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A SURVEY ON THE GENERAL CENTRAL LIMIT PROBLEM IN BANACH SPACES

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Let $\{X_{nj}: j=1,\ldots, n\in N\}$ be an infinitesimal array of real rv's i.e. such that $\max_j P\{|X_{nj}|>\varepsilon\}\to 0$ for every $\varepsilon>0$ (or $\max_j \rho(\mathcal{L}(X_{nj}),\delta_0)\to 0$, where ρ is Prokhorov's distance) and such that for each $n\in N$, $X_{n1},\ldots,X_{nk_n}$ are independent. And let $S_n=\Sigma_j X_{nj}$. The general central limit theorem (CLT) in the line is essentially the answer to the following question: what are the possible limits of $\{\mathcal{L}(S_n)\}$ and under what conditions does $\{\mathcal{L}(S_n)\}$ (perhaps suitably centered) converge to a given limit law? The possible limit laws are exactly the infinitely divisible, i.e. the probability measures which have n-th root with respect to convolution for every $n\in N$. In a sense the most natural infinitely divisible laws are the so called Poisson laws: if ν is a positive finite measure, $\mathrm{Pois}\,\nu=\exp(\nu-|\nu|\delta_0)$; then, $(\mathrm{Pois}\,\nu)^{1/n} = \exp\{(\nu-|\nu|\delta_0)/n\}$. If the total variation distance between $\mathcal{L}(X_{nj})$ and δ_0 is small, then a simple Banach algebra argument shows that $\mathcal{L}(S_n)$ is near in total variation to $\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj})$. What happens if the Prokhorov's distance between $\mathcal{L}(X_{nj})$ and δ_0 is small, or what is the same, if the system is infinitesimal? It turns out that in this case, if either $\{\mathcal{L}(S_n)\}$ or $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj})\}$ are relatively shift compact, the Prokhorov's distance between adequate shifts of the n-th terms of both sequences tends to zero as $n\to\infty$. Classically, the proof of the general CLT consists of: (i) this fact, together with (ii) necessary and sufficient conditions for convergence of Poisson (or more generally, infinitely divisible) measures. It turns out that one can prove the general CLT in the line (and its analogues in Banach spaces) using only elementary results about Poisson laws. However, the problem of the relation between $\{\mathcal{L}(S_n)\}$ and $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj})\}$ is interesting in its own right, and we will consider it as part of the general CLT in Banach. The measures $\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj})$ are called the accompanying Poisson laws for the triangular array $\{X_{nj}\}$.

In this note I will describe several results on: (a) w*-relative compactness and convergence of Poisson measures, (b) relation between relative compactness of row sums in triangular arrays and relative compactness of their accompanying Poisson laws, and (c) necessary and sufficient conditions for convergence of row sums of infinitesimal arrays of Banach space valued random variables.

Mainly, these are results contained in [3], and also in [14] and [7]. The point of departure of this work, done with de Acosta and Araujo, except for results contained in [14], is LeCam [14] and Hoffmann-Jorgensen and Pisier [12].

Notation. All the Banach spaces below will be separable except otherwise stated, and will be usually denoted by B. The measures will be positive and Borel. For each $\tau>0$, $B_\tau=\{x: \|x\|\le\tau\}$, if X_{nj} is a B-valued rv, then $X_{nj\tau}=X_{nj}I_{B_\tau}(X_{nj})$. We will write $\{X_{nj}\}$ for $\{X_{nj}: j=1,\ldots,k_n,\ n\in N\}$, $S_n=\Sigma_j X_{nj}$, $S_{n,\tau}=\Sigma_j X_{nj\tau}$ and $S_n^\tau=\Sigma_j X_{nj}^\tau=\Sigma_j(X_{nj}-X_{nj\tau})$.

1. Poisson probability measures. The accompanying laws Pois $\Sigma_j\mathcal{L}(X_{nj})$ of an infinitesimal system are exponentials of measures with total mass increasing to infinity. Thus, one needs to 'Poissonize' infinite measures.

1.1. Definition. A σ-finite measure μ on B is a Lévy measure if

(i) $\int|h_\tau(f,x)|d\mu(x)<\infty$ for every $f\in B'$ and some $\tau>0$, where
$$h_\tau(f,x)=e^{if(x)}-1-if(x)I_{B_\tau}(x),$$
(ii) the function $\varphi: B\to C$ defined as $\varphi(f)=\exp\{\int h_\tau(f,x)d\mu(x)\}$ is the characteristic function of a tight p.m. on B. This probability measure will be denoted by c_τ-Pois μ, the τ-centered Poisson p.m. with Lévy measure μ.

If μ is symmetric c_τ-Poisμ does not depend on τ and its ch.f. is $\exp\{\int(\cos f-1)d\mu\}$. It will be denoted by Pois μ.

The function h is not continuous, but one could equivalently define Lévy measure using the function $h(f,x)=e^{if(x)}-1-if(x)$ for $\|x\|\le1$, and $e^{if(x)}-1-if(x)/\|x\|$ for $\|x\|>1$.

The only result about Poisson measures needed in the proof of the general central limit theorem in Section 3 below, is the following. It explains Definition 1.1.

1.2. Theorem. For a σ-finite measure μ on B with $\mu\{0\}=0$, the following are equivalent:

(i) μ is a Lévy measure,

(ii) for every $\mu_n\uparrow\mu$ (setwise), μ_n finite, and every $\tau>0$, the sequence $\{c_\tau\text{-Pois}\,\mu_n\}$ converges (in the w*-topology, to c_τ-Pois μ),

(iii) there exists a sequence $\mu_n\uparrow\mu$, μ_n finite, such that $\{c_\tau\text{-Pois}\,\mu_n\}$ is relatively shift compact (for the w*-topology).

Proof. (Sketch; see [3] for details). One shows first that:

(1) If μ is a Lévy measure then $\mu(B_\delta^c)<\infty$ for every $\delta>0$ and the function $f \to \int_{B_1} f^2 d\mu$, restricted to any ball $B'_r=\{f\in B' : \|f\|\le r\}$ is w*- continuous.

(2) A family of p.m.'s $\{\nu_\alpha\}$ on B is relatively compact if and only if it is shift tight and $\{\hat{\nu}_\alpha | B'_r\}$ is w*-equicontinuous at zero for some $r>0$.

