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BASES IN THE SPACES C AND L^1

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This talk is devoted to certain properties of bases of C and L^1, related to the following result of Olevskii (see [5], ch. I, § 2, Th. 2 and Th. 9, also [6]) :

<u>Theorem 0</u> : No uniformly bounded orthonormal system is a basis of C or L^1. □

Some special cases of this theorem were discovered earlier. For example, by du Bois-Reymond theorem (the end of 19th century), the trigonometric system $(e^{int})_{n\in Z}$ is not a basis of the space of 2π-periodic continuous functions on R. By Cohen's result on the Littlewood conjecture, the same is true for any its permutation $(e^{in_k t})$.

In the sequel C and L^p will always mean C(0,1) or $L^p(0,1)$ with respect to the Lebesgue measure λ. However, all the results are valid for infinitely many dimensional C(K) or $L^p(S)$-spaces over finite measure (Theorems 5-8 hold even for abstract $\mathcal{L}^1$ and $\mathcal{L}^\infty$-spaces).

Olevskiǐ's result rises a number of natural questions, which I list below as theorems (if the answers are known) or problems (if they are still open).

<u>Theorem 1</u> [3] : Let (φ_n) be an orthonormal system, which is a basis o of C. Then there exists a constant $\alpha > 0$ such that

$$\max_{1\leq k\leq n} \|\varphi_k\|_\infty \geq \alpha \sqrt{n}$$

for all n. □

On the other hand, for the orthonormal Haar system (χ_n) (it is a basis of $C(\Delta)$, Δ -the Cantor set) we have $\sqrt{n/2} \leq \|\chi_n\|_\infty \leq \sqrt{n}$. Moreover, there exists an orthonormal basis of $C(\Delta)$ (ψ_n) (see [5], p. 22) such that $\|\psi_n\|_\infty \leq \sqrt{n}$ for all n, but $\liminf_n \|\psi_n\|_\infty = 1$.

<u>Theorem 2</u> [3], [4] : There is no normalized, uniformly bounded basis of L^1. □

Since every uniformly bounded orthonormal system remains uniformly bounded after normalization in L^1, Th. 2 generalizes Th. 0.

Theorems 1 and 2 suggest the following

Problem 3 : Let (f_n) ba a normalized basis of L^1. What is the smallest possible order of growth of $\|f_n\|_\infty$? □

The conjecture is that it is $O(n)$, the same as in the case of the Haar system (this time normalized in L^1).

It is easy to show (via change of measure) that Th. 2 implies the following fact :

Theorem 2' [4] : No normalized basis of L^1 satisfies

(*) $$\Phi \overset{df}{=} \sup_n |f_n| \in L^1 \quad . \qquad \square$$

On the other hand, we have

Theorem 4 : Given measurable function Φ with $\Phi > 0$ and $\Phi \notin L^1$ there exists a normalized basis of L^1 (f_n) satisfying (*). □

The formulation of all the statements above are not satisfactory from the point of view of the Banach space theory : they involve "alien" concepts of orthogonality, uniform boundedness, boundedness in order. The next three facts are stated in the pure Banach space theory language.

The following theorem answers the question asked by Pełczyński at this seminar six years ago :

Theorem 5 [8] : There is no Besselian basis of C (Hilbertian basis of L^1). □

Recall that a sequence (x_k) of elements of a Banach space is called Besselian if, for some constant c and all sequences of scalars (t_k) with finite number of nonzero elements, we have

$$c \left\|\sum_k t_k x_k\right\| \geq \left(\sum_k |t_k|^2 \|x_k\|^2\right)^{1/2} \quad ;$$

Hilbertian -the same with an inverse inequality. In particular a uniformly bounded orthonormal system is Besselian in C and Hilbertian in L^1.

It is easy to see that every bounded Hilbertian sequence tends weakly to 0. This leads to the following strengthening of Th. 5 :

<u>Theorem 6</u> [9] : There is no normalized relatively weakly compact basis of L^1. □

Th. 6 answers a question posed, in particular, in [7] (p. 296, Problem 7.1).

To make the assertion of Th. 6 more transparent, I mention the following well known

<u>Fact</u> : Let Z be a bounded subset of L^1. TFAE

(1) Z is relatively weakly compact

(2) Z is uniformly integrable

(3) no sequence of elements of Z is equivalent to the unit vector basis of ℓ^1. □

The above indicates obvious reformulations of Th. 6 :

<u>Theorem 6'</u> : No normalized basis of L^1 consists of uniformly integrable functions. □

<u>Theorem 6"</u> : Every normalized basis of L^1 contains a subsequence, which is equivalent to the unit vector basis of ℓ^1. □

The following result is dual, in a sense, to Th. 6 :

<u>Theorem 7</u> [9] : Given normalized basis (g_n) of C there exists an increasing sequence of integers (n_k) such that the operator

$$g = \sum_n t_n g_n \longmapsto (t_{n_k})_{k=1}^{\infty}$$

takes C <u>onto</u> c_o. □

See [9] for more problems and facts on related subjects (concerning, in particular, quantitative estimates of Lebesgue functions and formal Fourier series with respect to certain classes of biorthogonal systems).

