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OPERATORS INTO L_p WHICH FACTOR THROUGH ℓ_p

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In this seminar we prove the following theorem from [2].

Theorem A : Let T be a bounded linear operator from a Banach space X into L_p ($\equiv L_p[0,1]$), $2 < p < \infty$. Then T factors through ℓ_p if and only if T is compact when considered as an operator into L_2 .

The "only if" part is an immediate consequence of the fact that every operator from ℓ_p to ℓ_2 is compact when $p > 2$ (cf. Proposition 2.C.3 in [7]). The "if" part generalizes an earlier result of Johnson-Odell [5] which says that if X is a subspace of L_p ($p > 2$) which does not contain an isomorphic copy of ℓ_2 , then X embeds into ℓ_p , because it is an easy consequence of the results in [6] that the restriction to such an X of the injection from L_p into L_2 is compact.

Proof of Theorem A : We factor T through a space of the form $Y = (\sum (H_n, |\cdot|_n))_{\ell_p}$, where each space $(H_n, |\cdot|_n)$ is finite dimensional. We will observe that the spaces $(H_n, |\cdot|_n)$ are uniformly isomorphic to uniformly complemented subspaces of L_p , and hence Y is isomorphic to a complemented subspaces of ℓ_p . (Of course, this implies that Y is isomorphic to ℓ_p by a result of Pełczyński's [8], but we don't need this fact, since it is clear that if T factors through a complemented subspace of ℓ_p , then T factors through ℓ_p .)

The spaces (H_n) are chosen to be a blocking of the Haar basis for L_p . That is, $H_n = \text{span}(h_i)_{i=k(n)}^{k(n+1)-1}$, where (h_i) is the Haar basis for L_p and $1 = k(1) < k(2) < \dots$ is a suitably chosen sequence of positive integers. The operators $A: X \rightarrow Y$ and $B: Y \rightarrow L_p$ which factor T are defined in the natural way : for $x \in X$ with $Tx = \sum y_n$ ($y_n \in H_n$), we define $Ax = (y_n)_{n=1}^{\infty}$. For $y_n \in H_n$ with $(y_n)_{n=1}^{\infty} \in Y$, we define $B(y_n) = \sum y_n \in L_p$. Obviously we have $BA = T$, but of course we have to show that A and B are bounded if the $(H_n, |\cdot|_n)$ sequence is appropriately defined.

It is convenient to define $|\cdot|_n$ on all of L_p . For appropriate values of M_n , $1 \leq M_1 < M_2 < M_3 < \dots$, $|\cdot|_n$ is defined by

$$|f|_n = \max(M_n \|f\|_2, \|f\|_p),$$

where

$$\|f\|_2 \equiv \left(\int_0^1 |f(t)|^2 dt \right)^{1/2}, \quad \|f\|_p \equiv \left(\int_0^1 |f(t)|^p dt \right)^{1/p}$$

have their usual meaning. It is evident that each $|\cdot|_n$ is equivalent to $\|\cdot\|_p$ on L_p , but as $M_n \uparrow \infty$ the constant of equivalence tends to infinity.

We break the proof that T factors through Y if (H_n) and (M_n) are defined appropriately into three steps.

Step One. There is a constant $K = K(p)$ such that $(H_n, |\cdot|_n)$ is K -isomorphic to a K -complemented subspace of L_p .

Of course, this means that Y is isomorphic to a complemented subspace of ℓ_p no matter how M_n is defined.

Step one is easy, given a result of Rosenthal's [9]. Rosenthal proved that there is a constant $\lambda = \lambda(p)$ so that for any sequence $w = (w_1, w_2, \dots)$ of positive numbers the space $X_{p,w}$ is λ -isomorphic to a λ -complemented subspace of L_p . Here $X_{p,w}$ is the completion of $\mathbb{R}^\infty$ (or $\mathbb{C}^\infty$) under the norm $\|\cdot\|_w$ defined by

$$\|(\alpha_i)\|_w = \max\left((\sum |\alpha_i|^2 w_i^2)^{1/2}, \left(\sum_{i=1}^\infty |\alpha_i|^p\right)^{1/p}\right).$$