The proof of (2) is almost classical; we will give the proof of (1). Note that it is enough to prove (1) for symmetric measures (just take $\mu+\bar{\mu}$ instead of μ, with $\bar{\mu}(A)=\mu(-A)$ for every Borel set A). The second assertion follows from the fact that $(\mathrm{Pois}\,\mu)^\wedge$ is w*-sequentially continuous, so w*-continuous when restricted to B'_r. If $\mu_n\uparrow\mu$, μ_n finite, symmetric, then Ito-Nisio's theorem applied to $(\mathrm{Pois}\,\mu_1)*(\mathrm{Pois}(\mu_2-\mu_1))*\dots*(\mathrm{Pois}(\mu_n-\mu_{n-1}))$ implies $\mathrm{Pois}\,\mu_n \to_{w*} \mathrm{Pois}\,\mu$. Now, following [16], proof of Theorem IV.4.3, if $\{n'\}$ is a subsequence and N a neighborhood of zero such that $\mu_{n'}(N^c)\to 0$, define $\nu_{n'}=(\mu_{n'}(N^c))^{-1}\mu_n|N^c$. Then, $\mathrm{Pois}\,\nu_{n'}$ is a factor of $\mathrm{Pois}\,\mu$, and so is $\mathrm{Pois}\,k\nu_{n'}$ for every k and from some n'_k on. So, $\{\mathrm{Pois}\,\nu_{n'}\}$ is relatively compact ([16], Theorem III.2.2) and if λ is a limit point, then λ^k is also a factor of $\mathrm{Pois}\,\mu$ for every k. Hence, λ^k is relatively compact, which implies that $\lambda=\delta_0$. But then, if $\mathrm{Pois}\,\nu_{n''}\to_{w*}\lambda$ we will have $\mathrm{Pois}\,\nu_{n''}(N^c)\to\lambda(N^c)=0$ and at the same time, $\mathrm{Pois}\,\nu_{n''}(N^c)=e^{-1}\sum_{k=0}^{\infty}\nu_{n''}^k(N^c)/k! \ge e^{-1}\nu_{n''}(N^c)=e^{-1}$, contradiction.

Now we can prove the theorem. (i)$\Rightarrow$(ii): As seen in the proof of (1), $\{\mathrm{Pois}(\mu_n+\bar{\mu}_n)\}$ converges when $\mu_n\uparrow\mu$. So, $\{c_\tau\text{-}\mathrm{Pois}\,\mu_n\}$ is relatively shift compact for each $\tau>0$. Now, if K is a compact set,

$$\int|h_\tau(f,x)|d\mu_n(x)\le\int_{B_\tau}f^2d\mu+(\sup_{x\in K}|f(x)|)\mu(B^c)+2\mu(K^c\cap B^c)$$

and since the w*-topology and the topology of uniform convergence on compact subsets of B coincide on B'_r, it follows from (1) and (2) above that $\{c_\tau\text{-}\mathrm{Pois}\,\mu_n\}$ is relatively compact. Since $(c_\tau\text{-}\mathrm{Pois}\,\mu_n)^\wedge(f)\to(c_\tau\text{-}\mathrm{Pois}\,\mu)^\wedge(f)$ for each $f\in B'$, it turns out that $c_\tau\text{-}\mathrm{Pois}\,\mu_n\to_{w*}c_\tau\text{-}\mathrm{Pois}\,\mu$. (ii)$\Rightarrow$(iii): obvious. (iii)$\Rightarrow$(i): The proof of (1) shows that $\sup_n\mu_n(B_r^c)<\infty$ for every $r>0$ and that the functions $f\to\int_{B_r}f^2d\mu_n$, restricted to B'_s are w*-equicontinuous. This implies that $\int|h_\tau(f,x)|\,d\mu(x)<\infty$ and therefore also that $\{c_\tau\text{-}\mathrm{Pois}\,\mu_n\}$ converges to a tight p.m. with ch.f. $\exp\{\int h_\tau(f,x)\,d\mu(x)\}$. $\square$

Let us remark that on the line the conditions in Theorem 1.2 are all equivalent to:

(iv) $\int \min(1,x^2)\,d\mu(x) < \infty$.

This allows for a modification to the function h which classically is $h(t,x) = e^{itx} - 1 - itx/(1+x^2)$. The situation in Banach spaces is quite different (cf. [5]):

1.3. Theorem. B is of type 2 if and only if

$$\int_B \min(1, \|x\|^2)\,d\mu(x) < \infty \Rightarrow \mu \text{ is a Lévy measure.}$$

B is of cotype 2 if and only if

$$\mu \text{ is a Lévy measure} \Rightarrow \int_B \min(1, \|x\|^2)\,d\mu(x) < \infty .$$

Power, type and cotype 2 in the above theorem can be replaced by p ([5] and [18]).

It is easy to see that in the real line a family of σ-finite measures $\{\mu_\alpha\}$ yields a relatively compact family of Poisson measures $\{c_\tau\text{-Pois}\,\mu_\alpha\}$ if and only if the family of finite measures $\{\min(1,x^2)\,d\mu_\alpha(x)\}$ is relatively compact. This fails to be true even in Hilbert space: if $\{e_n\}$ is a cons of H, then $\mu_n = n^2(\delta_{e_n/n} + \delta_{-e_n/n})$ satisfy the second condition but not the first. The next few theorems describe the situation of this subject in Banach spaces. Roughly, there are necessary conditions for tightness of families of Poisson measures (in terms of the associated Lévy measures) in general, but sufficient only in type p spaces.

1.4. Theorem. Let $\{\mu_\alpha\}$ be a family of Lévy measures on B such that $\{c_\tau\text{-Pois}\,\mu_\alpha\}$ is relatively shift compact. Then:

(i) $\{\mu_\alpha | B_r^c\}$ is a family of relatively compact finite measures for every $r > 0$,

(ii) if $\psi_{\alpha,r}(f) = \int_{B_r} f^2\,d\mu_\alpha$, $f \in B'$, then for every r and $s > 0$, the family of functions $\{\psi_{\alpha,r} | B'_s\}$ is w^*-equicontinuous.

The proof is essentially contained in the proof of Theorem 1.2. See [3] for details. A useful corollary is:

1.5. Corollary. If $\{c_\tau\text{-Pois}\,\mu_\alpha\}$ is relatively shift compact, then it is relatively compact.

This fact was first observed in [4]. Next we give some partial converses to Theorem 1.4 (cf. [3]).

1.6. Theorem. Let B and E be Banach spaces, $u: B \to E$ a continuous linear map of type p, and $\{\mu_\alpha\}$ a family of σ-finite positive measures on B such that:

(i) $\mu_\alpha(B_r^c)<\infty$ for all $r>0$ and $\{\mu_\alpha | B_1^c\}$ is relatively compact,

(ii) for every $f \in B'$, $\sup_\alpha \int_{B_1} f^2 d\mu_\alpha < \infty$,

(iii) there exists a sequence $\{F_n\}$ of finite dimensional subspaces of E such that

$$\lim_n \sup_\alpha \int_{B_1} d^p(x, u^{-1}(F_n))\, d\mu_\alpha(x) = 0.$$

Then, $\mu_\alpha \circ u^{-1}$ is a Lévy measure for every α and $\{c_\tau\text{-Pois}(\mu_\alpha \circ u^{-1})\}$ is relatively compact.

