SÉMINAIRE D'ANALYSE FONCTIONNELLE
ÉCOLE POLYTECHNIQUE

N. GHOUSSOUB

On spaces with local unconditional structure

Séminaire d'analyse fonctionnelle (Polytechnique) (1979-1980), exp. n° 24, p. 1-8
<http://www.numdam.org/item?id=SAF_1979-1980____A21_0>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. : (1) 941.82.00 - Poste N°
Télex : ECOLEX 691596 F

S E M I N A I R E

D ' A N A L Y S E   F O N C T I O N N E L L E

1979-1980

ON SPACES WITH LOCAL UNCONDITIONAL STRUCTURE

N. GHOUSSOUB

(University of British Columbia, Vancouver)

Exposé No XXIV

16 Mai 1980

In this talk we want to emphasize on the "miraculous" effect that a lattice structure has on the geometry of a Banach space. We are mainly concerned by new characterizations of the Radon-Nikodym property which do not hold if the lattice structure is absent. We will deal first with spaces having local unconditional structure à la Gordon-Lewis [8] (l.u.s.t.). For this notion, we will only use the fact that they are exactly the spaces whose duals are complemented in a Banach lattice. The positive results obtained for this class of spaces are particularly useful for identifying the spaces with no l.u.s.t., namely the James-type of spaces.

We start by the following key proposition due to Johnson-Tzafriri [11].

<u>Proposition 1</u> : Let X be a complemented subspace of a Banach lattice L. If c_o does not embed in X, then there exists an order continuous Banach lattice $\tilde{L}$ which contains a complemented subspace $\tilde{X}$ isomorphic to X.

The idea of the proof is to renorm L by the semi-norm $|||x||| = \sup\{\|Pz\| \; ; \; |z| \leq |x|\}$ where P is the projection. Let I be the ideal of elements x in L so that $|||x||| = 0$, let $\tilde{L}$ be the completion of L/I for that norm and Q the quotient map from L to L/I. The following diagram then commutes

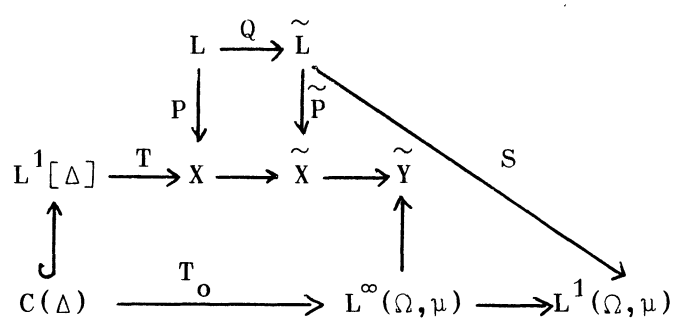
$$\begin{array}{ccc} L & \xrightarrow{\;Q\;} & \tilde{L} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \tilde{P}} \\ X & \longleftrightarrow & \tilde{X} \end{array}$$

The restriction of Q to X is an isomorphism and c_o is killed in $\tilde{L}$.

Using the above proposition, we can refine a result of Kalton [10] which extends a well known result of Enflo-Starbird in L^1 [3].

<u>Proposition 2</u> : Let X be a Banach space complemented in a Banach lattice L such that c_o does not embed in X. If X contains a subspace Z isomorphic to L^1, then Z contains a subspace Z_o isomorphic to L^1 and complemented in L.

Sketch of proof : Associate the diagram of proposition 1 and suppose T is an isomorphism from $L^1[\Delta]$ onto a subspace Z of X (Δ being the Cantor set). Let $\tilde{Y}$ be the smallest sublattice of $\tilde{L}$ containing $\tilde{X}$. Since $\tilde{L}$ does not contain c_o, it is a standard manner to associate a probability space (Ω,μ) such that the following diagram commutes. For details see [10].



