i=1}^j r_i(t) y_i \right\|^p dt \right)^{1/p}$$

holds for all j . Hence

$$\begin{aligned} \left(\sum_j \|y_j\|^q \right)^{1/q} &\leq \left(\sum_j \gamma_j^q \right)^{1/q} \left(\int_0^1 \left\| \sum_i r_i(t) y_i \right\|^p dt \right)^{1/p} \\ &\leq \left(\sum_j \gamma_j^q \right)^{1/q} \left(\int_0^1 \left\| \sum_i r_i(t) y_i \right\|^2 dt \right)^{1/2} \end{aligned}$$

and thus

$$b_q(X) \leq \left(\sum_j \gamma_j^q \right)^{1/q} \quad .$$

Let k be the first integer $\leq n$ such that

$$\gamma_{2^k} \leq a_{p,n}(X) k^{-1/p'} \leq 1/2 \quad .$$

Then

$$\begin{aligned} \left(\sum_{j=1}^{\infty} \gamma_j^q \right)^{1/q} &\leq 2^{k/q} \left(\sum_{i=0}^{\infty} 2^{ik} \gamma_{2^{ik}}^q \right)^{1/q} \\ &\leq 2^{k/q} \left(\sum_{i=0}^{\infty} 2^{ik} \gamma_{2^k}^{iq} \right)^{1/q} \\ &\leq 2^{k/q} \left(\sum_{i=0}^{\infty} \left(\frac{1}{2^{q-k}} \right)^i \right)^{1/q} \leq 2^{k/q} 2^{1/q} \leq 2 . \end{aligned}$$

Note here $k-1 \leq 2^{p'}$ $a_{p,n}(X)^{p'}$ and thus $k+1 \leq q$ as well as $q-k \geq 1$. Hence $b_q(X) \leq 2$ in the second case, too.

Example : The value of $q = 2 + 2^{p'}$ $a_p(X)^{p'}$ may be slightly improved : choosing $c = e^{1/p'}$ instead of $c = 2$ in $a_{p,n}(X) n^{-1/p'} \geq c^{-1}$, one gets that X is of type q for any $q > p' \ln 2 + (e p' \ln 2) a_p(X)^{p'}$. However, in general this value of q has to be at least as large as $q \geq e a_p(X)^{p'}$ as shown by $X = L_r(\mu)$, $r > 2$: clearly X is not of cotype q for $q < r$. On the other hand, $a_p(X) \leq a_2(X)^{2/p'} \leq B_r^{1/p'}$ where B_r is the Khintchin constant for Rademacher r -averages. By the estimates of Haagerup [3], $B_r \sim r/e$ as $r \rightarrow \infty$.

If $T: X \rightarrow Y$ factors through a Hilbert space, let $\gamma_2(T)$ denote the factorization norm.

Corollary 3 : Assume X is a Banach space of type 2. Let $\varepsilon = (6 a_2(X)^2)^{-1}$. Then onto any n -dimensional subspace X_n of X there is a projection P_n with

$$\gamma_2(P_n) \leq 8 a_2(X) n^{1/2-\varepsilon} .$$

In particular, one has for the relative projection constant of X_n in X and the Banach-Mazur distance to Hilbert space

$$\lambda_X(X_n) \leq 8 a_2(X) n^{1/2-\varepsilon}$$

$$d(X_n, X) \leq 8 a_2(X) n^{1/2-\varepsilon} .$$

Proof : It is known (cf. [4]) that there is a projection $P_n: X \rightarrow X_n$ with

$$\gamma_2(P_n) \leq 4 a_2(X) \beta_2(X_n) .$$

Since $\beta_2(X_n) \leq b_2(X_n) \leq n^{1/2 - 1/q} b_q(X_n)$, theorem 2 implies for $q = 6$ $a_2(X)^2 \geq 2 + 2^2 a_2(X)^2$ that $\beta_2(X_n) \leq 2 n^{1/2 - \varepsilon}$.

Remarks :

(1) The existence of an $\varepsilon > 0$ as in corollary 3 is an open problem in spaces of type $p > 1$. Theorem 2 yields a positive answer for small type p constants and values p near 2 : if

$$a_p(X) < \left(\frac{2(p-1) - (2-p) \ell_n 2}{e(2-p) \ell_n 2} \right)^{1/p'} \tag{10}$$

corollary 3 holds for X and $\varepsilon = \frac{1}{2} - (\frac{1}{p} - \frac{1}{r}) (> 0)$ where $r > p' \ell_n 2 + (e p' \ell_n 2) a_p(X)^{p'}$. This condition makes sense only if the right side of (9) is > 1 which means $p > 1.56$.

