

SÉMINAIRE D'ANALYSE FONCTIONNELLE ÉCOLE POLYTECHNIQUE

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Séminaire d'analyse fonctionnelle (Polytechnique) (1975-1976), exp. n° 25, p. 1-7

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QUASI-REFLEXIVE BANACH SPACES

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It is customary to use J to indicate any Banach space isometric to the space introduced in [2]. Thus J is isomorphic to a space that is isometric to its second dual [3], and J is quasi-reflexive of order one (i.e., the quotient of J^{**} and the natural image of J in J^{**} has dimension one). If J is given a norm for which J is isometric to J^{**} , then the ℓ_2 -product of J and J^* is quasi-reflexive of order two and isometric to its first dual.

However, J and J^* are not isomorphic [4, theorem 3]. The purpose of this discussion is to outline a proof of the existence of a Banach space that is quasi-reflexive of order one and isomorphic to its dual. It is not known whether there is a Banach space that is quasi-reflexive of order one and isometric to its dual.

We will be particularly interested in the norm for J given by

$$\|x\| = \sup \left\{ \sum_{k=1}^n [x(p_{2k-1}) - x(p_{2k})]^2 \right\}^{1/2}, \quad (1)$$

where the sup is over all positive integers n and all increasing (not necessarily strictly increasing) sequences $\{p_k\}$ of positive integers. The space J is the completion of the space of sequences with finite support, given this norm.

In order to describe the predual I of J , some conventions are needed. A bump is a sequence of real number $x = \{x(i)\}$ for which there is a bounded interval and a number a such that $x(i) = a$ if i is in this interval and $x(i) = 0$ otherwise. The altitude of a bump is a . Two bumps are disjoint if the intersection of their associated intervals is empty; they are strongly disjoint if these intervals are separated by at least one integer.

The space I is the completion of the normed linear space of sequences with finite support for which

$$\|x\| = \inf \left\{ \sum_{k=1}^n [\|x^k\|] : x = \sum_{k=1}^n x^k \right\}, \quad (2)$$

where $[\| \cdot \|]$ is the function defined by $[\|x\|] = (\sum a_i^2)^{1/2}$ if x is the sum of disjoint bumps whose altitudes are $\{a_i\}$. Note that $[\| \cdot \|]$ is a function of x

and a particular representation of x as a sum of disjoint bumps.

The spaces I and J have the properties that the natural basis $\{e_n\}$ for I is shrinking and the sequence of coefficient functionals $\{u_n\}$ is a boundedly complete basis for J (see [4, theorem 1] or [5, pg. 279], and [6, corollary 6.1, pg. 286]). If $\{\varepsilon_n\}$ is the natural basis for J with the norm (1), then $\{\varepsilon_n\}$ is shrinking and $u_n = \sum_{i=1}^n \varepsilon_i$ for each n , so

$$u_1 = \varepsilon_1 \quad \text{and} \quad u_n - u_{n-1} = \varepsilon_n \quad \text{if } n > 1. \quad (3)$$

If $x = \sum_{i=1}^{\infty} x(i) \varepsilon_i$, then $\|x\| = \sup \left\{ \sum_{k=1}^n \left[\sum_{p_{2k-1}}^{p_{2k}-1} x(i) \right]^2 \right\}^{1/2}$, where the sup is over

all positive integers n and all increasing sequences $\{p_k\}$ of positive integers. These properties of I and J suggested the following theorem [4, theorem 2], which will be used later to show that a certain space B , known to be isomorphic to B^* , is quasi-reflexive of order one.

Theorem 1 : Suppose a Banach space X has a basic $\{e_n\}$ with coefficient functionals $\{u_n\}$. If $\sum_{i=1}^{\infty} e_i$ is not norm-convergent and $\{u_1, u_2 - u_1, u_3 - u_2, \dots\}$ is a shrinking basis for X^* , then X is quasi-reflexive of order one.

For computational reasons, it will be useful to introduce the concept of double basis and several related concepts, some of which are extensions of familiar properties of bases.

A double basis for a Banach space X is a subset $\{e_n : -\infty < n < \infty\}$ such that each x in X has a unique representation as $\sum_{i=-\infty}^{\infty} x(i) e_i$ in the sense that

$$\lim_{m, n \rightarrow \infty} \left\| x - \sum_{i=-m}^n x(i) e_i \right\| = 0.$$

A bimonotone double basis is a double basis $\{e_n\}$ such that

$$\left\| \sum_p^s x(i) e_i \right\| \geq \left\| \sum_q^r x(i) e_i \right\| \quad \text{if} \quad p \leq q \leq r \leq s.$$

A shrinking double basis is a double basis $\{e_n\}$ such that each of the basic sequences $\{e_n : n < 0\}$ and $\{e_n : n > 0\}$ is shrinking.

