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D ' A N A L Y S E F O N C T I O N N E L L E

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GEOMETRY OF NUCLEAR SPACES

-I-

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These talks present my results and results of my colleagues for the last three-four years on the geometry of nuclear spaces.

I - NUCLEAR FRÉCHET SPACES WITHOUT BASIS.

The notion of a nuclear space was inspired by L. Schwartz' theorem on kernel [1] which states that any bilinear continuous form $B : \mathcal{D}(\mathbb{R}^{m_1}) \times \mathcal{D}(\mathbb{R}^{m_2}) \to \mathbb{C}^1$ generates the linear functional $B^* : \mathcal{D}(\mathbb{R}^m) \to \mathbb{C}^1$, $m = m_1 + m_2$ by the formula

$$B^*(\varphi(x_1,\ldots,x_{m_1}) \cdot \psi(x_{m_1+1},\ldots,x_{m_1+m_2})) = B(\varphi;\psi) \quad .$$

A. Grothendieck [2] developped the tensor-product theory and on this base he constructed the theory of nuclear spaces and especially the duality theory on these spaces.

From the very beginning the notion of a nuclear space was parallel to the notion of a nuclear operator. Recall that an operator $A : H_1 \to H_2$ in Hilbert spaces is called nuclear iff it is compact and

$$\Sigma\ \rho_k(A) < \infty \quad \text{where} \quad \rho_k(A) = \lambda_k(\sqrt{A^* \cdot A}) \quad ,$$

$k = 0,1,\ldots$, are monotonically ordering (with multiplicity) eigenvalues of the module $|A|$. The extension of this notion to the general case of Banach spaces was very fruitful ; it has been developed to the theory of normed operator-ideals and the theory of absolutely-summing operators of different types (see [3], [4] and refenreces there). I will not touch these topics.

I will not give different (equivalent) definitions of nuclear space and recall the simplest one which is sufficient for our further consideration.

<u>Definition 1</u> : A locally convex space is nuclear if it is a dense subspace of a projective limit of Hilbert spaces with nuclear maps.

A nuclear Fréchet space X is a projective limit of a sequence of nuclear maps on separable Hilbert spaces ; more detaily, there exists such a system of inner continuous (semi-) products $(x,y)_p$, $p \in \mathbb{N}$, on X, that (semi-) norms $\|x\|_p = (x,x)_p^{1/2}$, $p \in \mathbb{N}$, generate the topology of X and $\forall\, p\ \exists\, q \mid i_p^q : X_q \to X_p$ is nuclear.

Here X_p denotes as usually the Hilbert space $(\overline{X/N_p})$, $N_o = \{u \in X : \|u\| = 0\}$ with the norm $\|x\|_p$, and i_p^q denotes the induced "imbedding".

Example 1 : The space $C^\infty(\mathbb{T}^k)$ of all infinitely differentiable (real- or complex valued) functions on k-dimensional torus. The topology of the uniform convergence of all derivatives is generated by the system of norms

$$\|x\|_p = (x,x)_p^{1/2} \quad ; \quad (x,y) = \int_{\mathbb{T}^k} {}_{0\le|\alpha|\le p}\ \partial^\alpha x(t) \,.\, \overline{\partial^\alpha y(t)}\ dt \quad ,$$

dt be the Haar measure on $\mathbb{T}^k$ and ∂^α, $\alpha = (\alpha_1,\dots,\alpha_k)$, be the usual notion of partial derivatives.

The operator $i_p^q : W_q^2(\mathbb{T}^k) = X_q \to X_p = W_p^2(\mathbb{T}^k)$ is nuclear iff $q > p + k$.