Proof. By (i) and 1.5 we may assume μ_α symmetric and $\mu_\alpha(B_1^c) = 0$. Let $\mu_\alpha^r = \mu_\alpha | B_{1/r}^c$ and for each α and $r \in N$, $\{Z_{\alpha j}^r\}_{j=1}^\infty$ independent B-valued rv's such that $\mathcal{L}(Z_{\alpha j}^r) = \mu_\alpha^r / |\mu_\alpha^r|$ if $\mu_\alpha^r \neq 0$ and δ_0 otherwise, and let F be a finite dimensional subspace of E and $G = u^{-1}(F)$. Then, since the induced map $u: B/G \to E/F$ is of type p with the same type p constant C of u,

$$Ed^p(\Sigma_{j=1}^k u(Z_{\alpha j}^r), F) \leq C \Sigma_{j=1}^k Ed^p(Z_{\alpha j}^r, G) = CkEd^p(Z_{\alpha 1}^r).$$

Hence,

$$\int d^p(x,F)\, d\mathrm{Pois}(\mu_\alpha^r \circ u^{-1})(x) = \exp(-|\mu_\alpha^r|)\, \Sigma_{k=1}^\infty (k!)^{-1} |\mu_\alpha^r|^k Ed^p(\Sigma_{j=1}^k u(Z_{\alpha j}^r), F)$$
$$\leq \exp(-|\mu_\alpha^r|)\, CEd^p(Z_{\alpha 1}^r, G)\, \Sigma_{k=1}^\infty |\mu_\alpha^r|^k/(k-1)! = C \int d^p(x,G)\, d\mu_\alpha^r(x).$$

So, by Chebyshev's inequality, the family $\{\mathrm{Pois}\, \mu_\alpha^r \circ u^{-1}\}_{\alpha, r}$ is flatly concentrated (cf. [1]). Also, if $g \in E'$,

$$\int g^2 d\mathrm{Pois}(\mu_\alpha^r \circ u^{-1}) = \int g^2 d(\mu_\alpha^r \circ u^{-1}) = \int (g \circ u)^2 d\mu_\alpha$$

as one can show with computations similar to the above. Hence, $\{\mathrm{Pois}(\mu_\alpha^r \circ u^{-1}) \circ g^{-1}\}_{\alpha, r}$ is tight. Therefore, by [1], Theorem 2.3, $\{\mathrm{Pois}(\mu_\alpha^r \circ u^{-1})\}_r$ is tight and $\mu_\alpha \circ u^{-1}$ is Lévy by Theorem 1.2; again by [1] 2.3, $\{\mathrm{Pois}\, \mu_\alpha \circ u^{-1}\}$ is tight. $\square$

Remarks. (1) This theorem implies the first part of Theorem 1.3 in one direction; hence, if conditions (i)-(iii) for $u=I$ imply tightness of the Poisson measures, B is of type p. (2) Together with results in the next section this theorem also implies a general CLT in type p spaces (which for instance contains the direct part of the CLT in [12] **-the Gaussian** domain of normal attraction- and in [6]-the stable domains of attraction). (3) Assume that in Theorem

1.6, u is of type 2 from B into E_K, the Banach space generated by a compact convex symmetric subset of E; then conditions (ii) and (iii) there can be replaced by:

(ii)' $\sup_\alpha \int_{B_1} \|x\|^2 d\mu_\alpha(x) < \infty$,

and still have relative compactness of the Poisson measures. The proof is as that of 1.6 but one works with the Minkowski functional of K instead of the distances to subspaces. This type of result has application to the CLT in C(S), Gaussian and non-Gaussian convergence cases, for not necessarily identically distributed rv's. (cf. [3]).

Some complements to the previous results:

1.7. Theorem. Let $\{\mu_n\}$ be a sequence of Lévy measures on B such that c_τ-Pois $\mu_n \to_{w*} \nu$. Then:

(i) there exists a Lévy measure μ such that $\mu_n|B_\tau \to_{w*} \mu|B_\tau$ for every τ such that $\mu(\partial B_\tau)=0$,

(ii) there exists a centered Gaussian measure γ such that

$$\lim_{\delta\downarrow 0}\left\{\begin{matrix}\limsup \\ \liminf\end{matrix}\right\}_n \int_{B_\delta} f^2 d\mu_n = \int f^2 d\gamma$$

for every $f\in B'$,

(iii) $\nu=\gamma * c_\tau$-Pois μ .

The proof of this theorem is similar to that of 3.3 and so it is postponed. A simple corollary to 1.6 and 1.7 is the following result proved in [17] for the symmetric case; it is useful in the study of stable domains of attraction in Banach spaces.

1.8. Corollary. Let B be of type p, and let $\{\mu_n\}$ be a sequence of σ-finite measures on B which integrate $\min(1,\|x\|^p)$ and such that:

(i) there exists μ σ-finite such that $\mu_n|B_\tau \to_{w*} \mu|B_\tau$ whenever $\mu(\partial B_\tau)=0$,

(ii) $\lim_{\delta\downarrow 0} \limsup_n \int_{B_\delta} \|x\|^p d\mu_n(x) = 0$.

Then, μ is Lévy and for every $\tau>0$, c_τ-Pois $\mu_n \to_{w*} c_\tau$-Pois μ.

(It is easy to see that condition (ii) implies conditions (ii) and (iii) in 1.6, so that the Poisson p.m.'s are tight; then 1.7 together with (ii) identify the limit). Note that Theorem 1.3 for p instead of 2 proves that: if (i) and (ii) in 1.8 imply convergence of the Poisson measures, then B is of type p.

2. Tightness of row sums and their accompanying laws. As in Section 1 we start with the results needed in the CLT, and then continue to complete the theory as much as we can.

In the next theorem, $\|\cdot\|$ denotes the total variation norm.