As in [10], one proves that $S \circ Q \circ T$ is atomic in the sense of Kalton from $L^1[\Delta]$ into $L^1(\Omega)$. Hence, there exists a non-negligeable Borel subset B of Δ such that $S \circ Q \circ T$ is an isomorphism on $L^1[B]$ and $S \circ Q \circ T(L^1[B])$ is complemented in $L^1(\Omega)$. Let φ be such a projection. It is easy to see that $(S \circ Q)^{-1} \circ \varphi \circ (S \circ Q)$ projects L onto $T(L^1(B))$.

Proposition 3 : Under the hypothesis of Prop. 1, every X-valued Pettis integrable random variable is equivalent to a Bochner integrable r.v.

Proof : It is enough to prove it for a complemented subspace of an order continuous lattice by Proposition 1. According to [2] it is enough to show that such an X has the separable complementation property, that is every separable subspace Z of X, is contained in a separable subspace Y which is complemented in L. Since Z is separable, it is contained in a principal ideal V which is the rang of a band projection $P_1 : L \to V$. But V is W.C.G. hence there exists Y separable containing Z and Y complemented in V.

$$L \xrightarrow{P_1} V \xrightarrow{P_2} Y \supseteq Z \quad .$$

The following theorem is now a consequence of the above and some well known facts

Theorem 4 (Ghoussoub-Saab) : Let X be a Banach space with l.u.s.t. The following properties are equivalent :

1) $\ell_1 \not\hookrightarrow X$
2) $L_1 \not\hookrightarrow X^*$

3) $c_o \overset{}{\not\hookrightarrow} X^*$ and $L_1 \overset{c}{\not\hookrightarrow} X^*$

4) X^* has the weak Radon-Nikodym property.

5) Every operator from L^1 to X^* is Dunford-Pettis.

6) X^* has the Radon-Nikodym property.

An immediate consequence of the above theorem is that the James tree space and the James-Hagler space do not have l.u.s.t.
We also have the following

<u>Corollary 1</u> : If X has l.u.s.t. and if ℓ_1 embeds in X but not complementably, then $(\sum_n \oplus \ell_\infty^n)_{\ell_1}$ embeds in X.

This follows from Proposition 1, a result of Pełczynski [11] ($\ell_1 \overset{c}{\not\hookrightarrow} X \Longleftrightarrow c_o \hookrightarrow X^*$) and a theorem of Hagler-Stegall [9] ($(\sum_n \oplus \ell_\infty^n)_{\ell_1} \hookrightarrow X \Longleftrightarrow L^1 \overset{c}{\hookrightarrow} X^*$).

We do not know if the above corollary is true in general.

One can also prove the following [4]

<u>Theorem 5</u> (Ghoussoub-Johnson) : Let X be a separable Banach space with l.u.s.t. such that $c_o \not\hookrightarrow X^*$. Then every operator T from X into any Banach space Y sucht that $T^*(Y^*)$ is not separable, fixes a copy of $(\sum_n \oplus \ell_\infty^n)_{\ell_1}$.

Theorem 5 was proved by Hagler-Stegall [9], for X being a separable $\mathcal{L}^\infty$-space.

What about fixing $C(\Delta)$?? Is is clear that this is not true even for $\mathcal{L}^\infty$-spaces.
On the other hand, Rosenthal proved such a result for $X = C(K)$ and then Lotz-Rosenthal extended it to Banach lattices namely

<u>Theorem 6</u> (Lotz-Rosenthal [13]) : If X is a separable Banach lattice such that $c_o \not\hookrightarrow X^*$. Then every operator T from X into any Banach space Y such that $T^*(Y^*)$ is not separable fixes a complemented copy of $C(\Delta)$.

The above theorem extends immediatly to complemented subspaces of a Banach lattice via the following renorming theorem which is the predual version of Proposition 1.

