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Operators into L_p which factor through l_p

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OPERATORS INTO L_p WHICH FACTOR THROUGH ℓ_p

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In this seminar we prove the following theorem from [2].

Theorem A : Let T be a bounded linear operator from a Banach space X into L_p ($\equiv L_p[0,1]$), $2 < p < \infty$. Then T factors through ℓ_p if and only if T is compact when considered as an operator into L_2.

The "only if" part is an immediate consequence of the fact that every operator from ℓ_p to ℓ_2 is compact when $p > 2$ (cf. Proposition 2.C.3 in [7]). The "if" part generalizes an earlier result of Johnson-Odell [5] which says that if X is a subspace of L_p $(p > 2)$ which does not contain an isomorphic copy of ℓ_2, then X embeds into ℓ_p, because it is an easy consequence of the results in [6] that the restriction to such an X of the injection from L_p into L_2 is compact.

Proof of Theorem A : We factor T through a space of the form $Y = (\Sigma(H_n, |\cdot|_n))_{\ell_p}$, where each space $(H_n, |\cdot|_n)$ is finite dimensional. We will observe that the spaces $(H_n, |\cdot|_n)$ are uniformly isomorphic to uniformly complemented subspaces of L_p, and hence Y is isomorphic to a complemented subspaces of ℓ_p. (Of course, this implies that Y is isomorphic to ℓ_p by a result of Pełczynski's [8], but we don't need this fact, since it is clear that if T factors through a complemented subspace of ℓ_p, then T factors through ℓ_p.)

The spaces (H_n) are chosen to be a blocking of the Haar basis for L_p. That is, $H_n = \mathrm{span}(h_i)_{i=k(n)}^{k(n+1)-1}$, where (h_i) is the Haar basis for L_p and $1 = k(1) < k(2) < \dots$ is a suitably chosen sequence of positive integers. The operators $A : X \to Y$ and $B : Y \to L_p$ which factor T are defined in the natural way : for $x \in X$ with $Tx = \Sigma\, y_n$ $(y_n \in H_n)$, we define $Ax = (y_n)_{n=1}^{\infty}$. For $y_n \in H_n$ with $(y_n)_{n=1}^{\infty} \in Y$, we define $B(y_n) = \Sigma\, y_n \in L_p$. Obviously we have $BA = T$, but of course we have to show that A and B are bounded if the $(H_n, |\cdot|_n)$ sequence is appropriately defined.

It is convenient to define $|\cdot|_n$ on all of L_p. For appropriate values of M_n, $1 \le M_1 < M_2 < M_3 < \dots$, $|\cdot|_n$ is defined by

$$|f|_n = \max(M_n \|f\|_2, \|f\|_p) \quad ,$$

where

$$\|f\|_2 = (\int_0^1 |f(t)|^2 dt)^{1/2} \quad , \quad \|f\|_p \equiv (\int_0^1 |f(t)|^p dt)^{1/p}$$

have their usual meaning. It is evident that each $|\cdot|_n$ is equivalent to $\|\cdot\|_p$ on L_p, but as $M_n \uparrow \infty$ the constant of equivalence tends to infinity.

We break the proof that T factors through Y if (H_n) and (M_n) are defined appropriately into three steps.

Step One. There is a constant $K = K(p)$ such that $(H_n, |\cdot|_n)$ is K-isomorphic to a K-complemented subspace of L_p.

Of course, this means that Y is isomorphic to a complemented subspace of ℓ_p no matter how M_n is defined.

Step one is easy, given a result of Rosenthal's [9]. Rosenthal proved that there is a constant $\lambda = \lambda(p)$ so that for any sequence $w = (w_1, w_2, \ldots)$ of positive numbers the space $X_{p,w}$ is λ-isomorphic to a λ-complemented subspace of L_p. Here $X_{p,w}$ is the completion of $\mathbb{R}^\infty$ (or $\mathbb{C}^\infty$) under the norm $\|\cdot\|_w$ defined by

$$\|(\alpha_i)\|_w = \max((\Sigma\ |\alpha_i|^2\ w_i^2)^{1/2} \quad , \quad (\sum_{i=1}^{\infty} |\alpha_i|^p)^{1/p}) \quad .$$