2.1. Theorem. Let $\{X_i\}$ be a finite set of independent B-valued rv's and let $S=\Sigma_i X_i$. Then,

$$\|\mathcal{L}(S) - \operatorname{Pois}\Sigma_i\mathcal{L}(X_i)\| \le 2\,\Sigma_i P^2\{X_i \neq 0\}.$$

Proof. (Partial). We give a very simple proof of the inequality with a larger constant. For the real proof see LeCam [13]. By Fubini's theorem, $\|\mathcal{L}(X_1)*\dots*\mathcal{L}(X_n) - (\operatorname{Pois}\mathcal{L}(X_1))*\dots*(\operatorname{Pois}\mathcal{L}(X_n))\|$
$\le\Sigma_i \|\mathcal{L}(X_i) - \operatorname{Pois}\mathcal{L}(X_i)\|$, but
$\|\mathcal{L}(X_i) - \exp(\mathcal{L}(X_i)-\delta_0)\| \le \|\mathcal{L}(X_i)-\delta_0\|^2\,\Sigma_{k=2}^{\infty} 2^{k-2}/k! \le 2e^2P^2\{X_i\neq 0\}$. ▯

This theorem is basic, and is attributed to Khinchin by LeCam [13]. The next basic theorem is the weakest version of the classical Lindeberg theorem. For probability measures in the line, define $d_3(\mu,\nu) = \sup\{|\int f\,d(\mu-\nu)|:\ f\in C_b^3(\mathbb{R}),\ \Sigma_{i=0}^3\|f^{(i)}\|_\infty\le 1\}$. Then it is clear that d_3 metrizes weak-star convergence in the set of p.m.'s on $\mathbb{R}$. We have:

2.2. Theorem. Let $\{X_i\}$ be a finite set of independent, centered, real valued rv's such that ess sup $\|X_i\|\le C$ for each i, and let $\sigma_i^2=EX_i^2$, $\sigma^2=\Sigma_i\sigma_i^2=ES^2$. Then,

$$d_3(\mathcal{L}(S),N(0,\sigma^2)) \le 6^{-1}(1+(8/\pi)^{\frac{1}{2}})C\sigma^2$$

$$d_3(\mathcal{L}(S),\operatorname{Pois}\Sigma_i\mathcal{L}(X_i)) \le C\sigma^2/2.$$

Proof. Let Y_i be independent with $\mathcal{L}(Y_i)=N(0,\sigma_i^2)$ ($\mathcal{L}(Y_i)=\operatorname{Pois}\mathcal{L}(X_i)$), and $\Sigma_iY_i=T$. The first terms in the inequalities above are $d_3(\mathcal{L}(S),\mathcal{L}(T))$ by the well known composition properties of Normal and Poisson laws; by Fubini, they are bounded by $\Sigma_i d_3(\mathcal{L}(X_i),\mathcal{L}(Y_i))$, and since the first two moments of X_i and Y_i coincide, Taylor's formula gives $|E(f(X_i)-f(Y_i))| \le \|f^{(3)}\|_\infty(E|X_i|^3+E|Y_i|^3)$, and this yields the theorem. ▯

These two results are useful in the general case because there is a way of patching them together: under certain conditions (infinitesimality and shift compactness of the sums), $\mathcal{L}(S_n)\approx_{w*}\mathcal{L}(S_{n,\delta})*\mathcal{L}(S_n^\delta)$, as the next proposition shows.

2.3. Proposition. Let $\{X_{nj}\}$ be a triangular array of row-wise independent rv's. Then, for every $\delta>0$ and $n\in N$, there exist random variables $U_{n\delta}$, $V_{n\delta}$ and $W_{n\delta}$ such that:

(i) $\mathcal{L}(S_n)=\mathcal{L}(U_{n\delta}+V_{n\delta}+W_{n\delta})$,

(ii) $U_{n\delta}$ and $V_{n\delta}$ are independent and $\mathcal{L}(U_{n\delta})=\mathcal{L}(S_{n,\delta})$, $\mathcal{L}(V_{n\delta})=\mathcal{L}(S_n^{\delta})$,

(iii) $E\|W_{n\delta}\|\le \max_j\|EX_{nj\delta}\|\Sigma_j P\{\|X_{nj}\|>\delta\}$.

Proof. Take U'_{nj} and V'_{nj} independent with laws

$P\{U'_{nj}\in A\}=P\{X_{nj}\in A\cap B_\delta\}/P\{X_{nj}\in B_\delta\}$, $P\{V'_{nj}\in A\}=P\{X_{nj}\in A\cap B^c_\delta\}/P\{X_{nj}\in B^c_\delta\}$

if the denominators are different from zero, and if one of them is zero, take the corresponding variable equal to zero. Let ξ_{nj} and η_{nj} be independent real rv's, independent of the previous ones, Bernoulli with parameters $p_{nj}=P\{X_{nj}\in B_\delta\}$. Then it is easy to see that the variables

$U_{n\delta}=\Sigma_j\eta_{nj}U'_{nj}$, $V_{n\delta}=\Sigma_j(1-\xi_{nj})V'_{nj}$ and $W_{n\delta}=\Sigma_j(\xi_{nj}-\eta_{nj})U'_{nj}$

satisfy the required conditions.

This decomposition is due to LeCam [13], [14]. He calls it the découpage de Lévy.

Before studying the problem of accompanying laws in all its generality, we state without proof a theorem about necessary integrability conditions for shift compactness of sums and about centering shift compact sequences of sums. The proof, mainly based on the Lévy and converse Kolmogorov inequalities ([2]) and the tightness condition in [1], can be found in [3].

2.4. Theorem. Let $\{X_{nj}\}$ be a triangular array of row-wise independent B-valued rv's such that $\{\mathcal{L}(S_n)\}$ is relatively shift compact. Then:

(i) if the X_{nj} are centered and uniformly bounded, then

(i1) $\sup_n E\|S_n\|^p<\infty$ for all $p>0$,

(i2) if $F_k\subset B$ are finite dimensional subspaces such that $\overline{\cup F_k}=B$, $\lim_k\sup_n Ed^p(S_n,F_k)=0$ for all $p>0$;

(ii) if $\{\Sigma_j\mathcal{L}(X_{nj})|B^c_\delta\}$ is relatively compact for some $\delta>0$, then so are $\{\mathcal{L}(S_n^{\tau})\}$, $\{\mathcal{L}(S_{n,\tau}-ES_{n,\tau})\}$ and $\{\mathcal{L}(S_n-ES_{n,\tau})\}$ for every $\tau\ge\delta$.

For shifts of Poisson measures we have ([7], proof of 2.4):

2.5. Lemma. Let $\{X_{nj}\}$ be a triangular array of row-wise independent B-valued rv's such that for some $\delta>0$, $\max_j\|EX_{nj\delta}\|\to 0$. Then if $\{\mathrm{Pois}\Sigma_j\mathcal{L}(X_{nj})\}$ is shift tight, $\{\mathrm{Pois}\Sigma_j\mathcal{L}(X_{nj}-EX_{nj\delta})\}$ is relatively compact.

2.6. Theorem. Let X_{nj} be an infinitesimal system of B-valued rv's. If $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj}-EX_{nj\delta})\}$ and $\{\mathcal{L}(S_n-ES_{n,\delta})\}$ are relatively compact for some $\delta>0$, then

$$\lim_n d[\mathcal{L}(S_n-ES_{n,\delta}),\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj}-EX_{nj\delta})]=0 \tag{2.1}$$

for any distance d metrizing w*-convergence of p.m.'s on B.