<u>Proposition 7</u> (Ghoussoub-Johnson) : Let X be a complemented subspace of a Banach lattice L. If $c_o \not\hookrightarrow X^*$, then there exists a Banach lattice $\tilde{L}$, such that $c_o \not\hookrightarrow (\tilde{L})^*$, and which contains a complemented subspace $\tilde{X}$ isomorphic to X.

<u>Proof</u> : Let P be the projection from L onto X, and P^* its adjoint. Define the new norm on L by

$$|||x||| = \sup\{\, |f(x)| \text{ over all } f\text{'s such that } \|P^*g\| \leq 1 \text{ whenever } |g| \leq |f| \,\} \quad .$$

One may verify immediatly that $|||\ \ |||$ is a lattice norm and that for every x in L

$$|||Px||| \leq \|x\| \leq \|P\| \cdot |||x||| \quad .$$

Let now $\tilde{L} = \{x \in L \ ; \ |||x||| < \infty\}$. It is a Banach space and the following diagram commutes

$$\begin{array}{ccc} (\tilde{L}, |||\ \ |||) & \xrightarrow{\ Q\ } & (L, \|\ \ \|) \\ P \big\downarrow & & \big\downarrow P \\ \tilde{X} & \longleftrightarrow & X \end{array}$$

where Q is the canonical injection and $\tilde{X}$ is X equipped with the new norm $|||\ \ |||$ equivalent to $\|\ \ \|$ on X. Consider the dual diagram

$$\begin{array}{ccc} (L^*, \|\ \ \|) & \longrightarrow & ((\tilde{L})^*, |||\ \ |||) \\ P^* \big\downarrow & & \big\downarrow P^* \\ X^* & \longleftrightarrow & (\tilde{X})^* \end{array}$$

Since $c_o \not\hookrightarrow (\tilde{X})^*$, we apply the renorming of Proposition 1 to $(\tilde{L})^*$, $|||\ \ |||$ and P^*. That is for $f \in (\tilde{L})^*$

$$!f! = \sup\{\, |||P^*g||| \ ; \ |g| \leq |f| \,\} \quad .$$

An immediate verification shows that

$$|||f||| \leq \, !f! \, \leq \|P\| \ |||f||| \quad \text{on} \quad (\tilde{L})^*$$

which shows that $(\tilde{L})^*$ does not contain c_o.

Theorem 8 : If X is a separable complemented subspace of a Banach lattice and if $c_0 \not\hookrightarrow X^*$, then every operator T from X to any Banach space Y such that $T^*(Y^*)$ is not separable fixes a complemented copy of $C(\Delta)$.

Proof : Consider the diagram of Prop. 7 and apply the result of Lotz-Rosenthal to $T \circ Q \circ \tilde{P}$

$$\begin{array}{ccccc} (\tilde{L}, |||\ \ |||) & \xrightarrow{Q} & (L, \|\ \ \|) & & \\ \tilde{P}\big\downarrow & & \big\downarrow P & & \\ \tilde{X} & \overset{Q}{\longleftrightarrow} & X & \xrightarrow{T} & Y \end{array}$$

. In Theorem 4 we showed that for dual spaces complemented in a Banach lattice, the Radon-Nikodym property is equivalent to the property that every operator from L^1 into the space is Dunford-Pettis. It is still unknown if this remains true for non-dual complemented subspaces of L^1. For non-complemented subspaces of L^1, the counterexample of Bourgain-Rosenthal [1] gives a negative answer to this question. In the sequel, we shall prove that the answer is positive in the case of Banach lattices. For that we recall the notion of order dentability introduced in [5].