It is easy to see that $(H_n, |\cdot|_n)$ is isometric to a norm 2 complemented subspace of $X_{p,w}$ for some w. Indeed, since each element of H_n is a step function and dim $H_n < \infty$, there is a sequence (even finite) of disjoint intervals (A_i) so that $H_n \subseteq \mathrm{span}(\chi_{A_i})$. Let

$$w_i = (\mathrm{meas}\ A_i)^{\frac{1}{2} - \frac{1}{p}} \quad (= \|\chi_{A_i}\|_2 / \|\chi_{A_i}\|_p)$$

and set $f_i = (\mathrm{meas}\ A_i)^{-1/p}\ \chi_{A_i}$ (so that $\|f_i\|_p = 1$). Then for any choice (α_i) of scalars,

$$|\Sigma\ \alpha_i\ f_i|_n = \max(M_n(\Sigma\ \alpha_i^2\ w_i^2)^{1/2} \quad , \quad (\Sigma\ |\alpha_i|^p)^{1/p}) \quad ;$$

i.e., $\overline{\mathrm{span}}\ \chi_{A_i}$ is, in the $|\cdot|_n$ norm, isometric to $X_{p,w}$ when $w = (M_n w_1, M_n w_2, \ldots)$. Thus, by Rosenthal's theorem, we can complete the proof of step one by observing that $(H_n, |\cdot|_n)$ is norm 2 complemented in $(L_p, |\cdot|_n)$ and hence in span χ_{A_i}. But the orthogonal projection P onto H_n satisfies $\|P\|_2 = 1$ and (since the Haar functions are a monotone, orthogonal basis for L_p) $\|P\| \le 2$, hence $|P|_n \le 2$ by the definition of $|.|_n$.

<u>Step Two</u>. B <u>has norm</u> ≤ 5 <u>provided that</u>, <u>given</u> $H_1, H_2, \dots, H_n$, M_{n+2} <u>is chosen sufficiently large</u>.

Suppose that the blocking (H_n) of the Haar functions and numbers (M_n) are given. We want to compute that for $y_n \in H_n$, $\|\Sigma\, y_n\|_p \le 5(\Sigma\, |y_n|_n^p)^{1/p}$, as long as each M_{n+2} is big relative to the modulus of uniform integrability of $H_1 + \dots + H_n$.

Let $M = \{n : |y_n|_n \ge 2^n \|y_n\|_p\}$. Certainly $\|\Sigma\, y_n\|_p \le \|\sum_{n\notin M} y_n\|_p + \sum_{n\in M} \|y_n\|_p \le \|\sum_{n\notin M} y_n\|_p + (\Sigma\, |y_n|_n^p)^{1/p}$, so we need check only that

$$(*) \quad \|\sum_{2n\notin M} y_{2n}\| \le 2(\Sigma\, |y_n|_n^p)^{1/p} \quad , \quad \|\sum_{2n-1\notin M} y_{2n-1}\|_p \le 2(\Sigma\, |y_n|_n^p)^{1/p} \quad .$$

For $n \notin M$ we have that $M_n \|y_n\|_2 \le |y_n|_n \le 2^n \|y_n\|_p$, so that $\|y_n\|_2 / \|y_n\|_p < 2^n M_n^{-1}$. Now if $2^n M_n^{-1}$ is very small, this means that y_n is essentially supported on a set of very small measure, hence if y is a fairly flat function in L_p, then $\|y + y_n\|_p^p \approx \|y\|_p^p + \|y_n\|_p^p$. Thus if M_{n+2} is chosen big relative to the modulus of uniform integrability of $H_1 + \dots + H_n$, then $\|\sum_{2n\notin M} y_{2n}\|_p \approx (\sum_{2n\notin M} \|y_{2n}\|_p^p)^{1/p}$ and $\|\sum_{2n-1\notin M} y_{2n-1}\|_p \approx (\sum_{2n-1\notin M} \|y_{2n-1}\|_p^p)^{1/p}$; in particular, we can guarantee that (*) holds.

Recalling that the blocking $H_n = \mathrm{span}(h_i)_{i=k(n)}^{k(n+1)-1}$ is defined by the increasing sequence $1 = k(1) < k(2) < \dots$, we state

<u>Step Three</u>. A <u>has norm</u> $\le K\|T\|$ (<u>where</u> $K = K_p$ <u>is a constant which depends only on</u> p) <u>provided that</u>, <u>given</u> M_n $(n > 1)$, $k(n)$ <u>is sufficiently big relative to</u> M_n.