Proof. It is enough to prove that both sequences have the same limits through the same subsequences. For this, it is enough to prove the same for $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(f(X_{nj}-EX_{nj\delta}))\}$ and $\{\mathcal{L}(f(S_n-ES_{n,\delta}))\}$ for every $f\in B'$. The theorems 2.1,2,3 give that for $0<\delta\le\tau$:

$$\|\mathcal{L}(f(S_n^\delta-ES^\delta_{n,\tau}))-\mathrm{Pois}\,\Sigma_j\mathcal{L}(f(X^\delta_{nj}-EX^\delta_{nj\tau}))\|\le 2\,\Sigma_j P^2\{|f(X_{nj})|>\delta-f(EX^\delta_{nj\tau})\},$$

$$d_3[\mathcal{L}(f(S_{n,\delta}-ES_{n,\delta})),\mathrm{Pois}\,\Sigma_j\mathcal{L}(f(X_{nj\delta}-EX_{nj\delta}))]\le\delta\|f\|\Sigma_j Ef^2(X_{nj\delta}-EX_{nj\delta})$$

$$d_{pr}[f(S_n-ES_{n,\tau}),\ f(U_{n\delta}-ES_{n,\delta})+f(V_{n\delta}-ES^\delta_{n,\tau})]$$
$$\le[\|f\|\max_j\|EX_{nj\delta}\|\Sigma_j P\{\|X_{nj}\|>\delta\}]^{\frac12}$$

where $U_{n\delta}$ and $V_{n\delta}$ are as in 2.3, and d_{pr} is the distance in probability $d_{pr}(X,Y)=\inf\{\varepsilon:P\{|X-Y|>\varepsilon\}\le\varepsilon\}$. Noting that d_3 is smaller that $\|\cdot\|$ and d_{pr}, and that by Theorem 1.4 and by infinitesimality the last terms in the three inequalities above give zero if one takes first $\limsup_n$ and then $\lim_{\delta\downarrow 0}$, we get

$$d_3[\mathcal{L}(f(S_n-ES_{n,\tau})),\mathrm{Pois}\,\Sigma_j\mathcal{L}(f(X_{nj}-EX_{nj\tau}))]\to 0 \quad\text{as}\quad n\to\infty.\ \square$$

The general problem of the accompanying Poisson laws is reduced, by 2.4,5,6, to a question on the relation between shift tightness of row sums and their exponentials. This simplifies the proof of the main theorem (which collects results in [14], [7] and [3]):

2.7. Theorem. Let $\{X_{nj}\}$ be a triangular array of row-wise independent B-valued random variables. Then:

(i) If $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj})\}$ is relatively shift compact, then $\{\mathcal{L}(S_n-ES_{n,\delta})\}$ is relatively compact for every $\delta>0$; if moreover $\max_j\|EX_{nj\delta}\|\to 0$ as $n\to\infty$ for some $\delta>0$, then also $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj}-EX_{nj\delta})\}$ is relatively compact; and if the system is infinitesimal, then the limit (2.1) holds.

(ii) If c_0 is not finitely representable in B, then there exist symmetric infinitesimal systems $\{X_{nj}\}$ in B such that $\{\mathcal{L}(S_n)\}$ is relatively compact but $\{\mathrm{Pois}\,\Sigma_j\mathcal{L}(X_{nj})\}$ is not.

(iii) Let B be a Banach space such that for some $q\ge 0$ (≥ 2) and some sequence of finite dimensional subspaces $F_k\subset B$, $F_k\uparrow$, with $\overline{\cup_k F_k} \subset B$, the spaces B/F_k are of cotype q with constants c_k satisfying $\sup_k c_k < \infty$; then, if for some $\delta > 0$, $\max_j \|EX_{nj\delta}\| \to 0$, $\{\Sigma_j \mathcal{L}(X_{nj}) | B_\delta^c\}$ is relatively compact and $\{\mathcal{L}(S_n)\}$ is relatively shift compact (hence $\{\mathcal{L}(S_n - ES_{n,\delta})\}$ relatively compact), we have that $\{\mathrm{Pois}\, \Sigma_j \mathcal{L}(X_{nj} - EX_{nj\delta})\}$ is relatively compact. If moreover $\{X_{nj}\}$ is infinitesimal, then (2.1) is satisfied.

Proof. We will outline the main steps. To prove (i), in view of 2.4-2.6, it is enough to see that shift tightness of the Poisson laws implies shift tightness of the sums. We take the proof of this from LeCam [14]. If $X_{nj0}=0$, X_{nji}, $j=1,\dots,k_n$, $i\in N$, and N_{nj}, $j=1,\dots,k_n$ are independent, $\mathcal{L}(N_{nj}) = \mathrm{Pois}\,\delta_1$ and $\mathcal{L}(X_{nji}) = \mathcal{L}(X_{nj})$, then

(2.2) $$\mathrm{Pois}\, \Sigma_j \mathcal{L}(X_{nj}) = \mathcal{L}(\Sigma_j \Sigma_{i \le N_{nj}} X_{nji}).$$

Using this representation one easily sees that $\mathrm{Pois}\, \Sigma_j \mathcal{L}(X_{nj} - X'_{nj})$, $n\in N$, where X'_{nj} is independent of and distributed like X_{nj}, are the laws of the differences of two variables with laws the original Poisson, and therefore make a relatively shift compact sequence. So, we may assume the X_{nj} symmetric. Let $a = \log 2$ and N'_{nk} independent and independent of the X_{nji}, with law $\mathrm{Pois}(a\,\delta_1)$, and let $\xi_{nk} = \min(1, N'_{nk})$ (hence, ξ_{nk} are Bernoulli with expectation $p=\frac{1}{2}$). Then, if $T'_n = \Sigma_j \Sigma_{i \le N'_{nk}} X_{nji}$, $\{\mathcal{L}(T'_n)\}$ is tight. Define $S'_n = \Sigma_j \Sigma_{i \le \xi_{nj}} X_{nj}$ and $R'_n = T'_n - S'_n$. By symmetry, for every compact convex symmetric set K we have

$$P\{S'_n + R'_n \in K^c | N'_{nj} = \tau_{nj}, j=1,\dots,k_n\} = P\{S'_n - R'_n \in K^c | N'_{nj} = \tau_{nj},\ j=1,\dots,k_n\}$$
$$\ge \tfrac{1}{2} P\{S'_n \in K^c | N'_{nj} = \tau_{nj},\ j=1,\dots,k_n\}.$$

Therefore, $\{\mathcal{L}(S'_n)\}$ is also tight. But $S_n = \Sigma_j \xi_{nj} X_{nj} + \Sigma_j (1-\xi_{nj}) X_{nj}$ and both sums have the distribution of S'_n, so $\{\mathcal{L}(S_n)\}$ is tight.

(ii) If c_0 is finitely representable in B, it is possible to construct an infinitesimal system of symmetric rv's such that ess sup $\|S_n\| \to 0$ and that for every $r>0$, $(\mathrm{Pois}\, \Sigma_j \mathcal{L}(X_{nj}))(B_r^c) \to 1$. We refer to [7], [14] for such an example.

