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WEAKLY COMPACT SETS IN L^1/H_o^1

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By A we mean a uniform algebra in the sense of T.W. Gamelin [4], i.e. there is a compact Hausdorff space X such that $A \subset C(X)$, $1 \in A$ and A separates the points of X. If $\phi : A \to \mathbb{C}$ is a nonzero, multiplicative, linear functional then M_ϕ denotes the set of positive representing measures on X. More precisely $M_\phi = \{\mu \mid \mu$ a positive measure on X and $\int f \, d\mu = \phi(f)$ for all $f \in A\}$.
We will suppose that M_ϕ is a weakly compact set in the space of all measures on X. In this case it is easily seen that there is $m \in M_\phi$ such that all other measures in M_ϕ are absolutely continuous with respect to m (f.i. a slight modification of the proof given in Dunford-Schwartz [3] p. 307 already gives this result).
By H^∞ we mean the Hardy space which is the weak star closure of A in $L^\infty(m)$ where m is the dominant measure mentioned before. The predual of H^∞ is $L^1(m)/N$ where N is the space of functions annihilating H^∞ for the bilinear form $\langle f,g \rangle = \int fg \, dm$.
Since M_ϕ is weakly compact in $L^1(m)$, all the results of [1] and [2] apply. Of course we identify M_ϕ with the set $\{\frac{d\mu}{dm} \mid \mu \in M_\phi\} \subset L^1(m)$.

Given an element $\phi \in L^1(m)/N$ then we can restrict ϕ to the space A and obtain an element $\phi|_A \in A^*$. It follows immediately from the results of Ahern and Sarason that $||\phi|_A|| = ||\phi||$. ([4] Theorem VI.5.2., p. 152-153).
It follows that $L^1(m)/N$ can be identified with a closed subspace of A^*.

<u>LEMMA</u> : Let $\phi \in L^1/N$ and let μ be a measure on X such that

i) $||\mu|| = ||\phi||$

ii) $\mu|_A = \phi$

then $\mu \in L^1(m)$.

<u>Proof</u> : The existence of μ is given by the Hahn-Banach theorem. Let now $\nu \in L^1(m)$ such that $\nu|_A = \phi$ then $(\nu - \mu) \perp A$.
If $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m then by the abstract F. and M. Riesz theorem $(\nu - \mu_a) \perp A$ and $\mu_s \perp A$ ([4] p. 44).

Since $||\phi|| = ||\mu|| = ||\mu_a|| + ||\mu_s||$ and $\mu_a/_A = \nu/_A = \phi$ we obtain that $||\mu_s|| = 0$ and hence $\mu = \mu_a \in L^1(m)$.
We will need the following results of [1] and [2].

LEMMA (Chaumat [2], lemme 2) : Let f_n be a bounded sequence in $L^1(m)$ and let μ be an element of $(L^\infty)^*$ adherent to the sequence f_n (for the topology $\sigma((L^\infty)^*, L^\infty)$. Let $\mu = \mu_a + \mu_s$ where μ_a is the σ-additive part of μ and μ_s is the purely finitely additive part of μ (Hewitt-Yosida [5]). If μ_s is not orthogonal to H^∞ then there is a subsequence f_{n_k} such that

$$H^\infty \to \ell^\infty$$
$$g \mapsto \left(\int g\, f_{n_k}\, dm\right)_k$$

is onto, i.e. f_{n_k} is an interpolating sequence.

LEMMA ([1] and [2]) : If K is a bounded subset of $L^1/_N$ then are equivalent

i) K is weakly relatively compact

ii) $\forall \varepsilon > 0$; $\exists \delta > 0$ such that $f \in H^\infty$; $||f||_\infty \leq 1$ and $||f||_1 \leq \delta$ imply $\sup_{\phi \in K} |\phi(f)| \leq \varepsilon$

iii) K does not contain an interpolating subsequence.

The preceding lemmas give following corollary ($\tilde{f}$ denotes the class of $f \in L^1$ in the quotient L^1/N).