(2) In general estimates of the form $b_2(X_n) \leq c f(a_2(X_n)) n^{1/2 - \varepsilon}$, $\varepsilon > 0$, and $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ increasing, f cannot be of polynomial growth because necessarily $n^\varepsilon \leq d f(\sqrt{\ell_n n})$ for some $d > 0$: take $X_n = \ell_n^n$, $r = \ell_n n > 2$ to see this, $a_2(X_n) \leq \sqrt{r}$, $n^{1/2 - 1/r} \leq b_2(X_n)$.

Proof of proposition 2 : The proof is a combination of ideas of Maurey-Pisier [7], p. 68 and the proof of theorem 2. We will show for $\dim X_n = n$, $q > 2$, $n > 2^q b_{q,n}(X_n)^q$ and any finite sequence $(z_i)_{i=1}^m \subseteq X$:

$$\left(\int_{\Omega} \left\| \sum_{i=1}^m g_i(\omega) z_i \right\|^2 dP(\omega) \leq (c_q^2 \log(b_{q,n}(X_n) + 1)) \left(\int_{\Omega} \left\| \sum_{i=1}^m r_i(t) z_i \right\|^2 dt \right) \right) \tag{11}$$

Theorem 1 and inequalities (1) and (11) yield

$$\begin{aligned} b_q(X_n) &\leq c_q \{ \log(b_{q,n}(X_n) + 1) \}^{1/2} \beta_q(X_n) \\ &\leq c'_q \{ \log(b_{q,n}(X_n) + 1) \}^{1/2} \beta_{q,n}(X_n) \\ &\leq c''_q \{ \log(b_{q,n}(X_n) + 1) \}^{1/2} b_{q,n}(X_n) . \end{aligned}$$

In the case $n \leq 2^q b_{q,n}(X_n)^q$ the same estimate is trivial, since necessarily $b_q(X_n) \leq n^{1/q}$ and thus

$$b_q(X_n) \leq 2 b_{q,n}(X_n) .$$

Thus it suffices to prove (11). Let $(z_i)_{i=1}^m \subseteq X$ be given,

$Z = \text{span}\{r_i(t)z_i\}_{i=1}^m \subseteq L_2(X_n)$. Define $(\gamma_j)_{j=1}^m$ as in the proof of theorem 2. Let $k = \lceil 2^q b_{q,n}(X_n)^q \rceil + 1$. Then for any set $(v_i)_{i=1}^k \subseteq Z$ of pairwise disjoint vectors

$$\left(\inf_{1 \leq i \leq k} \|v_i\| \right) k^{1/q} \leq \left(\sum_{i=1}^k \|v_i\|_Z^q \right)^{1/q} \leq b_{q,n}(X_n) \left\| \sum_{i=1}^k v_i \right\|_Z$$

since $b_{q,n}(Z) \leq b_{q,n}(X_n)$. Thus

$$\gamma_k \leq b_{q,n}(X_n) k^{-1/q} \leq 1/2 \quad .$$

Let $r = 2 \log_2 k$. Then, with $\gamma_j = 0$ for $j > m$

$$\begin{aligned} \left(\sum_{j=1}^m \gamma_j^r \right)^{1/r} &\leq k^{1/r} \left(\sum_{j=0}^{\infty} k^j \gamma_k^{rj} \right)^{1/r} \\ &\leq k^{1/r} \left(\sum_{j=0}^{\infty} [k/2^r]^j \right)^{1/r} \leq 2 \quad . \end{aligned}$$

This implies for arbitrary pairwise disjoint vectors $(v_j)_{j=1}^m$ in Z ,

$$\left(\sum_{j=1}^m \|v_j\|_Z^r \right)^{1/r} \leq 2 \left\| \sum_{j=1}^m v_j \right\|_Z \quad . \quad (12)$$