Example 2 : The space $H(\mathbb{D}^k)$ of all holomorphic functions on the open unit polydisc

$$\mathbb{D}^k = \{z = (z_1,\dots,z_k) \in \mathbb{C}^k \ ;\ |z_j| < 1,\ 1 \le j \le k\} \quad .$$

The topology of uniform convergence on all compacta in $\mathcal{D}^k$ is generated by the system of Hilbert norms with inner products

$$(x,y) = \int_{\mathbb{T}^k} x((1-\tfrac{1}{p})\zeta) \,.\, \overline{y((1-\tfrac{1}{p})\zeta)}\ dt \quad ,$$

$$\zeta = (e^{it_1},\dots,e^{it_k}) \quad .$$

In the terms of Taylor coefficients

$$(x,y)_p = {}_{n\in\mathbb{Z}_+^k}\ \tilde{x}(n) \,.\, \overline{\tilde{y}(n)} \,.\, (1-\tfrac{1}{p})^{2|n|} \quad ,$$

where $n = (n_1, \ldots, n_k)$, $|n| = n_1 + \ldots + n_k$,

$$\tilde{x}(n) = \int_{\mathbb{T}^k} x(\zeta)\ \exp(-i \cdot \langle n,t\rangle)\ dt \quad .$$

Hence $X_p = H^2(r_p \cdot \mathbb{T}^k)$ is Hardy space, $r_\rho = 1 - \frac{1}{\rho}$; the operator $i_r^{r'} : H^2(r'\ \mathbb{T}^k) \to H^2(r\ \mathbb{T}^k)$ is (ultra) nuclear for any pair γ', γ, $\gamma' > \gamma$.

<u>Example 3</u> : The Köthe space [5]

$$K(a) = \{x = (x_\nu)_{\nu \in \mathfrak{N}}\ ,\ x_\nu \in \mathbb{C}^1\ :\ \sum_\nu a^2_{\nu p} \cdot |x_\nu|^2 < \infty\ ,\ \forall\ p\}$$

where $a = (a_{\nu p})$ is a matrix with non-negative (positive) scalar terms is nuclear iff

$$\forall\ p\ \exists\ q\ \mid\ \sum_\nu a_{\nu p} / a_{\nu q} < \infty \quad .$$

The last example is very important because of

<u>Theorem AB</u> (on absoluteness of bases -[6], [7]) : In a nuclear Fréchet space X any basis $\{e_n, e'_n\}_0^\infty$ is absolute, i.e. for any (semi) norm $\|\cdot\|_p$

$$\sum |e'_n(x)| \cdot \|e_n\|_p < \infty \quad , \quad \forall\ x \in X \quad .$$

Hence the space X is isomorphic to the Köthe space K(a), $a = (a_{np})$, $a_{np} = \|e_n\|_p$, $n, p = 0, 1, \ldots$.

(Recall that the biorthogonal system $\{e_n, e'_n\}$ is a basis in a linear topological space E if every element $x \in E$ has an expansion $x = \sum e'_n(x)\ e_n$.)

It is easy to see that the exponentials $e_n = \exp\ i\langle n,t\rangle$, $n \in \mathbb{Z}^k$, give a (absolute) basis in $C^\infty(\mathbb{T}^k)$, and that $\{e_n, n \in \mathbb{Z}^k_+\}$ is an absolute basis in $H(\mathbb{D}^k)$ so we have the isomorphisms $I : x \to \tilde{x}(n)$, $C(\mathbb{T}^k) \approx K(a)$, $a_{np} = (1 + |n|^2)^p$, $n \in \mathbb{Z}^k$,

$$H(\mathbb{D}^k) \approx K(b) \quad , \quad b_{np} = \exp(-\frac{1}{p}\,|n|) \quad , \quad n \in \mathbb{Z}^k_+$$

or

$$\approx K(c) \quad , \quad c_{np} = \exp(-\frac{n^{1/k}}{p}) \quad , \quad n \in \mathbb{Z}_+ \quad .$$

For any compact C^{∞}-manifold M, the space $C^{\infty}(\mathbb{T}^k)$, $k = \dim_{\mathbb{R}} M$; eigenfunctions $u_n(x)$, $n = 0,1,\ldots$, of Laplace-Beltrami operator, $Lu_n = \lambda_n u_n$, $\lambda_n \le \lambda_{n+1}$, give a basis of this space, and Fourier coefficients of any C^{∞}-function decrease faster than any power of 1/n. For any compact F, $F \subset \mathbb{R}^k$, we define $C^{\infty}(F)$ as the quotient space $C^{\infty}(\mathbb{R}^k)/Z(F)$, $Z(F)$ is the closure of the linear manifold $\{f \in C^{\infty}(\mathbb{R}^k) \mid f \equiv 0$ for some neighbourhood of $F\}$ so $C^{\infty}(F)$ is nuclear.