It is easy to see that $(H_n, |\cdot|_n)$ is isometric to a norm 2 complemented subspace of $X_{p,w}$ for some w . Indeed, since each element of H_n is a step function and $\dim H_n < \infty$, there is a sequence (even finite) of disjoint intervals (A_i) so that $H_n \subseteq \text{span}(\chi_{A_i})$. Let

$$w_i = (\text{meas } A_i)^{\frac{1}{2} - \frac{1}{p}} \quad (= \|\chi_{A_i}\|_2 / \|\chi_{A_i}\|_p)$$

and set $f_i = (\text{meas } A_i)^{-1/p} \chi_{A_i}$ (so that $\|f_i\|_p = 1$). Then for any choice (α_i) of scalars,

$$\|\sum \alpha_i f_i\|_n = \max(M_n (\sum \alpha_i^2 w_i^2)^{1/2}, (\sum |\alpha_i|^p)^{1/p});$$

i.e., $\overline{\text{span } \chi_{A_i}}$ is, in the $|\cdot|_n$ norm, isometric to $X_{p,w}$ when $w = (M_n w_1, M_n w_2, \dots)$. Thus, by Rosenthal's theorem, we can complete the proof of step one by observing that $(H_n, |\cdot|_n)$ is norm 2 complemented in $(L_p, |\cdot|_n)$ and hence in $\text{span } \chi_{A_i}$. But the orthogonal projection P onto H_n satisfies $\|P\|_2 = 1$ and (since the Haar functions are a monotone, ortho-

gonal basis for L_p) $\|P\| \leq 2$, hence $|P|_n \leq 2$ by the definition of $| \cdot |_n$.

Step Two. B has norm ≤ 5 provided that, given $H_1, H_2, \dots, H_n, M_{n+2}$ is chosen sufficiently large.

Suppose that the blocking (H_n) of the Haar functions and numbers (M_n) are given. We want to compute that for $y_n \in H_n$, $\|\sum y_n\|_p \leq 5(\sum |y_n|_n^p)^{1/p}$, as long as each M_{n+2} is big relative to the modulus of uniform integrability of $H_1 + \dots + H_n$.

Let $M = \{n : |y_n|_n \geq 2^n \|y_n\|_p\}$. Certainly $\|\sum y_n\|_p \leq \|\sum_{n \notin M} y_n\|_p + \sum_{n \in M} \|y_n\|_p \leq \|\sum_{n \notin M} y_n\|_p + (\sum_{n \in M} |y_n|_n^p)^{1/p}$, so we need check only that

$$(*) \quad \|\sum_{2n \notin M} y_{2n}\|_p \leq 2(\sum_{n \notin M} |y_n|_n^p)^{1/p}, \quad \|\sum_{2n-1 \notin M} y_{2n-1}\|_p \leq 2(\sum_{n \notin M} |y_n|_n^p)^{1/p}.$$

For $n \notin M$ we have that $M_n \|y_n\|_2 \leq |y_n|_n \leq 2^n \|y_n\|_p$, so that $\|y_n\|_2 / \|y_n\|_p < 2^n M_n^{-1}$. Now if $2^n M_n^{-1}$ is very small, this means that y_n is essentially supported on a set of very small measure, hence if y is a fairly flat function in L_p , then $\|y + y_n\|_p^p \approx \|y\|_p^p + \|y_n\|_p^p$. Thus if M_{n+2} is chosen big relative to the modulus of uniform integrability of $H_1 + \dots + H_n$, then $\|\sum_{2n \notin M} y_{2n}\|_p \approx (\sum_{2n \notin M} \|y_{2n}\|_p^p)^{1/p}$ and $\|\sum_{2n-1 \notin M} y_{2n-1}\|_p \approx (\sum_{2n-1 \notin M} \|y_{2n-1}\|_p^p)^{1/p}$; in particular, we can guarantee that $(*)$ holds.

Recalling that the blocking $H_n = \text{span}(h_i)_{i=k(n)}^{k(n+1)-1}$ is defined by the increasing sequence $1 = k(1) < k(2) < \dots$, we state

Step Three. A has norm $\leq K\|T\|$ (where $K = K_p$ is a constant which depends only on p) provided that, given M_n ($n > 1$), $k(n)$ is sufficiently big relative to M_n .