Now I am passing to a more detailed study of some of the results listed above. I am going to sketch a simple proof of Th. 2 and a construction from Th. 4.

Proof of Theorem 2 : Suppose not ; let (f_n) be a normalized basis of L^1, which is uniformly bounded, i.e.

(j) $$\|f_n\|_\infty \le M \quad \text{for some } M \text{ and all } n .$$

Denote by (g_n) the sequence of basis functionals. Then

(jj) $$\|g_n\|_\infty \le K \quad \text{for some } K \text{ and all } n .$$

It is well known that if (f_n) is a basis of L^1, then

(jjj) $$\operatorname{supess}_y \int \Big| \sum_{k=1}^{m} g_k(y)\, f_k(x) \Big| dx \overset{df}{=} L_m \le L$$

for some constant L and all m.

Indeed, to see (jjj) it suffices to observe that L_m is equal to the norm of the operator of m-th partial sum $P_m : \sum_{k=1}^{\infty} t_k f_k \longmapsto \sum_{k=1}^{m} t_k f_k$ and the norms $\|P_m\|$ are, by the Banach-Steinhaus theorem, uniformly bounded.

To complete the proof of Th. 2 we shall need the following two facts :

I . (Kashin's version [2] of a Bočkariev type inequality [1]). There exists a universal constant c such that, for every n and every choice of scalars $a_1, a_2, \dots, a_n$, we have

(i) $$\Big(\max_{1\le k\le n} |a_k| \Big) \Big(\sum_{m=1}^{n} \Big| \sum_{k=1}^{m} a_k \Big| \Big) \ge c \sum_{n\ge\mu>\nu\ge 0} \frac{\Big| \sum_{k=\nu+1}^{\mu} a_k \Big|^2}{(\mu-\nu)^2} . \qquad \square$$

II . Let $(f_i, g_i)_{i\in\Lambda}$ be a biorthogonal system of functions (i.e. $\int f_i\, g_j = \delta_{ij}$ for all $i, j \in \Lambda$). Then

(ii) $$\iint \Big| \sum_{i\in\Lambda} f_i(x)\, g_i(y) \Big|^2 dx\, dy \ge \operatorname{card} \Lambda . \qquad \square$$

Let us first show how (j), (jj), (jjj) and I, II lead to a contradiction.

Substituing $a_k = f_k(x)\ g_k(y)$ in (i), integrating the both sides with respect to x and y and using (ii) one gets

$$\left(\max_{1\le k\le n} \|f_k\|_\infty \|g_k\|_\infty\right)\left(\sum_{m=1}^{n} \iint \left|\sum_{k=1}^{m} g_k(y)\ f_k(x)\right| dx\ dy\right)$$

$$\ge c \sum_{n\ge\mu>\nu\ge 0} \frac{\iint \left|\sum_{k=\nu+1}^{\mu} g_k(y)\ f_k(x)\right|^2 dx\ dy}{(\mu-\nu)^2} \ge c \sum_{n\ge\mu>\nu\ge 0} \frac{1}{\mu-\nu} \ge c'n\ \ell n\ n$$

for all n (c' does not depend on n). Hence, by (j) and (jj),

$$\frac{1}{n} \sum_{m=1}^{n} \iint \left|\sum_{k=1}^{m} g_k(y)\ f_k(x)\right| dx\ dy \ge c'\ K^{-1}\ M^{-1}\ \ell n\ n \quad ,$$

which, for n sufficiently large, contradicts (jjj). □

Thus to prove Th. 2 it remains to show I and II.

Proof of II :

$$\iint \left|\sum_{i\in\Lambda} f_i(x)\ g_i(y)\right|^2 dx\ dy = \left(\iint \left|\sum_{i\in\Lambda} f_i(x)\ g_i(y)\right|^2 dx\ dy\right)^{1/2}$$

$$\left(\iint \sum_{j\in\Lambda} f_j(y)\ g_j(x)|^2 dx\ dy\right)^{1/2} \ge \iint \sum_{i,j\in\Lambda} f_i(x)\ g_j(x)\ g_i(y)\ f_j(y)\ dx\ dy =$$

$$= \operatorname{card} \Lambda \ . \quad □$$

Sketch of the proof of I : Observe first that (i) is a discrete version of the following inequality

$$\text{(iii)} \qquad \gamma \|f'\|_\infty \|f\|_1 \ge \|f^{(1/2)}\|_2^2 \overset{df}{=} \iint_{s<t} \left|\frac{f(t)-f(s)}{t-s}\right|^2 dt\ ds$$

for functions $f \in C^1(R) \cap L^1(R)$.