<u>Question 1</u> : Is it true that for any compact $F \subset \mathbb{R}^k$ the space $C^{\infty}(F)$ has a base ?

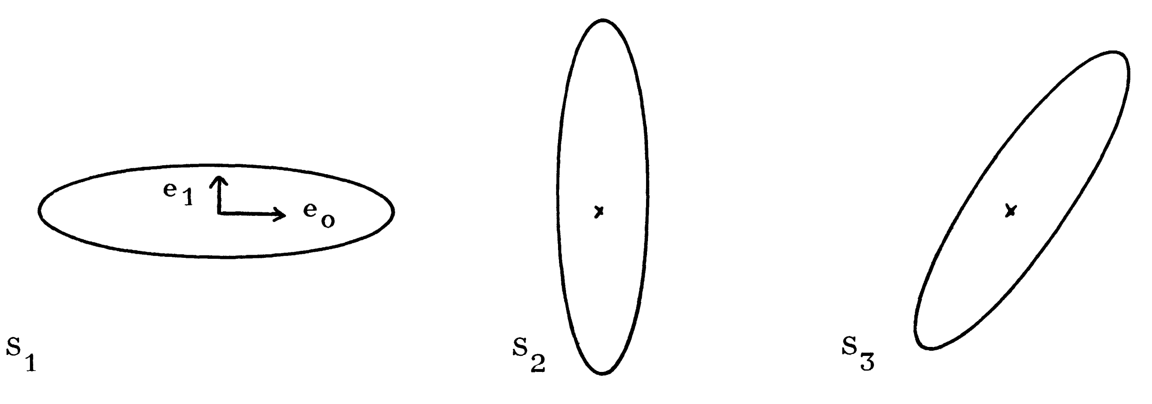
For any domain G of holomorphy, $G \subset \mathbb{C}^k$ (or Stein manifold) the space H(G) of all holomorphic functions on G with the topology of uniform convergence on compact sets is nuclear.

<u>Question 2</u> : Is it true that for any domain $G \subset \mathbb{C}^k$ of holomorphy (or for any Stein manifold) the space H(G) has a basis ?

The answer is unknown even for the case $k = 1$.

We do not know any concrete example of nuclear functional space without basis, although I believe there exist such counterexamples to Que. 1 and 2. Now I present series of general nuclear Fréchet spaces without basis (after Mityagin-Zobin [8]-[10] and Djakov-Mityagin [11]).

"Two-dimensional case" (after [11], § 2, and [12]).
Let us consider three ellipses



$$S_1 = \{x = (\xi_o, \xi_1) : a_1^2 \, |\xi_o|^2 + b_1^2 \, |\xi_1|^2 \le 1\}$$

$$S_2 = \{x \in \mathbb{C}^2 : a_2^2 \, |e_o^*(x)|^2 + b_2^2 \, |e_1^*(x)|^2 \le 1\}$$

$$S_3 = \{x \in \mathbb{C}^2 : a_3^2 \, |w_o^*(x)|^2 + b_3^2 \, |w_1^*(x)|^2 \le 1\}$$

where $w_o = \frac{1}{\sqrt{2}} (e_o + e_1)$, $w_1 = \frac{1}{\sqrt{2}} (-e_o + e_1)$, $f^*(x) = \langle x, f \rangle = \xi_o \, \overline{f}_o + \xi_1 \, \overline{f}_1$.

Then

$$|e_1^*|_1 \cdot |e_o|_1 = a_1/b_1 \quad , \quad |e_o^*|_2 \cdot |e_1|_2 = b_2/a_2 \quad , \tag{1}$$

$$|w_o^*|_3 \cdot |w_1|_3 = b_3/a_3 \quad .$$

More accurately we have to write $|x|_{\varepsilon,(a_\varepsilon, b_\varepsilon)}$ for the norms

$|x|_\varepsilon = (a_\varepsilon^2 \, |\xi_o|^2 + b_\varepsilon^2 \, |\xi_1|^2)^{1/2}$, $\varepsilon = 1,2,3$, or analoguously for the dual norms. If we consider (see below (4)) homothetic ellipses the relations (1) do not change.