. Let X be a Banach lattice. For every convex closed bounded set C in the positive cone X_+ and any element u in X_+, define

$$\mathcal{a}_1(C,u) = \bigcap_n \overline{\text{conv}}\{y \in C \ ; \ \|y \wedge u\| \leq \frac{1}{n}\}$$

and by transfinite induction on ordinals

$$\mathcal{a}_\alpha(C,u) = \mathcal{a}_1(\mathcal{a}_\beta(C,u),u) \quad \text{if} \quad \alpha = \beta + 1$$

and

$$\mathcal{a}_\alpha(C,u) = \bigcap_{\beta<\alpha} \mathcal{a}_\beta(C,u) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Definition 1 : A set C in X_+ is said to be order dentable with respect to u if for all closed convex subsets B of C, we have

$$\mathcal{a}_1(B,u) \subsetneq B \ .$$

. Let now $T : L^1 \to X$ be a positive operator. The smallest closed sublattice of X containing $T(L^1)$ contains a quasi-interior point u.

Definition 2 : A positive operator $T : L^1 \to X$ is said to be order dentable if the closure of the image of the positive part of the unit ball of L^1 is order dentable with respect to u.

Theorem 9 : A Banach lattice X has the Radon-Nikodym property if and only if $c_0 \not\hookrightarrow X$ and every positive operator from L^1 to X is order dentable.

Sketch of proof : If X has R.N.P., the results of [5] gives the first implication. For the reverse implication, note first that since $c_0 \not\hookrightarrow X$, every operator from $L^1 \to X$ is the difference of two positive operators, hence it is enough to show that every positive operator is representable or equivalently that the martingale

$$T(X_n) = 2^{-n} \sum_k T(\chi_{I_{n,k}})\chi_{I_{n,k}} \quad \text{converges a.s.}$$

The $\chi_{I_{n,k}}$'s being the characteristic functions of the diadic intervals $I_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$.

. For every $k \in \mathbb{N}$, the supermartingale $(TX_n \wedge k)_n$ is in $[0,ku]$ which is weakly compact, hence it converges to $Z_\infty \wedge ku$ where $Z_\infty \in L^1[Y]$. Moreover, we have

$$0 \leq E^n[Z_\infty] \leq TX_n \quad \text{a.s.}$$

. Define the operator $T'(f) = \int f\, Z_\infty\, d\lambda$ and the martingale $E^n[Z_\infty]$ is actually equal to $2^{-n} \sum_k T'(\chi_{n,k})\chi_{n,k} = T'(X_n)$. One then verifies that the martingale $TX_n - T'X_n \in \mathcal{A}_\alpha(\overline{(T-T')(B)},u)$ a.s. for every countable ordinal α.

. Since $T-T'$ is order dentable, the family $\mathcal{A}_\alpha$ is strictly decreassing unless it reaches $\{0\}$. Since $T(L^1)$ is separable, $(\mathcal{A}_\alpha)$ must become stationary before the first uncountable ordinal. It follows that $T = T'$ which is representable.

Theorem 10 : If every positive operator from L^1 into a Banach lattice X is Dunford-Pettis then every positive operator is order dentable.

Sketch of proof : If T is not order dentable, there exists a positive bounded finitely valued martingale $X_n : [0,1] \to X$ such that $\|X_n \wedge u\| \leq \frac{1}{n}$, $\forall$ n. Define $T' : L^1 \to X$ by $T'(f) = \lim_{n\to\infty} \int f X_n\, d\lambda$. That is, for every n, there exists a finite family I of positive reals $(\alpha_i)_{i\in I}$, $\sum_{i\in I} \alpha_i = 1$ and

$(f_i)_{i\in I}$ in B such that

$$x = T'1 = \sum_{i\in I} \alpha_i T' f_i \quad \text{and} \quad \|T'1 \wedge T' f_i\| \le \frac{1}{n} \quad \forall\, i \in I .$$

- Let x^* be in X^*_+ such that $x^*(x) = \alpha > 0$.
- Let B_i be the support of $(f_i - 1)^+$, we get

$$x^*(T\chi_{B_i}) \le \frac{1}{n} \quad \text{and} \quad x^*(\sum_{i\in I} \alpha_i T'(f_i - 1)^+) \ge \alpha - \frac{1}{n} .$$

