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S E M I N A I R E
D ' A N A L Y S E F O N C T I O N N E L L E
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VOLUME ESTIMATES AND NEARLY EUCLIDEAN
DECOMPOSITIONS FOR NORMED SPACES

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The purpose of this talk is to present a new isomorphic invariant of a finite dimensional normed space, so called "volume ratio" (introduced in [8]). We set

$$\text{vr}(E) = \left(\frac{\text{vol } B_E}{\text{vol } \mathcal{E}} \right)^{1/n},$$

where B_E is the unit ball of an n -dimensional real normed space E , \mathcal{E} -the ellipsoid of maximal volume contained in B_E (so called "John's ellipsoid of E ") and $\text{vol } A$ stands for volume of a set A .

It follows directly from the definition that

$$(1) \quad \text{vr}(E) \leq \text{vr}(F) d(E, F),$$

where E and F are normed spaces of the same dimension, d -the Banach-Mazur distance.

To explain the motivation for introducing such an invariant let us mention the following :

Theorem 1 (Kashin [6]) : There is a universal constant C such that, given n , there exist two n -dimensional subspaces E_1, E_2 of L_{2n}^1 , orthogonal (in ℓ_{2n}^2) satisfying

$$d(E_i, \ell_n^2) \leq C \quad \text{for } i = 1, 2 \quad \blacksquare$$

Theorem 1 solved some problems from the approximation theory and was used later (see [3]) to construct an n -dimensional space, whose constant of local unconditional structure is of order \sqrt{n} (the largest possible). However, Kashin's original proof was very complicated. A simple proof of Th. 1 appeared in [7]. It depends essentially on the following two observations.

Proposition 2 : $\text{vr}(\ell_n^1) \leq \sqrt{2 e/\pi}$ for $n = 1, 2, \dots$ \blacksquare

Proposition 3 : Let C and $\theta < 1$ be positive constants. Then, for any normed space E with $\text{vr}(E) < C$ and positive integer $k \leq \theta \dim E$, "most of" k -dimensional subspaces F of E satisfy

$$(2) \quad d(F, \ell_k^2) \leq C',$$

where C' depends only on C and θ . More precisely : if $G = G(k, n)$ is the Grassmann manifold of k -dimensional subspaces of E , μ - a normalized invariant measure on G , generated by the John's ellipsoid. Then

$$(3) \quad \mu(\{F \in G : F \text{ satisfies (2)}\}) > \frac{1}{2} \quad . \quad \blacksquare$$

Deducing Th. 1 from Prop. 2 and Prop. 3 is immediate, one must only remember that the map $F \mapsto F^\perp$ (the orthogonal complement of F), acting on $G(n, 2n)$, is measure-preserving.

Proof of Prop. 2 : By direct computation. \blacksquare

Proof of Prop. 3 : Let $E = (\mathbb{R}^n, \|\cdot\|)$. We may assume that the John's ellipsoid of E is equal to the Euclidean unit ball $B^n = \{\|x\|_2 \leq 1\}$. Denote by m the normalized Haar measure on S^{n-1} . Then

$$(4) \quad C^n > \text{vol}(E)^n = \int_{S^{n-1}} \|x\|^{-n} m(dx)$$

(one gets the equality by representing $\text{vol } A$ as $\int_{\mathbb{R}^n} \chi_A$ and passing to polar coordinates).

Given $r \in (0, 1)$ define $A_r = \{x \in S^{n-1} : \|x\| < r\}$. Then one gets from (4) that

$$m(A_r) < (Cr)^n \quad .$$

On the other hand, we have

$$\begin{aligned} m(A_r) &= \int_{S^{n-1}} \chi_{A_r} dm = \int_G \mu(dF) \int_{S_F} \chi_{A_r \cap F} dm_F = \\ &= \int_G m_F(A_r \cap F) \mu(dF) \quad , \end{aligned}$$

where m_F is the normalized Haar measure on $S_F = F \cap S^{n-1}$. The last two formulae show that

$$\mu(\{F \in G : m_F(A_r \cap F) < 2(Cr)^n\}) > \frac{1}{2} \quad ;$$

in other words, for "most of" $F \in G$ we have

$$m_F(\{x \in S_F : \|x\| < r\}) < 2(Cr)^n \leq (2Cr)^n \quad .$$

We show that every such F is "close" to ℓ_k^2 in the Banach-Mazur sense, thus proving Prop. 3.