(iii) Using (2.2), Fubini and that if $\{X_i\}$ are equidistributed and independent of X_0, then $E\|\Sigma_{i=0}^n X_i\|^2 \le E\|X_0 + nX_1\|^2$, it is easy to prove that in any Banach space,

$$\int \|x\|^2 d(\mathrm{Pois}\, \Sigma_j \mathcal{L}(X_j))(x) \le E\|\Sigma_j N_j X_j\|^2$$

if the X_j, N_j are independent and $\mathcal{L}(N_j) = \mathrm{Pois}\,\delta_1$. Now, Corollary 1.3 in [15] shows that if B is of cotype q for some q, and the random variables X_j are independent and symmetric,

$$\int \|x\|^2 d(\mathrm{Pois}\,\Sigma_j \mathcal{L}(X_j))(x) \leq CE\|\Sigma_j X_j\|^2$$

where C does not depend on the X_j, and depends on B only through its cotype q constant. This inequality can be desymmetrized. If B satisfies the hypothesis stated in (iii), it is clear that this inequality together with 2.4(i) will imply tightness of $\{\mathrm{Pois}\,\Sigma_j \mathcal{L}(X_{nj\delta}-EX_{nj\delta})\}$ if $\{\mathcal{L}(S_n)\}$ is relatively shift compact. But it turns out that this is enough to yield tightness of $\{\mathrm{Pois}\,\Sigma_j \mathcal{L}(X_{nj}-EX_{nj\delta})\}$ by virtue of 1.4(i) and 2.1. $\square$

Contained in the previous proof is the following characterisation of spaces where c_0 is not finitely representable ([7]):

2.8. Theorem. c_0 is not finitely representable in B if and only if for every finite set $\{X_i\}$ of symmetric B-valued rv's,

$$\int \|x\|^2 d(\mathrm{Pois}\,\Sigma_j \mathcal{L}(X_i))(x) \leq CE\|\Sigma_j X_j\|^2$$

for some $C<\infty$ independent of the X_i.

And also ([7]):

2.9. Corollary. Let B be a Banach space with a Schauder basis. Then c_0 is not finitely representable in B if and only if

$\{\mathcal{L}(S_n)\}$ relatively compact $\Leftrightarrow$ $\{\mathrm{Pois}\,\Sigma_j \mathcal{L}(X_{nj})\}$ relatively compact

for triangular arrays of row-wise independent symmetric B-valued rv's $\{X_{nj}\}$.

2.9 can be stated for infinitesimal arrays.

2.10. Problem. Is 2.9 true without any additional assumption on B?

The solution of this problem would give possibly a complete picture of the subject of accompanying laws in Banach spaces.

3. The general central limit theorem. The general limit theorem that we will give in this section has the disadvantage that one of the conditions depends on the truncated sums rather than on the individual variables directly, but the advantage that many known limit theorems in Banach follow from it quite directly. All the ingredients for the proof of these theorems have been given above except for the following result of LeCam [14] which is basic in the converse CLT (but is not needed for the direct part). The proof here is as in [3].

3.1. Theorem. If $\{X_{nj}\}$ is a triangular array of row-wise independent B-valued rv's and $\{\mathcal{L}(S_n)\}$ is relatively shift compact, then for every $\varepsilon>0$ there exist a compact set $K_\varepsilon\subset B_\varepsilon$ and $\{x_{nj\varepsilon}\}\subset B$ such that $\{\Sigma_j\mathcal{L}(X_{nj}-x_{nj\varepsilon})|K_\varepsilon^c\}$ is relatively compact.

Proof. Here, as in the 'converse' tightness theorem 2.4(i), the Lévy and the converse Kolmogorov inequalities are the basic tools. Let $\{\tilde{X}_{nj}\}$ be independent symmetrisations of the X_{nj}, $\tilde{S}_n=\Sigma_j\tilde{X}_{nj}$, and let K be a compact symmetric set such that $P\{\tilde{S}_n\in K^c\}<\alpha<\frac{1}{2}$. Then, the Lévy inequality applied to the Minkowski functional of K yields

$$(3.1)\quad \sup_n \Sigma_j P\{\tilde{X}_{nj}\in K_\varepsilon^c\}\leq -\log(1-2\sup_n P\{\tilde{S}_n\in K^c\})< -\log(1-2\alpha)\doteq T<\infty.$$

We will show that $K_\varepsilon=B_\varepsilon\cap K$ satisfy the conditions of the theorem. First we must see that

$$(3.2)\qquad \sup_n \Sigma_j P\{\tilde{X}_{nj}\in K_\varepsilon^c\}<\infty$$

for every $\varepsilon>0$. By (3.1) it is enough to prove this for $Y_{nj}=\tilde{X}_{nj}I_K(\tilde{X}_{nj})$. Observing that the open sets $\{x: |f(x)|>\varepsilon/2\}$, $f\in B_1'$, are an open cover of $B^c_{2\varepsilon/3}\cap K$, we obtain that for some finite subset $F\subset B_1'$,

$$\Sigma_j P\{Y_{nj}\in K_\varepsilon^c\}\leq\Sigma_j\Sigma_{f\in F}P\{|f(Y_{nj})|>\varepsilon/2\}\leq(\varepsilon/2)^2\Sigma_{f\in F}Ef^2(\Sigma_jY_{nj}).$$

Now (3.2) follows from the converse Kolmogorov inequality because $\{\Sigma_j\mathcal{L}(Y_{nj})\}$ is relatively compact (it is easy to see that for every convex symmetric set Q, $P\{\Sigma_jY_{nj}\in Q^c\}\leq 2P\{\tilde{S}_n\in Q^c\}$, as observed in [10]).

If $J_n=\{j\in\{1,\dots,k_n\}: P\{\tilde{X}_{nj}\in K_\varepsilon\}<3/4\}$, then (3.2) implies $\sup_n\mathrm{Card}(J_n)<\infty$; therefore, by [16], Theorem III.2.2, there exists $\{x_{nj\varepsilon}\}\subset B$ such that $\{\Sigma_{j\in J_n}\mathcal{L}(X_{nj}-x_{nj\varepsilon})\}$ is tight. So we need only prove that for some $\{x_{nj\varepsilon}\}$ the sequence $\{\Sigma_{j\in J_n^c}\mathcal{L}(X_{nj}-x_{nj\varepsilon})|K_\varepsilon^c\}$ is relatively compact. By Fubini's theorem there exist points $x_{nj\varepsilon}$ such that $P\{X_{nj}-x_{nj\varepsilon}\in K_\varepsilon^c\}<\frac{1}{2}$ and $\sup_n\Sigma_{j\in J_n^c}P\{X_{nj}-x_{nj\varepsilon}\in K_\varepsilon^c\}<\infty$. If given $r\in N$ we apply (3.1) and Fubini to $\mathcal{L}(S_n)*\overset{r}{\cdots}*\mathcal{L}(S_n)$ we obtain that there exist $\{z_{nj}^r\}\subset B$ such that $\Sigma_jP\{X_{nj}-z_{nj}^r\in K_r^c\}\leq T/r$ for some compact convex set K_r. Hence, if r is big enough, $z_{nj}^r-x_{nj}\in K_r+K_\varepsilon$ and

$$\Sigma_{j\in J_n^c}P\{X_{nj}-x_{nj\varepsilon}\in(2K_r+K_\varepsilon)^c\}\leq\Sigma_{j\in J_n^c}P\{X_{nj}-z_{nj}\in K_r^c\}\leq T/r.\ \square$$