The following elementary lemma holds.

<u>Lemma A</u> : Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be of rank < 2 ; then

$$|t_{oo}| = |e_o^*(Te_o)| \le |e_o^*(Te_1)| + |e_1^*(Te_o)| + 2|w_1^*(Tw_o)| \quad . \tag{2}$$

Indeed by homogeneity we can assume that $t_{oo} = 1$, and so

$$T = \begin{pmatrix} 1 & \alpha \\ \beta & \alpha\beta \end{pmatrix} \quad , \quad w_1^*(Tw_o) = \frac{1}{2} \, (-(1 + \alpha) + (\beta + \alpha\beta))$$

and (2) has the form

$$1 \le |\alpha| + |\beta| + |1 - \beta| \cdot |1 + \alpha| \quad . \tag{2'}$$

It is evident for $|\alpha| \ge 1$ or $|\beta| \ge 1$. Otherwise

$$|1 - \beta| \cdot |1 + \alpha| \ge (1 - |\beta|)(1 - |\alpha|) = 1 - |\beta| - |\alpha| + |\alpha\beta| \ge 1 - |\alpha| - |\beta|$$

and (2'), and (2), is true also.

Lemma B : Let $1_{\mathbb{C}^2} = \Sigma\, T_k$ on $\mathbb{C}^2$, rank $T_k < 2$, and $\Sigma\, |T_k\, x|_\varepsilon \le A\, |x|_\varepsilon$, $\forall\, x \in \mathbb{C}^2$, $\varepsilon = 1,2,3$. Then

$$(3) \qquad A \ge \alpha \quad , \quad \alpha = \frac{1}{4} \min(a_2/b_2\,,\ b_1/a_1\,,\ a_3/b_3) \quad .$$

Indeed by (1) and (2), Lemma A,

$$1 = e_o^*(e_o) = e_o^*(\Sigma\, T_k\, e_o) \le \Sigma\, |e_o^*(T_k\, e_o)| \le$$

$$\le \Sigma\, |e_1^*(T\, e_o)| + \Sigma\, |e_o^*(T\, e_1)| + 2\, \Sigma\, |w_o^*(T\, w_1)| \le$$

$$\le A\, |e_1^*|_1 \cdot |e_o|_1 + A\, |e_o^*|_2 \cdot |e_1|_2 + 2 \cdot A \cdot |w_o^*|_3 \cdot |w_1|_3$$

$$= A\, (a_1/b_1 + b_2/a_2 + 2 \cdot b_3/a_3) \le \frac{A}{\alpha} \quad ,$$

and it implies (3).

Example 4 : Generalized Köthe space

$$K(a) = \{x = (x_n)_o^\infty\ ,\ x_n \in \mathbb{C}^2 \ :\ \|x\|_p^2 = \Sigma\, |A_{np}\, x_n|^2 < \infty\}$$

by the definition is a space of vector sequences ; its topology is determined by the fundamental system of (semi) norms $\|x\|_p$, $p = 0,1,\ldots$, where (a_{np}) is a matrix with two-dimensional positive self-adjoint operators as its terms. Under the particular choice of a matrix, $a = (A_{np})$ the generalized Köthe space has no base.