Indeed, since, for given $x_0 \in S_F$ and $\delta \leq \frac{1}{2}$,

$$m_F(\{x \in S_F : \|x - x_0\|_2 \leq \delta\}) \geq \left(\frac{\delta}{4}\right)^k,$$

the previous estimate shows (remember that $k \leq \theta n$) that $S_F \setminus A_r$ is an $r/2$ -net (in ℓ_n^2 metric) for S_F , provided $r = r(\theta, C)$ is small enough (precisely, if $r \leq (2^{3\theta+1} C)^{1/(1-\theta)}$). Fix such r . Then, for any $y \in S_F$, there is a $y_0 \in S_F \setminus A_r$ (i.e. $\|y_0\| \geq r$) such that $\|y - y_0\|_2 \leq r/2$. Since (by $B^n \subset B_E$) $\|x\|_2 \geq \|x\|$ for all $x \in E$, we have also $\|y - y_0\| \leq \frac{r}{2}$. Therefore

$$\|y\| \geq \|y_0\| - \|y - y_0\| \geq r - \frac{r}{2} = \frac{r}{2}.$$

So, by homogeneity,

$$\frac{r}{2} \|y\|_2 \leq \|y\| \leq \|y\|_2$$

for all $y \in F$. Hence $d(F, \ell_k^2) \leq 2r^{-1} = 2 r(\theta, C)^{-1}$.

This ends the proof of Prop. 3. ■

In the sequel, we shall frequently use the following concepts. We say that (e_i) is an unconditional basis of a B-space E provided

$$\text{ubc}(e_i) \stackrel{\text{def}}{=} \sup_{|\varepsilon_i| \leq 1, \|\sum_i t_i e_i\| \leq 1} \left\| \sum_i \varepsilon_i t_i e_i \right\| < \infty.$$

We say that a B-space E is of cotype q ($q \geq 2$) if there is a constant K such that, for every finite sequence $x_1, x_2, \dots \in E$, we have

$$\int \left\| \sum_i r_i x_i \right\| \geq K^{-1} \left(\sum_i \|x_i\|^q \right)^{1/q},$$

where (r_i) is the sequence of Rademacher functions. The smallest such constant K is called the cotype q constant of E and denoted by $K_q(E)$.

It was proved in [4] that given K there exist $C, \theta > 0$ such that, for every finite dimensional E with $K_2(E) \leq K$, one can find a subspace of E , say F , with $\dim F = k \geq \theta \dim E$ and $d(F, \ell_k^2) \leq C$. Prop. 2 and Prop. 3 strengthen this result in the special case $E = \ell_n^1$. This raises

the following problems :

Problem 4 : Given $\theta \in (0,1)$, does every normed space E contain a $[\theta \dim E]$ -dimensional subspace F with $d(\ell_{\dim F}^2, F) < C$, where C depends only on $K_2(E)$? ■

Problem 5 : Does there exist a function $C(\cdot)$ such that $\text{vr}(E) \leq C(K_2(E))$ for every E ? ■

Of course a positive solution of Problem 5 implies a positive solution of Problem 4. We have two partial results in this direction.

Theorem 6 [8] : Let E be a finite dimensional space, (e_i) -its basis. Then

$$\text{vr}(E) \leq C K_2(E) \text{ubc}(e_i)$$

where C is a universal constant. ■

Theorem 7 [8] : There is a universal constant C such that

$$\text{vr}(\ell_n^2 \hat{\otimes} \ell_n^2) \leq C \text{ for all } n. \quad \blacksquare$$

Recall that $\ell_n^2 \hat{\otimes} \ell_n^2$ is the tensor product $\ell_n^2 \otimes \ell_n^2$ equipped with the largest tensor norm (in other words : the space of nuclear operators on ℓ_n^2). It is known that $\text{ubc}(\omega_i)$ is of order \sqrt{n} for every basis (ω_i) of $\ell_n^2 \hat{\otimes} \ell_n^2$, while $K_2(\ell_n^2 \hat{\otimes} \ell_n^2) \leq K$, where K does not depend on n .

Theorem 7 can be generalized to a large class of tensor products and unitary ideals. In particular, a unitary ideal \mathfrak{U} on ℓ_n^2 has "small" volume ratio if the associated n -dimension symmetric space $\ell_{\mathfrak{U}}$ has (in the case of Th. 7 we have $\ell_{\mathfrak{U}} = \ell_n^1$; see e.g. [5] for definitions).

Now I present a sketch of the proof of Th. 6. We shall need two lemmas.