3.2. Corollary. If $\{X_{nj}\}$ is infinitesimal and $\{\mathcal{L}(S_n)\}$ relatively shift compact, then for every $\varepsilon>0$ there exists $K_\varepsilon\subset B_\varepsilon$ compact such that $\{\Sigma_j\mathcal{L}(X_{nj})|K_\varepsilon^c\}$ is a relatively compact sequence.

The proof follows easily from the tightness of $\{\mathcal{L}(X_{nj})\}_{n,j}$ and Theorem 3.1. 3.2 is observed in [14].

Finally we give what may be considered as a general CLT in Banach. It is taken from [3] with only minor modifications in the proof.

3.3. Theorem. Let X_{nj} be infinitesimal. Then, $\{\mathcal{L}(S_n)\}$ is shift convergent if and only if:

(i) there exists a σ-finite measure μ on P with $\mu\{0\}=0$ such that $\Sigma_j\mathcal{L}(X_{nj})|B_\delta^c \to_{w^*} \mu|B_\delta^c$ whenever $\delta>0$ and $\mu(\partial B_\delta)=0$,

(ii) the limit

$$\phi(f)=\lim_{\delta\downarrow 0}\begin{Bmatrix}\lim\sup\\ \lim\inf\end{Bmatrix}_n \Sigma_j Ef^2(X_{nj\delta}-EX_{nj\delta})$$

exists for every $f\in W\subset B'$, W weak-star total in B' (for every $f\in B'$),

(iii) There exists a (for all) sequence $\{F_k\}$ of finite dimensional subspaces of B with $\overline{\cup_k F_k}=B$, $F_k\uparrow$, and $\beta>0$ (for all $\beta>0$) such that

$$\lim_k \sup_n Ed^p(S_{n,\beta}-ES_{n,\beta},F_k)=0$$

for some (for all) $p>0$.

And then,

(1) μ is a Lévy measure and there exists a centered Gaussian p. m. γ such that $\int f^2 d\gamma=\phi(f)$ for every $f\in W$ $(f\in B')$,

(2) $w^*\text{-}\lim_n\mathcal{L}(S_n-ES_{n,\delta})=\gamma * c_\delta \mathrm{Pois}\,\mu$ for every $\delta>0$ such that $\mu(\partial B_\delta)=0$,

(3) for these same values of δ, $w^*\text{-}\lim_n\mathcal{L}(S_n^\delta)=\mathrm{Pois}(\mu|B_\delta^c)$ and $w^*\text{-}\lim_n\mathcal{L}(S_{n,\delta}-ES_{n,\delta})=\gamma * c_\delta \mathrm{Pois}(\mu|B_\delta^c)$.

Proof. a) The direct part. Assume i)-iii) hold. First we will prove that $\{\mathcal{L}(S_n-ES_{n,\delta})\}$ is relatively compact and then will identify the limit. By infinitesimality, $\max_j E\|X_{nj\delta}\|\to 0$ for every $\delta>0$, hence (i) and 2.3 implies that $\{\mathcal{L}(S_n-ES_{n,\delta})\}$ is relatively compact if and only if $\{\mathcal{L}(S_{n,\delta}-ES_{n,\delta})*\mathcal{L}(S_n^\delta)\}$ is tight, and both sequences have the same limits. Theorem 2.1 together with condition (i) give the tightness of $\{\mathcal{L}(S_n^\delta)\}$ for every $\delta>0$, and proves also that if $0<\delta<\tau$ and $\mu(\partial B_\tau)=\mu(\partial B_\delta)=0$ then

$$\mathcal{L}(S_n^\delta)\to_{w^*}\mathrm{Pois}(\mu|B_\delta^c) \text{ and } \mathcal{L}(S_n^\delta-ES_{n,\tau}^\delta)\to_{w^*} c_\tau\text{-}\mathrm{Pois}(\mu|B_\delta^c). \tag{3.3}$$

Hence, part of (3) is proved. On the other hand, $\{\mathcal{L}(S_{n,\beta}-ES_{n,\beta})\}$ is flatly concentrated by (iii), and (ii) easily gives (by infinitesimality and (i)) that $\sup_n Ef^2(S_{n,\beta}-ES_{n,\beta})<\infty$ for every $f\in W$. So, by [1] Theorem 2.3, $\{\mathcal{L}(S_{n,\beta}-ES_{n,\beta})$ is relatively compact. Hence, so is $\{\mathcal{L}(S_n-ES_{n,\delta})\}$ for any $\delta>0$ (condition (i) together with Theorem 2.4 (ii)).

Next we identify the limits. Given $\tau>0$ with $\mu(\partial B_\tau)=0$, let $\tau>\delta_n\downarrow 0$ be such that $\mu(\partial B_{\delta_n})=0$ and

$$(3.4)\qquad \begin{cases} d\,[\mathcal{L}(S_n^{\delta_n}-ES_{n,\tau}^{\delta_n}, c_\tau\text{-Pois}(\mu|B_{\delta_n})] \to 0 \\ \max_j E\|X_{nj\delta_n}\|\,\Sigma_j P\{\|X_{nj}\|>\delta_n\} \to 0, \end{cases}$$

where d metrizes weak-star convergence of probability measures. Such a sequence $\{\delta_n\}$ exists by (i), 2.1, and the infinitesimality assumption. Hence, by Proposition 2.3 and [16] Theorem III.2.2, the relative compactness of $\{\mathcal{L}(S_n-ES_{n,\tau})\}$ implies that the sequences $\{\mathcal{L}(S_n^{\delta_n}-ES_{n,\tau}^{\delta_n})\}$ and $\{\mathcal{L}(S_{n,\delta_n}-ES_{n,\delta_n})\}$ are relatively shift compact. Now Theorem 1.2 proves that

$$(3.5)\qquad \begin{cases} \mu \text{ is a Lévy measure} \\ \mathcal{L}(S_n^{\delta_n}-ES_{n,\tau}^{\delta_n}) \to_{w*} c_\tau\text{-Pois}\,\mu \\ \{\mathcal{L}(S_{n,\delta_n}-ES_{n,\delta_n})\} \text{ is relatively compact.} \end{cases}$$