COROLLARY : If f_n is a bounded sequence of positive elements then f_n is a weakly relatively compact in $L^1(m)$ if and only if $\tilde{f}_n$ is weakly relatively compact in $L^1(m)/N$.

Proof : If μ is adherent to f_n in $(L^\infty)^*$, $\sigma((L^\infty)^*, L^\infty)$ and $\mu = \mu_a + \mu_s$ is the Hewitt-Yosida decomposition then μ_s is positive. It follows that $\mu_s = 0$ if and only if μ_s is orthogonal to H^∞ i.e. if and only if $\tilde{f}_n$ does not contain an interpolating subsequence.

<u>THEOREM</u> : If $K \subset L^1(m)/N$ is weakly compact then there is K' in $L^1(m)$ such that the quotient $L^1(m) \to L^1(m)/N$ maps K' onto K.

<u>Proof</u> : Let $\mu_\phi \in C(X)^*$ such that $||\mu_\phi|| = ||\phi||$. By the first lemma $\mu_\phi \in L^1(m)$.
Let $d\mu_\phi = g\, d|\mu_\phi|$ be the polar decomposition of μ_ϕ. It is well known that $|g_\phi| = 1$, $|\mu_\phi|$ a.e.. Since $\phi \in L^1/N$ there is $h_\phi \in H^\infty$, $||h_\phi||_\infty = 1$ such that $||\phi|| = \phi(h_\phi)$.
So $||\phi|| = \int h_\phi d\mu_\phi = \int h_\phi\, g\, d|\mu_\phi| = ||\mu|| = \int d|\mu_\phi|$ and hence $g = \bar{h}_\phi$, $|\mu_\phi|$ almost everywhere and $d\mu_\phi = \bar{h}_\phi\, d|\mu_\phi|$.
We now claim that $K_1' = \{|\mu_\phi| \mid \phi \in K\}$ is weakly relatively compact in $L^1(m)$. By the corollary we only have to prove that the image of K_1' in L^1/N is weakly relatively compact.
So let $\varepsilon > 0$ and take $\delta > 0$ such that $\sup_{\phi \in K} |\phi(f)| \leq \varepsilon$ as soon as $f \in H^\infty$, $||f||_\infty \leq 1$ and $||f||_1 \leq \delta$. But if f is a function satisfying these inequalities then $f.h_\phi$ also satisfies these inequalities and hence

$$\sup_{\phi \in K} \left|\int f\, d|\mu_\phi|\right| = \sup_{\phi \in K} \left|\int f\, h_\phi \bar{h}_\phi d|\mu_\phi|\right| = \sup_{\phi \in K} \left|\int f h_\phi\, d\mu_\phi\right| \leq \varepsilon .$$

The lemma above implies now that K_1' is relatively weakly compact and hence is equally integrable in $L^1(m)$ ([3] p. 294).
Let now $K_2' = \{\mu_\phi \mid \phi \in K\}$ then K_2' is obtained from K_1' by multiplying the elements of K_1' by functions bounded by 1. It is then obvious that K_2' is also equally integrable and hence weakly relatively compact ([3] p. 294). Define K_3' as the weak closure of K_2' in $L^1(m)$ and let $K' = K_3' \cap q^{-1}(K)$ where q is the quotient map $q : L^1(m) \to L^1(m)/N$.

[1] CHAUMAT : Une généralisation d'un théorème de Dunford-Pettis, Analyse Harmonique d'Orsay. Paris XI (preprint, 1974)

[2] CNOP-DELBAEN : A Dunford-Pettis theorem for L^1/H^∞. **Journal of Functional Analysis 24, 4 (1977) 364-378.**

[3] DUNFORD-SCHWARTZ : Linear Operators, Part 1, Interscience, New York (1958)

[4] GAMELIN : Uniform Algebras. Prentice Hall, Englewood Cliffs (1969)

[5] HEWITT-YOSIDA : Finitely additive measures. Trans. Amer. Math. Soc. 72 (1952) p. 46-66.