Let $\|S\|_2$ be the norm of operator S when considered as an operator into L_2. Let R_n be the orthogonal projection from L_2 onto $\overline{\mathrm{span}}(h_i)_{i=n}^{\infty}$ in L_2. Our hypothesis that T is compact as an operator into L_2 implies that $\|R_n T\|_2 \to 0$ as $n \to \infty$. Suppose now that $\|R_{k(n)} T\|_2 < 2^{-n} M_n^{-1} \|T\|$ for $n = 2,3,\dots$ For $x \in X$ with $Tx = \sum_{n=1}^{\infty} y_n$ $(y_n \in H_n)$, we need to show

$$\|Ax\| = (\sum_{n=1}^{\infty} |y_n|_n^p)^{1/p} \le K\|T\| \, \|x\| \quad .$$

Let $M = \{n : |y_n|_n = \|y_n\|_p\}$. Since the Haar system forms an unconditional basis for L_p and L_p has cotype p, there is a constant $0 < \lambda = \lambda(p)$ so that

$$\|\Sigma\, y_n\|_p \geq \lambda^{-1}\, (\Sigma\, \|y_n\|_p^p)^{1/p}$$

thus

$$(\sum_{n\in M} |y_n|^p)^{1/p} = (\sum_{n\in M} \|y_n\|_p^p)^{1/p} \leq \lambda\, \|\sum_{n=1}^{\infty} y_n\|_p = \lambda\, \|Tx\| \leq \lambda\, \|T\|\, \|x\| \quad .$$

Observing that $1 \in M$ (since $M_1 = 1$), we have that

$$(\sum_{n\notin M} |y_n|_n^p)^{1/p} \leq \sum_{n\notin M} M_n \|y_n\|_2 \leq \sum_{n\notin M} M_n \|\sum_{k=n}^{\infty} y_k\|_2 \leq$$
$$\sum_{n=2}^{\infty} M_n \|R_{k(n)}\, Tx\|_2 \leq \|T\|\, \|x\| \quad .$$

Thus

$$(\sum_{n=1}^{\infty} |y_n|_n^p)^{1/p} \leq (\lambda + 1)\, \|T\|\, \|x\| \quad ,$$

as desired.

Of course, to complete the proof that T factors through ℓ_p, we only have to make the obvious observation that the sufficient conditions in steps two and three for the boundedness of B and A are not mutually exclusive.

We conclude this seminar by giving acounter example to a conjecture made in [2]. Recall that a Banach space X is said to be of <u>type p-Banach-Saks</u> (where $1 < p < \infty$) provided there is a constant λ so that every normalized weakly null sequence in X has a subsequence $\{x_n\}_{n=1}^{\infty}$ which satisfies for $n = 1,2,\ldots$

$$\|\sum_{i=1}^{n} x_i\| \leq \lambda\, n^{1/p} \quad .$$

In [2] we conjectured (in a stronger form) that every operator from L_p $(2 < p < \infty)$ into a space which is of type p-Banach-Saks factors through ℓ_p. This conjectured had been verified in [3] in case T has closed range.

The counter example X can be taken to be the dual of a space (say, X^*) which is the q-convexification $(1/p + 1/q = 1)$ of the space constructed in [4]. (In fact, one could use the simpler space from [1].) X^* is a reflexive space which is q-convex relative to its natural basis, the unit vector basis $\{\delta_n\}_{n=1}^{\infty}$. The space ℓ_q does not embed into X^*, but X^* has the following property for each $n = 1,2,\ldots$:

(*) If $\{y_i\}_{i=1}^{2^n}$ are disjointly supported unit vectors in X^* and and $y_i \in \operatorname{span}(\delta_k)_{k=n+1}^{\infty}$ for $i \le i \le 2^n$, then $\{y_i\}_{i=1}^{2^n}$ is 2-equivalent to the unit vector basis for $\ell_q^{2^n}$.

Property (*) implies that the basis $\{\delta_n\}_{n=1}^{\infty}$ for X^* admits a lower ℓ_r estimate for all $r > q$; consequently, the formal identity map $I : \ell_2 \to X$ is a bounded operator. Now X is p-concave relative to the unit vector basis and no subsequence of this basis can be equivalent to the unit vector basis for ℓ_p, so a routine gliding hump argument shows that I cannot factor through ℓ_p. Since ℓ_2 embeds into L_p as a complemented subspace, there is also an operator from L_p into X which does not factor through ℓ_p.

Finally, to verify that X is of type p-Banach-Saks, it is enough to observe that if $\{x_i\}_{i=1}^{\infty}$ are disjointly supported unit vectors in X, $x_k \in \operatorname{span}(\delta_i)_{i=k+1}^{\infty}$ for $k = 1,2,\ldots$, then for every n, we have from (*) that

$$\left\|\sum_{i=1}^{2^n} x_i\right\| \le n + \left\|\sum_{i=n+1}^{2^n} x_i\right\| \le n + 2.2^{n/p} \le 3.2^{n/p} .$$

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