A neighborly (double) basis is a (double) basis $\{e_n\}$ such that $\|\sum_{-\infty}^{\infty} x(i) e_i\|$ is not increased if some $x(k)$ is replaced by either $x(k-1)$ or $x(k+1)$. Clearly, neighborliness implies bimonotonicity. Also, neighborliness implies what we will call repetition-invariance, namely, for all r ,

$$\left\| \sum_{-\infty}^{\infty} x(i) e_i \right\| = \left\| \sum_{-\infty}^r x(i) e_i + \sum_r^{\infty} x(i) e_{i+1} \right\|, \quad (4)$$

and also implies translation-invariance for double basis, namely,

$$\left\| \sum_{-\infty}^{\infty} x(i) e_i \right\| = \left\| \sum_{-\infty}^{\infty} x(i-1) e_i \right\|, \quad (5)$$

since it is possible to transform $\sum_{-\infty}^{\infty} x(i) e_i$ into the vector of the second member of (4) or (5) and back again by successive replacements of components, each being replaced by one of the two neighboring components. [Note that this process cannot be completed for (5) and a basis $\{e_n : n \geq 1\}$ unless $x(1) = 0$.]

An inversion-invariant (double) basis is a (double) basis such that, whenever x has finite support, there is an n such that

$$\left\| \sum x(i) e_i \right\| = \left\| \sum x(n-i) e_i \right\|$$

and, when $\{e_n\}$ is a basis (and not a double basis), $x(i) = 0$ if $i \geq n$. The importance of inversion-invariance at several places in proofs in [4] is due to the fact that if $\{e_n\}$ is a double basis with coefficient functionals $\{u_n\}$ and if $x = \sum_{-\infty}^{\infty} x(i) e_i$ and $f = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$, then

$$(x, f) = \sum_{-\infty}^{\infty} [f(i) - f(i+1)] x(i) = \sum_{-\infty}^{\infty} f(i) [x(i) - x(i-1)] .$$

For a while, let us concentrate our attention on the linear space of functions (double sequences) defined on the set of all integers and having finite support, and on norms we might give this space (and then complete the space). For example, suppose we let

$$I_1(x) = 2^{1/2} \|x\|_I \quad \text{and} \quad J_1(x) = 2^{-1/2} \|x\|_J ,$$

where $\| \cdot \|_I$ and $\| \cdot \|_J$ are the norms described in (2) and (1), respectively,

applied to double sequences with finite support. By the same methods used for the spaces I and J , it follows that the natural basis is shrinking for each of the norms I_1 and J_1 . If x is the sum of disjoint bumps with altitudes $\{a_n\}$, then

$$J_1(x) \leq 2^{-1/2}(\sum 4a_n^2)^{1/2} = 2^{1/2}(\sum a_n^2)^{1/2}.$$

Since the unit ball for the norm I_1 is the closure of the convex span of such x 's for which $2^{1/2}(\sum a_n^2)^{1/2} = 1$, it follows that

$$J_1(x) \leq I_1(x), \quad (6)$$

for all double sequences x with finite support.

If $\|\cdot\|$ is a norm for which the natural basis $\{e_n\}$ is basic and if $\{u_n\}$ is the sequence of coefficient functionals, then a new norm $\|\cdot\|$ is determined by $\|\cdot\|$ if we let

$$\left\| \sum_{-\infty}^{\infty} x(i) e_i \right\| = \left\| \sum_{-\infty}^{\infty} x(i) (u_i - u_{i-1}) \right\|, \quad (7)$$

for all double sequences $\{x(i)\}$ with finite support. This process is extremely critical for what we will be doing. Clearly, it is closely related to the relation (3) between the shrinking bases for I and J . The use in the next theorem of the process described in (7) is the backbone of our entire construction. This theorem is lemma 5 of [4].

Theorem 2 : Let X be the linear space of double sequences with finite support, let I_n be a norm for X , and let J_n be the norm determined from I_n by use of (7). Also, let

$$I_{n+1} = [1/2(I_n^2 + J_n^2)]^{1/2},$$

and J_{n+1} be determined from I_{n+1} by use of (7). If there is a number $\bar{M}$ for which I_n and J_n have properties (A) through (C) of the following discussion, and if $J_n \leq I_n$, then I_{n+1} and J_{n+1} have properties (A) through (C) for the same number $\bar{M}$, and

$$J_n \leq J_{n+1} \leq I_{n+1} \leq I_n.$$

We know that $J_1 \leq I_1$. By the same methods used for the spaces I and J , it can be shown that J_1 is determined from I_1 by use of (7). Thus if I_1 and J_1 have properties (A) through (C), we will have two sequences of norms for which

$$\lim_{n \rightarrow \infty} I_n(x) = \lim_{n \rightarrow \infty} J_n(x) = \|x\|, \quad (9)$$

for all double sequences x with finite support.

(A) The natural basis $\{e_n\}$ is neighborly. It is easy to check that $\{e_n\}$ is neighborly for each of the norms I_1 and J_1 . Also, it is clear from (8) that $\{e_n\}$ is neighborly for I_{n+1} if it is for both I_n and J_n . That $\{e_n\}$ is neighborly for J_{n+1} if it is neighborly for I_{n+1} is not so clear, but the proof is short (see [4, lemma 4]). The property of neighborliness was introduced both because it assures that $\{e_n\}$ is basic for the norm defined by (9), and because it implies both repetition-invariance and translation-invariance. These properties are useful in several proofs.