Lemma A : Let $(E, \|\cdot\|)$ be a B-space of cotype 2 with an unconditional basis (e_i) . Then there exists a norm $\|\cdot\|^{(1)}$ such that

a) $\|x\| \leq \|x\|^{(1)} \leq C K_2(E) \text{ubc}(e_i) \|x\|$ for $x \in E$

(C is an absolute constant).

b) $\text{ubc}(e_i) = 1$ in $(E, \|\cdot\|^{(1)})$

c) the dual norm $\|\cdot\|$ on E^* is 2-convex ; in other words a functional defined by $\|(\sum_j |x_j| e_j^*)\| = (\sum_j \sqrt{|x_j|} e_j^*)^{(1)}$ is a basic sequence E^* dual to (e_j) is a norm (then, of course, unconditional). ■

Lemma A is well known (see e.g. [1]).

Lemma B : Let $(F, \|\cdot\|)$ be an n -dimensional normed space, (f_i) -its basis with $\text{ubc}(f_i) = 1$. Then there exists a sequence of positive numbers $\beta_1, \beta_2, \dots, \beta_n$ such that, for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$,

$$(5) \quad \frac{1}{n} \sum_i |\lambda_i| \leq \left\| \sum_i \beta_i \lambda_i f_i \right\| \leq \max_i |\lambda_i| \quad . \quad \blacksquare$$

Proof of lemma B : Some variants of lemma B are known in a more general setting of B-lattices. I present a proof, which is essentially due to T.K. Carne.

Consider $f : B_E \rightarrow \mathbb{R}$ defined by $f(\sum_i b_i f_i) = \prod_i b_i$. Let $\beta = \sum_i \beta_i f_i$ be a point, where f attains its maximum. Of course one can choose β to satisfy $\beta_i \geq 0$ for $i = 1, 2, \dots, n$. Clearly $\|\beta\| = 1$; this implies immediately the right hand inequality of (5), because $\text{ubc}(f_i) = 1$. By the same reason, to prove the left hand inequality of (5) it is enough to show that the functional $\varphi : \sum_i \lambda_i \beta_i f_i \mapsto \frac{1}{n} \sum_i \lambda_i$ is of norm at most 1.

It is easy to see that φ is the only functional satisfying

$$(i) \quad \varphi(\beta) = 1, \\ \text{and } (ii) \quad \varphi(x) \geq 1 \text{ if } x \in Q = \left\{ x = \sum_i x_i f_i : x_i \geq 0 \text{ for } i = 1, 2, \dots, n \right. \\ \left. \text{and } \prod_i x_i > \prod_i \beta_i \right\}.$$

But it is clear that the functional ψ separating disjoint (by definition of β) and convex sets B_E and Q (i.e. $\psi(B_E) \leq 1$, $\psi(Q) > 1$) satisfies (i) and (ii) ; hence $\varphi = \psi$ and $\varphi(B_E) \leq 1$, in other words $\|\varphi\| \leq 1$. This proves lemma B.

Now we shall derive th. 6 from lemmas A and B.

Clearly, by lemma A and (1), it is enough to prove that if $(E, \|\cdot\|^{(1)})$ satisfies conditions (b) and (c) of lemma A, then $\text{vr}(E) \leq C$, where C is a universal constant. On the other hand, this estimate will immediately follow from existence of a sequence (α_k) such that

$$(*) \quad \sum_k |x_k| \leq \left\| \sum_k \alpha_k x_k e_k \right\|^{(1)} \leq \sqrt{n} (\sum_k |x_k|^2)^{1/2}$$

for all $x_1, x_2, \dots, x_n \in \mathbb{R}$. Indeed, defining an ellipsoid

$$\mathcal{E} = \left\{ x = \sum_k \alpha_k x_k e_k : \sqrt{n} \sqrt{\sum_k |x_k|^2} \leq 1 \right\}$$

we get $\mathcal{E} \subset B = B_{(E, \|\cdot\|)}(1) \subset B_{\ell_n^1}$. Hence

$$\text{vr}(E) \leq \left(\frac{\text{vol } B}{\text{vol } \mathcal{E}} \right)^{1/n} \leq \left(\frac{\text{vol } B_{\ell_n^1}}{\text{vol } \mathcal{E}} \right)^{1/n} = \text{vr}(\ell_n^1) \leq \sqrt{\frac{2e}{\pi}}$$

by proposition 2.