To make this choice, or to define two-dimensional Hilbert norms

$$\|x\|_{np} = |A_{np}\, x| \quad , \quad n,p = 0,1,\ldots,$$

let us choose a 1-1-correspondence

$$\sigma : \mathbb{N} \longrightarrow \pi \quad , \quad \pi = \{(p_o, p_1, \ell) \in \mathbb{N}^3 \ :\ 0 < p_o < p_1\}$$

and put $\qquad \mathbb{N}_p = \sigma^{-1}(\pi_p) \quad , \quad \pi_p = \{(p_o, p_1, \ell) \ :\ \ell \in \mathbb{N}\}$,

$p = (p_o, p_1)$ is fixed,

so $$|\mathbb{N}_p| = \infty \quad \text{for any pair } p,\ p_o < p_1 \ .$$

Then if $n \in \mathbb{N}_p$ we put

$$(4) \qquad |A_{nq}\, x| = (|\xi_o|^2 + |\xi_1|^2)^{1/2} \ , \quad q = 0 \ ,$$

$$= \lambda_{nq}\ |x|_{\varepsilon,(a_\varepsilon, b_\varepsilon)} \ , \quad \varepsilon = 1,\ 1 \le q \le p_o \ , \quad = 2,\ p_o < q \le p_1 \ , \quad = 3,\ p_1 < q \ .$$

Let the following condition MN (monotonicity and nuclearity) hold

$$(5) \quad \lambda_{n1}\, a_{n1} \ge n^2 \ ; \ \lambda_{n,p_o+1}\, b_{2n} \ge n^2\, \lambda_{np_o}\, b_{1n} \ ; \ \lambda_{np_1+1}\, b_{3n} \ge n^2\, \lambda_{np_1}\, a_{2n} \ .$$

Then $\|x\|_q \le \|x\|_{q+1}$ and the space $K(a)$ under the choice (4) is nuclear. We say that the baseless condition BL holds if

$\forall\ p = (p_o, p_1),\ \forall\ p_2 > p_1\ \exists\ N\ \forall\ n \in \mathbb{N}_p\ ,\ n \ge N$:

$$(6) \qquad n^2 \cdot \max\left\{ \frac{\lambda_{np_o}}{\lambda_{n1}} \ , \ \frac{\lambda_{np_1}}{\lambda_{np_o+1}} \ , \ \frac{\lambda_{np_2}}{\lambda_{np_1+1}} \right\} \le \min\left\{ \frac{b_{1n}}{a_{1n}} \ ; \ \frac{a_{2n}}{b_{2n}} \ ; \ \frac{a_{3n}}{b_{3n}} \right\} \ .$$

Both conditions MN and BL hold for example if for $n \in \mathbb{N}_p$

$$b_{1n} = a_{2n} = a_{3n} = 2^n \ ; \quad a_{1n} = b_{2n} = b_{3n} = 1 \ ;$$

$$\lambda_{ni} = n^{2i} \ , \quad 1 \le i \le p_o$$

$$n^{2i} \cdot 2^n \ , \quad p_o < i \le p_1$$

$$n^{2i} \cdot 2^{2n} \ , \quad p_1 < i \ .$$

<u>Remark 1</u> : It is useful to pay attention that the conditions (5) involves nontrivial restrictions on ratio $\lambda_{ni+1}/\lambda_{ni}$ only for $i = 0, p_o, p_1$, $n \in \mathbb{N}_p$, and the condition (6) involves these ratios for <u>other</u> indices i ; for example,

$$\lambda_{np_1}/\lambda_{np_o+1} = \prod_{i=p_o+1}^{p_1-1} \lambda_{ni+1}/\lambda_{ni} \ .$$

This remark makes conditions MN and BL practically independent and gives possibility to construct spaces without basis with "any given" properties.

<u>Theorem BL</u> (on baseless space) : If the conditions MN and BL hold under the choice (4) then the generalized Köthe space K(a) has no basis.

<u>Proof</u> : If the space K(a) has a base (f_k, f_k^*) then by theorem AB (and by the open-mapping theorem) $\forall\, p\ \exists\, q,\ C \mid \sum |f_k^*(x)| \cdot \|f_k\|_p \le C\ \|x\|_q$. In particular, $\exists\ q_o,\ q_1,\ q_2,\ C \mid$

(7.1) $$\sum |f_k^*(x)| \cdot \|f_k\|_1 \le C \cdot \|x\|_{q_o} \quad ,$$

(7.2) $$\sum |f_k^*(x)| \cdot \|f_k\|_{q_o+1} \le C \cdot \|x\|_{q_1} \quad ,$$

(7.3) $$\sum |f_k^*(x)| \cdot \|f_k\|_{q_1+1} \le C \cdot \|x\|_{q_2} \quad , \quad \forall\ x \in K(a) \quad .$$