Inequalities of this type are well known in the theory of Sobolev spaces, the only trouble is a derivative of fractional order, which appears here. The proof of (iii) (given by O.V. Besov, see [2]) is based on the formula

$$\|g\|_p^p = p \int_0^\infty t^{p-1}\ \lambda_g(t)\ dt \quad ,$$

where $\lambda_g(t) = \lambda(\{x : |g(x)| > t\})$ (λ -the Lebesgue measure). It is quite simple, but I will not present it here.

To justify the notation appearing in (iii) notice that the integral at the right hand side of (iii) is equal to $\delta \int |\hat{f}(x)|^2\ |x|\ dx$, where $\hat{f}$ is the Fourier transform of f and δ does not depend on f. □

Proof of Theorem 4 : Let Φ satisfy the assumptions of Theorem 4. Define $d\mu = \Phi\, d\lambda$. Then $L^1((0,1),\lambda)$ is isometric to $L^1(\mu) = L^1((0,1),\mu)$ via $T : L^1 \to L^1(\mu)$ defined by $Tf = f/\Phi$. Observe that f is dominated by Φ if and only if Tf is bounded by 1. Thus the problem reduces to construction of a normalized basis of $L^1(\mu)$, consisting of functions bounded by 1. On the other hand, since $((0,1),\mu)$ is equivalent (as a measure space) to $((0,\infty),\lambda)$, it is enough to construct a normalized basis of $L^1(0,\infty)$ uniformly bounded by 1.

To do this, we shall need some properties of certain matrices $A^{(s)}$ of order 2^s (see [5], p. 97), defined by

$$A_{i1} = A_{i1}^{(s)} = 2^{-s} \quad \text{for } i = 1,2,\dots,2^s$$

and, if $j = 2^p + r$ for some $p = 0,1,\dots,s-1$ and $r = 1,2,\dots,2^p$, then

$$A_{ij} = \begin{cases} 2^{p-s} & \text{if } (r-1)2^{s-p} < i \le (r-\frac{1}{2})\, 2^{s-p} \\ -2^{p-s} & \text{if } (r-\frac{1}{2})2^{s-p} < i \le r\, 2^{s-p} \\ 0 & \text{otherwise} \end{cases} .$$

In other words, A is a matrix of the "Haar type", i.e. A_{ij} is equal to the value of the j-th Haar function on the dyadic interval $((i-1)2^{-s}, i \cdot 2^{-s})$.

Now let us consider the vectors $A_i = (A_{ij})_{j=1}^{2^s}$, $i = 1,2,\dots,2^s$. We shall need the following properties of these vectors :

1° $\quad \|A_i\|_{\ell^1_{2^s}} = 1$ for all i

2° $\quad |A_{ij}| \le 1$ for all i,j

3° $\quad A_{i1} = 2^{-s}$ for all i

4° the basis constant of the sequence (A_i) in $\ell^1_{2^s}$ does not exceed 2.

Properties 1°-3° are obvious ; to verify 4° one must find a dual basis $(\tilde{A}_i)$ and then estimate

$$\max_{k,m} \sum_{j=1}^{2^s} \left| \sum_{i=1}^{m} A_{ij} \tilde{A}_{ik} \right| .$$

Now split $(0,\infty)$ into intervals $(n-1,n]$ and consider the Haar basis $(\chi_i^n)_{i=1}^{\infty}$ of $L^1(n-1,n)$ (normalized in L^1-norm). Then the system $(\chi_i^n)_{i,n=1}^{\infty}$ is a monotone basis of $L^1(0,\infty)$ provided it is ordered in such a way that χ_i^n precedes χ_j^n if $i<j$. We choose some particular ordering of this kind. Namely, each χ_I^N with $\|\chi_I^N\|_\infty > 1$ will be the first element of a block of 2^s of χ_i^ns, say $\chi(1) = \chi_I^N$, $\chi(2),\ldots,\chi(2^s)$, such that

(a) $\|\chi(1)\|_\infty = 2^s$

(b) all $\chi(j)$ have disjoint supports

(c) all $\chi(j)$ with $j \geq 2$ are of form χ_1^m for some m, in particular

$$\|\chi(j)\|_\infty = 1 \quad \text{for} \quad j = 2,3,\ldots,2^s .$$

It is easy to see that it is possible.

Now, having such a block of 2^s functions as above fixed, consider the subspace of L^1 spanned by $\chi(1),\ldots,\chi(2^s)$. By (b), it is isometric to $\ell^1_{2^s}$ and $(\chi(j))$ is equivalent to the unit vector basis. Now we change basis via the matrix $A^{(s)}$, i.e. we pass to $F_i = \sum_{j=1}^{2^s} A_{ij}\,\chi(j)$, $i = 1,2,\ldots,2^s$. Then, by 1^o and (b), $\|F_i\|_1 = 1$ for all i. By 2^o, 3^o and (a), (b), (c), $\|F_i\|_\infty \leq 1$ for all i. By 4^o, the basis constant of (F_i) does not exceed 2.

Proceeding in the same way with all blocks, we obtain a basis of $L^1(0,\infty)$ with desired properties.

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