To show (*) consider its dual version

$$(**) \quad \max_k |y_k| \geq \left\| \sum_k \frac{y_k}{\alpha_k} e_k^* \right\|_* \geq \frac{1}{\sqrt{n}} \left(\sum_k |y_k|^2 \right)^{1/2} .$$

Of course it is enough to prove (**) for nonnegative sequences (y_k) only. Substituting $y_k = \sqrt{\lambda_k}$ and $\alpha_k = 1/\sqrt{\beta_k}$ one gets

$$(***) \quad \frac{1}{n} \sum_k \lambda_k \leq \left(\left\| \sum_k \sqrt{\beta_k \lambda_k} e_k^* \right\|_* \right)^2 \leq \max_k \lambda_k .$$

Now existence of (β_k) satisfying (***) follows immediately from condition (c) of lemma A (i.e. the fact that the term in the centre of (***) is equal to $\|(\beta_k \lambda_k)\|$ for some unconditional norm $\|\cdot\|$) and lemma B. ■

Let us introduce another invariant :

$$\text{hvr}(E) \stackrel{\text{def}}{=} \sup_{F \subset E, \dim F < \infty} \text{vr}(E) ,$$

where E is a Banach space, not necessarily of finite dimension. Using some methods from [4], one can easily derive from Prop. 3 the following :

Theorem 8 : If $\text{hvr}(E) < \infty$, then E is of cotype $2 + \varepsilon$ for every $\varepsilon > 0$.
In general ε cannot be omitted. ■

Finally I am going to present :

Theorem 9 : There exists a function $(0,1) \ni \theta \rightarrow C(\theta)$ such that for any k -dimensional subspace E of ℓ_n^∞ we have

(+) $d(E, \ell_k^2) > C(k/n) \sqrt{k}$. ■

Remark : Our proof gives $C(\theta) = \sqrt{\pi/2} e^{\frac{3}{2}\theta}$.

Recently Figiel and Johnson proved th. 9 with $C(\theta) = \sqrt{\theta/2}$. ■

Proof of theorem 9 : Since $d(E, \ell_k^2) = d(E^*, \ell_k^2)$, it is enough to prove (+) with E replaced by E^* .

To say that E is a subspace of ℓ_n^∞ is the same as to say that the unit ball of E^* has at most $2n$ extreme points, say $x_1, x_2, \dots, x_n, -x_1, -x_2, \dots, -x_n$. Let \mathcal{E} be an ellipsoid contained in the unit ball of E^* . We must show that, for some i , $x_i \notin C(k/n) \sqrt{k} \mathcal{E}$. Thus the proof reduces to the following fact :

Let $B = \text{abs conv}(y_i)_{i=1}^n \subset \mathbb{R}^k$ and let the Euclidean unit ball $B^k = \{x \in \mathbb{R}^k : \|x\|_2 \leq 1\}$ be contained in B . Then $\max_{1 \leq i \leq n} \|y_i\|_2 \geq \sqrt{k} C(k/n)$.

To see the above consider all sets of the form

$B_A = \text{abs conv}(y_i)_{i \in A}$, $A \subset \{1, 2, \dots, n\}$, $\text{card } A = k$.

Clearly $\bigcup_A B_A = B$. Choose A so that $\text{vol } B_A$ is maximal. Then

$$\binom{n}{k} \text{vol } B_A \geq \text{vol } B \geq \text{vol } B^k .$$

On the other hand

$$\text{vol } B_A \leq \prod_{i \in A} \|y_i\|_2 \text{vol } B_{\ell_k^1} .$$

Combining these two estimates one gets

$$\begin{aligned} \prod_{i \in A} \|y_i\|_2 &\geq \binom{n}{k}^{-1} \frac{\text{vol } B^k}{\text{vol } B_{\ell_k^1}} = \binom{n}{k}^{-1} (\sqrt{k})^k \frac{\text{vol } B^k}{\text{vol } (\sqrt{k} B_{\ell_k^1})} = \\ &= \binom{n}{k}^{-1} (\sqrt{k})^k [\text{vr}(\ell_k^1)]^{-k} . \end{aligned}$$

Hence

$$\max_{i \in A} \|y_i\|_2 \geq \left[\binom{n}{k}^{1/k} \text{vr}(\ell_k^1) \right]^{-1} \sqrt{k} \geq \frac{k}{en} \sqrt{\frac{\pi}{2e}} \sqrt{k} .$$

This ends the proof of theorem 9. ■

Let us mention finally some easy observations, which may indicate another application of concepts introduced here. Namely, we have

$$[\text{vr}(E)]^\theta [\text{vr}(F)]^{1-\theta} = \text{vr}(E \oplus_{\ell_2} F)$$

where $\theta = \dim E / (\dim E + \dim F)$ and $E \oplus_{\ell^2} F$ is a direct sum of E and F in the sense of ℓ^2 .

One can hope that this may help in investigating complemented subspaces of a normed space.

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