Put $p = (q_o, q_1)$ and consider indices $n \in \mathbb{N}_p$ only. Let us define the operators in $\mathbb{C}^2$

$$T_k = T_k^n = r_n \circ (f_k^*(.) f_k) \circ j_n \quad ,$$

where

(8) $$\mathbb{C}^2 \xrightarrow{j_n} K(a) \xrightarrow{f_k^*(.)f_k} K(a) \xrightarrow{r_n} \mathbb{C}^2 \quad ,$$

and $j_n(y) = (0,\dots,0,\underset{n^{\underline{th}}}{y},0,\dots)$, $r_n(x) = x_n$.

Then $1_{\mathbb{C}^2} = \sum T_k$ and by (7.1-3)

$$\sum \|T_k\ x\|_1 = \lambda_{n1} \sum |T_k\ x|_1 \le C\ \|j_n\ x\|_{q_o} = C\ \lambda_{nq_o}\ |x|_1 \quad ,$$

$$\sum \|T_k\ x\|_{q_o+1} = \lambda_{nq_o+1} \sum |T_k\ x|_2 \le C\ \|j_n\ x\|_{q_1} = C\ \lambda_{nq_1}\ |x|_2 \quad ,$$

$$\sum \|T_k\ x\|_{q_1+1} = \lambda_{nq_1+1} \sum |T_k\ x|_3 \le C\ \|j_n\ x\|_{q_2} = C\ \lambda_{nq_2}\ |x|_3 \quad ,$$

and by Lemma B

$$C \max\left\{\frac{\lambda_{nq_o}}{\lambda_{n1}}, \frac{\lambda_{nq_1}}{\lambda_{nq_o+1}}, \frac{\lambda_{nq_2}}{\lambda_{nq_1+1}}\right\} \geq$$

$$\geq \frac{1}{4}\min\left\{\frac{b_{1n}}{a_{1n}}, \frac{a_{2n}}{b_{2n}}, \frac{a_{3n}}{b_{3n}}\right\}$$

and this contradicts to BL-condition.

Remark 2 : We could repeat the same argument replacing (8) by the analoguous sequence of mapping

$$\mathbb{C}^2 \xrightarrow{j_n} K(a)\times Y \xrightarrow{f_k^*(.)f_k} K(a)\times Y \xrightarrow{r_n} \mathbb{C}^2$$

if $\{f_k; f_k^*\}$ were a basis in $K(a)\times Y$.
Hence the space $K(a)\times Y$ has no basis for any nuclear Fréchet space Y if K(a) is as in Theorem BL.

Additional constructions give the following examples.

There exists a continuum of pairwise-non-isomorphic nuclear Fréchet spaces without basis [10], [11].

Any nuclear Fréchet space (except $\mathbb{C}^\infty$) has

- a subspace without base [11], [13] ;
- a quotient space without base [14] .

In all these cases the structure of spaces without base is of the above type, i.e. of Example 4 with different choices of norms (4) and modification of the BL-condition.

The further modifications use the generalized Köthe spaces of the type

$$(9) \qquad K(b) = \{x = (x_n)_o^\infty , \ x_n \in \mathbb{C}^{N(n)} : \ \Sigma\ |B_{np}\ x_n|^2 < \infty , \ \forall\ p\}$$

where N(n) is a sequence of integers and $B_{np}: \mathbb{C}^{N(n)} \to \mathbb{C}^{N(n)}$, $n,p = 0,1,\ldots,$ are positive operators under certain conditions (see [11], Sect. 4-5, and [15]). In particular,

there exists a nuclear Fréchet space $X = K(b)$ of (9) without strongly finite-dimensional decomposition, i.e. X has no system of projection $\{P_t\}$ such that

a) $$P_t\, P_{t'} = 0 \quad , \quad t \neq t' \quad ;$$

b) $$x = \sum P_t x \quad , \quad \forall\, x \in X \quad ;$$

c) $$\sup_t \dim P_t < \infty \quad .$$

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