In other words, if $\mathcal{B}$ is the ring generated by the B_i's and if H is the greatest elements of $\mathcal{B}$, we have $x^*(T'\chi_H) \ge \alpha - \frac{1}{n}$ and if P in an atom of $\mathcal{B}$ $x^*(T'\chi_P) \le \frac{1}{n}$. Using this fact, one can construct by induction an increasing sequence of finite algebras $(\mathcal{U}_n)$ and a sequence of sets (A_n) with $A_n \in \mathcal{U}_n$ for every n, such that for every $p \le n$

$$x^*(T'\chi_{A_p}) \ge \frac{\alpha}{4} \quad \text{and} \quad x^*(|T'\chi_{A_p} - T'\chi_{A_n}|) \ge \frac{\alpha}{12}$$

which shows that the image of the sequence (χ_{A_p}) which is valued in the weakly compact order interval [0,1] in L^1, does not have a cluster point in X which contradicts the fact that T'is Dunford-Pettis.

<u>Theorem 11</u> : For a Banach lattice X, the following conditions are equivalent :

1) X has R.N.P.
2) X contains no bounded δ-tree.
3) Every operator from L^1 to X is Dunford-Pettis.

The above theorem shows that there exists a positive operator from L^1 into the Banach lattice MT constructed by M. Talagrand [16] which is not Dunford-Pettis and yet this operator does not fix a copy of L^1. However, by extending the results of Rosenthal [14] on L^1, via the embedding technique of Kalton [10] one can show :

<u>Theorem 12</u> : If X is a Banach lattice such that $c_0 \not\hookrightarrow X$, then every positive operator from L^1 to X which is not Dunford-Pettis fixes a copy of L^2.

Now we have the immediate

Corollary 13 : A Banach lattice X not containing isomorphic copies of c_o and L^2 has the Radon-Nikodym property.

REFERENCES

[1] J. Bourgain, H.P. Rosenthal : Martingales valued in certain subspaces of L^1, Preprint (1979).

[2] J. Diestel, J. Jr. Uhl : Vector measures, A.M.S. Surveys 15, Providence (1977).

[3] P. Enflo, T. Starbird : Subspaces of L^1 containing L^1, Preprint.

[4] N. Ghoussoub and W. Johnson : To appear (1980).

[5] N. Ghoussoub and M. Talagrand : Order dentability and the Radon-Nikodym property in Banach lattices, Math. Annalen 243 (1979), 217-225.

[6] N. Ghoussoub and E. Saab : On the weak Radon-Nikodym property, P Proc. of A.M.S., (to appear) (1980).

[7] N. Ghoussoub : Dunford-Pettis and order dentable operators on L^1, (to appear) (1980).

[8] Y. Gordon and D.R. Lewis : "Absolutely summing operators and local unconditional structure", Acta Math.

[9] J. Hagler and C. Stegall : Banach spaces whose duals contain complemented subspaces isomorphic to $C[0,1]^*$, J. Funct. Anal. 13, 233-251 (1973).

[10] N. Kalton : Embedding L^1 in a Banach lattice", Preprint (1978).

[11] J. Lindenstrauss and L. Tzafriri : Classical Banach spaces I - Sequence spaces, Springer Verlag No 92 (1977).

[12] J. Lindenstrauss and L. Tzafriri : Classical Banach spaces II - Function spaces, Springer Verlag (1979).

[13] Lotz, H.P. Rosenthal : Embeddings of $C(\Delta)$ and $L^1[0,1]$ in Banach lattices, Israel J. Math. (to appear) (1979).

[14] H.P. Rosenthal : Convolution by a biased coin, Altgeld Book (1975-76).

[15] H.P. Rosenthal : On factors of C[0,1] with non-separable dual, Israel J. Math. 13 (1975) 361-378.

[16] M. Talagrand : Sur la propriété de Radon-Nikodym dans les espaces de Banach réticulés, C. R. Acad. Sc. (1979).