Define a map $T: \ell_\infty^m \rightarrow Z$ by $(a_i)_{i=1}^m \mapsto \sum_{i=1}^m a_i r_i(t) z_i$. Then T is a positive map relative to the lattice structure on Z inherited from the 1-unconditional basis $(r_i(t)z_i)_{i=1}^m$ of Z . By Lindenstrauss-Tzafriri [5], p. 84 and 55, (12) implies for $a^{(j)} \in \ell_\infty^m$

$$\begin{aligned} \left(\sum_{j=1}^m \|Ta^{(j)}\|_Z^r \right)^{1/r} &\leq 2 \left\| \sum_{j=1}^m |Ta^{(j)}| \right\|_Z \\ &\leq 2 \|T\| \left\| \sum_{j=1}^m |a^{(j)}| \right\|_\infty \end{aligned}$$

and thus $\pi_{r,1}(T: \ell_\infty^m \rightarrow Z) \leq 2 \|T\|$. Since T is defined on ℓ_∞^m , one has by Maurey [6]

$$\pi_{2r}(T) \leq c/2 \pi_{r,1}(T)$$

for some absolute constant c (even independent of r). Thus

$$\pi_{2r}(T) \leq c \|T\|.$$

The Pietsch factorization theorem yields the existence of a sequence

$$\delta_i \geq 0 \text{ with } \sum_{i=1}^m \delta_i = 1 \text{ and}$$

$$\begin{aligned} \|T(a_i)_{i=1}^m\| &\leq \pi_{2r}(T) \left(\sum_{i=1}^m |a_i|^{2r} \delta_i \right)^{1/2r} \\ &\leq c \|T\| \left(\sum_{i=1}^m |a_i|^{2r} \delta_i \right)^{1/2r} . \end{aligned}$$

Integrating this inequality with $a_i = g_i(\omega)$ gives

$$\begin{aligned} \left(\int_{\Omega} \int_0^1 \left\| \sum_{i=1}^m g_i(\omega) r_i(t) z_i \right\|^2 dt dP(\omega) \right)^{1/2} \\ \leq c \|T\| \left(\int_{\Omega} \left(\sum_{i=1}^m |g_i(\omega)|^{2r} \delta_i \right)^{2/2r} dP(\omega) \right)^{1/2} \\ \leq c \|T\| \left(\int_{\Omega} \sum_{i=1}^m |g_i(\omega)|^{2r} \delta_i dP(\omega) \right)^{1/2r} \\ \leq c \|T\| \|g_1\|_{2r} . \end{aligned} \quad (13)$$

Now $\|g_1\|_{2r} \leq c' \sqrt{r}$ and $\|T\| = \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\|^2 dt \right)^{1/2}$. The left side of (13) is nothing but the Gaussian average

$\left(\int_{\Omega} \left\| \sum_{i=1}^m g_i(\omega) x_i \right\|^2 dP(\omega) \right)^{1/2}$. This proves (11) since $r \sim q \log b_{q,n}(X_n)$.

Remark : Inequality (6) of proposition 2 is asymptotically optimal, in general : let $X_n = \ell_{\infty}^n$ and $q = 2$. Then

$$b_q(X_n) \sim b_{q,n}(X_n) \sim n^{1/2} , \quad \beta_{q,n}(X_n) \sim (n/\log n)^{1/2} ,$$

cf. [2]. It seems unknown, however, whether (7) could be improved to $b_q(X_n) \leq c_q b_{q,n}(X_n)$.

REFERENCES

- [1] J. Bergh, J. Löfström : Interpolation spaces, Springer, 1976.
- [2] T. Figiel, J. Lindenstrauss, V. Milman : The dimension of almost spherical sections of convex bodies, Acta Math., 139 (1977), 53-94.
- [3] U. Haagerup : The best constants in the Khintchine inequality, Odense Universitet, preprint 5, 1979.
- [4] H. König, J.R. Retherford, N. Tomczak-Jaegermann : Eigenvalues of $(p,2)$ -summing operators and constants associated with normed spaces, to appear in J. Funct. Anal.
- [5] J. Lindenstrauss, L. Tzafriri : Classical Banach spaces, II, Springer, 1978.

- [6] B. Maurey : Une nouvelle caractérisation des applications (p,q) -sommantes, Séminaire Maurey-Schwartz 1973-74, Exposé XII.
- [7] B. Maurey, G. Pisier : Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, *Studia Math.* 58 (1976), 45-90.
- [8] A. Pełczyński : Geometry of finite dimensional Banach spaces and operator ideals, in : Notes in Banach spaces, Univ. of Texas Press (1980).
- [9] G. Pisier : Types des espaces normés, Note aux C. R. Acad. Sc. Paris, t. 276 (1973), 1673-1674.
- [10] N. Tomczak-Jaegermann : Computing 2-summing norm with few vectors, *Arkiv för Math.* 17 (1979), 273-277.