(B) The natural basis $\{e_n\}$ is inversion-invariant. It is trivial that $\{e_n\}$ is inversion-invariant for the norms I_1 and J_1 . To prove that $\{e_n\}$ is inversion-invariant for I_{n+1} if it is for both I_n and J_n , one needs to use the translation-invariance of $\{e_n\}$ for I_n and J_n . Inversion-invariance of $\{e_n\}$ for J_{n+1} then follows easily from inversion-invariance for I_{n+1} .

(C) The natural basis $\{e_n\}$ is "strongly somewhat Euclidean", in the sense that there is a number M such that

$$\|z\| \leq M \left[\sum_1^S \|z^k\|^2 \right]^{1/2} \quad (10)$$

if $z = \sum_1^S z^k$ and the supports of the z^k 's are in intervals each two of which are separated by at least one integer n for which $z(n) = 0$, and M also satisfies

$$\frac{1}{M} \left[\sum_1^S \|x^k\|^2 \right]^{1/2} \leq \|z\| \quad (11)$$

if $z = \sum_{-\infty}^{\infty} (z_i) e_i$, $\{p_k\}$ is a strictly increasing sequence of integers,

and $\{x^k\}$ is a sequence such that $x^k(i) = 0$ if $i \leq p_{2k-1}$ or $i > p_{2k}$, and

$x^k(i) = z(i) - z(p_{2k-1})$ if $p_{2k-1} \leq i \leq p_{2k}$. The inequality (11) is formally stronger than if the x^k 's were replaced by the z^k 's used in (10). If I_n and J_n have property (C) for the same number $\bar{M}$, then it follows easily from (8) that I_{n+1} has property (C) for this $\bar{M}$. Also, (10) for I_{n+1} implies (11) for J_{n+1} , and (11) for I_{n+1} implies (10) for J_{n+1} (for this, repetition-invariance is needed : see lemma 3 of [4]). Property (C) was introduced because a property was needed that would be inherited by the norm $\| \cdot \|$ defined by (9) and which also would imply the natural basis $\{e_n\}$ for $\| \cdot \|$ is shrinking. It is easy to show that J_1 has property (C) for $M = 2^{1/2}$. Intuitively, it seems natural that I_1 also should have property (C). In fact, I_1 does have property (C) and M can be $2(1+2^{1/2})$, but this is not as easy to establish (see the proof of lemma 6 of [4]).

It follows from theorem 2 that, for the norm $\| \cdot \|$ defined by (9), $\{e_n\}$ is a basis that is inversion-invariant, neighborly, and satisfies (10) and (11) with $M = 2(1+2^{1/2})$. Because of (10), $\{e_n\}$ is shrinking. Also, if $\{u_n\}$ are the coefficient functionals, then, for all double sequences $\{x(i)\}$ with finite support,

$$\|\sum x(i) e_i\| = \|\sum x(i)(u_i - u_{i-1})\| \quad . \quad (12)$$

Now let B be the space spanned by the basic sequence $\{e_n : n \geq 1\}$. This basis for B is shrinking, and (12) implies that $\{u_1, u_2 - u_1, u_3 - u_2, \dots\}$ is a shrinking basis for B^* . Thus it follows from theorem 1 that B is quasi-reflexive of order one. Because of (12), the map T defined by

$$T\left[\sum_1^\infty x(i) e_i\right] = x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1})$$

is an isometry of $\{x : x \in B \text{ and } x(1) = 0\}$ onto the closure of the linear span of $\{u_i - u_{i-1} : i \geq 2\}$. As a map of B onto B^* , $\|T\| \leq 2$ and $\|T^{-1}\| \leq 3$ (see the proof of theorem 4 of [4]). Although $\| \cdot \|$ is not known explicitly, several other interesting properties of B are known (see [4, theorem 4]) :

(i) The basic $\{e_n\}$ is inversion-invariant, translation-invariant, and neighborly, with $1 \leq \|e_n\| \leq (3/2)^{1/2}$.

(ii) $[\sum a_n^2]^{1/2} \leq \|x\| \leq 2^{1/2} [\sum a_n^2]^{1/2}$ if x is the sum of strongly disjoint bumps with altitudes $\{a_n\}$.

(iii) If $z = \sum_1^s z^k$ and the supports of the z^k 's are in intervals each two

of which are separated by at last one integer n for which $z(n) = 0$, then

$$[2(1+2^{1/2})]^{-1} [\sum \|z^k\|^2]^{1/2} \leq \|z\| \leq 2(1+2^{1/2}) [\sum \|z^k\|^2]^{1/2} .$$

Finally, $\{e_n\}$ being neighborly implies the sequence of coefficient functionals is equal-signs-additive, which with [1, theorem 4] implies the existence of a Banach space X that is quasi-reflexive of order one, isomorphic to X^* , and isometric to X^{**} .

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