Suppose now that $\{\mathcal{L}(S_{n',\delta_{n'}}-ES_{n',\delta_{n'}})\}$ converges. By the converse Kolmogorov inequality, for every $f\in B'$ and $p>0$, $\sup_n E|f(S_{n',\delta_{n'}}-ES_{n',\delta_{n'}})|^p<\infty$ and therefore, $\lim_{n'} Ef^2(S_{n',\delta_{n'}}-ES_{n',\delta_{n'}}) \doteq \phi'(f)<\infty$ and, by Theorem 2.2, $\mathcal{L}(f(S_{n',\delta_{n'}}-ES_{n',\delta_{n'}})) \to_{w*} N(0,\phi'(f))$. So, there exists a centered Gaussian measure γ' on B such that $\int f^2 d\gamma' = \phi'(f)$. By the previous argument and (3.5) we have then

$$\mathcal{L}(S_{n'}-ES_{n',\tau}) \to_{w*} \gamma' * c_\tau\text{Pois}\,\mu .$$

Now, for the direct part of the theorem we only need to see that $\Phi(f)=\phi'(f)$ for every $f\in W$ (hence for every $f\in B'$). By previous arguments, $\mathcal{L}(S_{n',\tau}-ES_{n',\tau}) \to_{w*} \gamma' * c_\tau\text{-Pois}(\mu|B_\tau)$, so that (again justifying limits under the integral sign by Kolmogorov inequality)

$$\begin{aligned}\Phi(f) &= \lim_{\tau\downarrow 0,\,\mu(\partial B_\tau)=0} \lim_{n'} Ef^2(S_{n',\tau}-ES_{n',\tau}) \\ &= \lim_{\tau\downarrow 0}(\phi'(f)+\int f^2 d(\mu|B_\tau)) = \phi'(f)\end{aligned}$$

for every $f\in W$.

b) The converse part. If $\{\mathcal{L}(S_n)\}$ is shift convergent, then $\{\Sigma_j\mathcal{L}(X_{nj})|B_\delta^c\}$ is relatively compact for every $\delta>0$ by Corollary 3.2. If $\{\Sigma_j\mathcal{L}(X_{n'j})|B_\delta^c\}$ converges then, by a diagonal procedure, we can find a subsequence $\{m'\}\subset\{n'\}$ and a σ-finite measure μ with $\mu\{0\}=0$ such that $\Sigma_j\mathcal{L}(X_{m'})|B_\tau^c \to_{w*} \mu|B_\tau^c$ for every $\tau>0$ with $\mu(\partial B_\tau)=0$ and for $\tau=\delta$.

Hence (i) is satisfied for the sequence $\{m'\}$. By infinitesimality and 2.3, $\mathcal{L}(S_{n,\delta}-ES_{n,\delta})$ and $\{\mathcal{L}(S_n^\delta)\}$ are relatively shift compact, hence relatively compact by 3.2 and 2.4(ii). In particular 2.4(i) implies that condition (iii) is satisfied. Also, whenever $\mu(\partial B_\tau)=0$, $\{\mathcal{L}(S_{m'}^\tau)\}$ converges by 2.1, and therefore so does $\{\mathcal{L}(S_{m',\tau}-ES_{m',\tau})$ (2.4(ii)). This implies that condition (ii) is satisfied for the sequence $\{m'\}$ and for every $f\in B'$ ($\lim_{m'} Ef^2(S_{m',\tau}-ES_{m',\tau})$ exists for every $\tau>0$ with $\mu(\partial B_\tau)=0$ by the Kolmogorov converse inequality, and, as a simple computation shows, $\lim\sup_{m'} Ef^2(S_{m',\tau}-ES_{m',\tau})$ (lim inf) is an increasing function of τ). Then, the direct limit theorem implies that the limit of $\{\mathcal{L}(S_{m'}-ES_{m',\tau})\}$ is $\gamma * c_\tau\text{-Pois}\,\mu$, $\tau\le\delta$, where γ is determined as before. By [16], p. 110, if $\gamma * c_\tau\text{-Pois}\,\mu = \gamma' * c_\tau\text{-Pois}\,\mu'$ then $\gamma=\gamma'$ and $\mu=\mu'$ (outside the origin), and from this it follows that (i) and (ii) hold in fact for the whole sequence $\{n\}$. ∎

Remarks. (1) For type p spaces, the direct part of the theorem is true with condition (iii) replaced by

(iii)' $\quad \lim_k \sup_n \sum_j Ed^p(X_{nj\delta}-EX_{nj\delta},F_k)=0.$

In this case the theorem simply results from putting together the theorems 1.6, 1.7 and 2.7(i). This result contains the direct part of the Hoffman-Jorgensen and Pisier CLT [12] and of the theorems on domains of attraction in [6] and in [18] (which can be desymmetrized).

(2) Assume B satisfies: there exist $F_k\subset B$ finite dimensional with $\overline{\bigcup_k F_k}=B$, $F_k\uparrow$, B/F_k of cotype p for some $p\ge 0$ (≥ 2) and constant c_k^p such that $\sup_k c_k^p<\infty$. Then the converse part of the theorem is true with condition (iii) replaced by condition (iii)'. Again, in this case the theorem can also be proved putting together 1.7 and 2.7(iii).

(3) If B is of cotype q, another necessary condition for the CLT can be added, namely that $\sup_n \sum_j E\|X_{nj\delta}-EX_{nj\delta}\|^q<\infty$ (Theorem 2.4(i)). This theorem implies the well known fact that $X\in$ CLT in cotype 2 $\Rightarrow$ $E\|X\|^2<\infty$.

(4) Hilbert space can be characterized as the only Banach space where Theorem 3.3 is true with condition (iii) replaced by (iii)'. In H this condition takes the form $\lim_k \sup_n E\sum_j r_k^2(X_{nj\delta}-EX_{nj\delta})=0$, where $r_k^2(x)=\sum_{n=k+1}^\infty \langle x,e_k\rangle^2$, $\{e_k\}$ a cons. (A somewhat similar approach to the CLT in Hilbert space is given in [10]; the theorems in section 1 and 2 were proved in Hilbert space by Varadhan [17]).

(5) A Corollary to the previous theorem, converse part, is the Lévy-Khinchin representation in Banach: if ρ is an infinitely divisible p.m. on B, then there exists a centered Gaussian p.m. γ, a vector $a \in B$, and a Lévy measure μ such that $\rho = \delta_a * \gamma * c_\tau\text{-Pois}\,\mu$. For a direct approach, similar to the above and independent of the one-dimensional case, see [6]. This theorem was proved first by Araujo [4] and Dettweiler [8].

(6) Theorem 1.7 can be proved similarly to the converse part of Theorem 3.3, but the proof is simpler. It is omitted.

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