Let $\|S\|_2$ be the norm of operator S when considered as an operator into L_2 . Let R_n be the orthogonal projection from L_2 onto $\overline{\text{span}(h_i)_{i=n}^\infty}$ in L_2 . Our hypothesis that T is compact as an operator into L_2 implies that $\|R_n T\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Suppose now that

$$\|R_{k(n)} T\|_2 < 2^{-n} M_n^{-1} \|T\| \quad \text{for } n = 2, 3, \dots \quad \text{For } x \in X \text{ with } Tx = \sum_{n=1}^\infty y_n \quad (y_n \in H_n),$$

we need to show

$$\|Ax\| = \left(\sum_{n=1}^\infty |y_n|_n^p \right)^{1/p} \leq K\|T\| \|x\|.$$

Let $M = \{n : |y_n|_n = \|y_n\|_p\}$. Since the Haar system forms an unconditional basis for L_p and L_p has cotype p , there is a constant $0 < \lambda = \lambda(p)$ so that

$$\|\sum y_n\|_p \geq \lambda^{-1} (\sum \|y_n\|_p^p)^{1/p}$$

thus

$$(\sum_{n \in M} |y_n|^p)^{1/p} = (\sum_{n \in M} \|y_n\|_p^p)^{1/p} \leq \lambda \left\| \sum_{n=1}^{\infty} y_n \right\|_p = \lambda \|Tx\| \leq \lambda \|T\| \|x\|.$$

Observing that $1 \in M$ (since $M_1 = 1$), we have that

$$\begin{aligned} (\sum_{n \notin M} |y_n|^p)^{1/p} &\leq \sum_{n \notin M} M_n \|y_n\|_2 \leq \sum_{n \notin M} M_n \left\| \sum_{k=n}^{\infty} y_k \right\|_2 \leq \\ &\sum_{n=2}^{\infty} M_n \|R_{k(n)} Tx\|_2 \leq \|T\| \|x\|. \end{aligned}$$

Thus

$$(\sum_{n=1}^{\infty} |y_n|^p)^{1/p} \leq (\lambda + 1) \|T\| \|x\|,$$

as desired.

Of course, to complete the proof that T factors through ℓ_p , we only have to make the obvious observation that the sufficient conditions in steps two and three for the boundedness of B and A are not mutually exclusive.

We conclude this seminar by giving a counter example to a conjecture made in [2]. Recall that a Banach space X is said to be of type p -Banach-Saks (where $1 < p < \infty$) provided there is a constant λ so that every normalized weakly null sequence in X has a subsequence $\{x_n\}_{n=1}^{\infty}$ which satisfies for $n = 1, 2, \dots$

$$\left\| \sum_{i=1}^n x_i \right\| \leq \lambda n^{1/p}.$$

In [2] we conjectured (in a stronger form) that every operator from L_p ($2 < p < \infty$) into a space which is of type p -Banach-Saks factors through ℓ_p . This conjecture had been verified in [3] in case T has closed range.

The counter example X can be taken to be the dual of a space (say, X^*) which is the q -convexification ($1/p + 1/q = 1$) of the space

constructed in [4]. (In fact, one could use the simpler space from [1].) X^* is a reflexive space which is q -convex relative to its natural basis, the unit vector basis $\{\delta_n\}_{n=1}^\infty$. The space ℓ_q does not embed into X^* , but X^* has the following property for each $n = 1, 2, \dots$:

$$(*) \quad \left\{ \begin{array}{l} \text{If } \{y_i\}_{i=1}^{2^n} \text{ are disjointly supported unit vectors in } X^* \text{ and} \\ \text{and } y_i \in \text{span}(\delta_k)_{k=n+1}^\infty \text{ for } i \leq i \leq 2^n, \text{ then } \{y_i\}_{i=1}^{2^n} \text{ is 2-equivalent} \\ \text{to the unit vector basis for } \ell_q^{2^n}. \end{array} \right.$$

Property (*) implies that the basis $\{\delta_n\}_{n=1}^\infty$ for X^* admits a lower ℓ_r estimate for all $r > q$; consequently, the formal identity map $I: \ell_2 \rightarrow X$ is a bounded operator. Now X is p -concave relative to the unit vector basis and no subsequence of this basis can be equivalent to the unit vector basis for ℓ_p , so a routine gliding hump argument shows that I cannot factor through ℓ_p . Since ℓ_2 embeds into L_p as a complemented subspace, there is also an operator from L_p into X which does not factor through ℓ_p .

Finally, to verify that X is of type p -Banach-Saks, it is enough to observe that if $\{x_i\}_{i=1}^\infty$ are disjointly supported unit vectors in X , $x_k \in \text{span}(\delta_i)_{i=k+1}^\infty$ for $k = 1, 2, \dots$, then for every n , we have from (*) that

$$\left\| \sum_{i=1}^{2^n} x_i \right\| \leq n + \left\| \sum_{i=n+1}^{2^n} x_i \right\| \leq n + 2 \cdot 2^{n/p} \leq 3 \cdot 2^{n/p}.$$

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