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THE DIMENSION OF ALMOST SPHERICAL SECTIONS OF CONVEX BODIES

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This is a report on some joint work with T. Figiel and V. Milman. It is a preliminary version of a part of a joint paper that is now under preparation.

The plan of this report is the following. First we present a general result on the dimension of spherical sections. This is basically a revised version of the proof given by V. Milman to Dvoretzky's theorem [8]. Then we show how this general result combined on the one hand with duality and on the other hand with the notion of cotype leads to quite strong and rather surprising results. We illustrate these results in some concrete examples. All these results and examples are of the following nature : if $\dim X = n$ then for some integer $k(X)$ it is true that "most" subspaces $Y \subset X$ with $\dim Y = k(X)$ satisfy $d(Y, \ell_2^{k(X)}) \leq 2$. We conclude by a result in the converse direction, namely we investigate under what condition on k the assumption that $d(Y, \ell_2^k) \leq 2$ for <u>all</u> subspaces Y of X of dimension k implies that X itself is close to a Euclidean space. This result has several applications. In particular by combining it with the results of the first part of this paper it enables the solution of the local version of the complemented subspaces problem.

The starting point of our approach is the isoperimetric inequality for subsets on the surface of balls in R^n.
Let $S^{n-1} = \{t = (t_1, \ldots t_n) \ , \ \sum_{i=1}^n t_i^2 = 1\}$, let μ_{n-1} be the unique normalized rotation invariant measure on S^{n-1}, and let d be the geodesic distance of S^{n-1}. For a subset $A \subset S^{n-1}$ we let $A_\varepsilon = \{s \ ; \ d(s,A) \leq \varepsilon\}$. A <u>cap</u> is a subset of S^{n-1} of the form $B(s_0, r) = \{t \ ; \ d(t, s_0) \leq r\}$. The isoperimetric inequality for subsets of S^{n-1} is the following :

<u>Theorem 1</u> : Let A be a closed subset of S^{n-1} and let B be a cap in S^{n-1} so that $\mu_{n-1}(A) = \mu_{n-1}(B)$. Then for every $\varepsilon > 0$, $\mu_{n-1}(A_\varepsilon) \geq \mu_{n-1}(B_\varepsilon)$.

Theorem 1 is due to E. Schmidt [9]. We shall present a quite simple complete proof of it in the definitive version of this paper which is now being written. We apply Theorem 1 in the following form (cf. Lévy [6]).

<u>Proposition 2</u> : Let $f : S^{n-1} \to R$ be a continuous function and let M_f be its median. Let $A^f = \{t, f(t) = M_f\}$. Then

$$\mu_{n-1}((A^f)_\varepsilon) \geq 1 - 2e^{-n\varepsilon^2/4} \quad . \tag{1}$$

<u>Proof</u> : Let us recall that M_f is the unique number for which

$$\mu_{n-1}(A_+^f) = \mu_{n-1}\{t \,;\, f(t) \geq M_f\} \geq 1/2 \quad ,$$

$$\mu_{n-1}(A_-^f) = \mu_{n-1}\{t \,;\, f(t) \leq M_f\} \geq 1/2 \quad .$$

Observe that for every $\varepsilon > 0$, $(A^f)_\varepsilon = (A_+^f)_\varepsilon \cap (A_-^f)_\varepsilon$ and that by Theorem 1 both $\mu_{n-1}((A_+^f)_\varepsilon)$ and $\mu_{n-1}((A_-^f)_\varepsilon)$ are larger or equal to the measure of a cap of radius $\pi/2 + \varepsilon$ i.e. to $\gamma_n \cdot \int_{-\pi/2}^{\varepsilon} \cos^{n-2} t \, dt$ where $\gamma_n = (\int_{-\pi/2}^{\pi/2} \cos^{n-2} t \, dt)^{-1} \approx (n/2\pi)^{1/2}$. Hence

$$\mu_{n-1}((A^f)_\varepsilon) \geq 1 - 2\gamma_n \int_{\varepsilon}^{\pi/2} \cos^{n-2} t \, dt \geq 1 - 2\gamma_n \cos^{n/2} \varepsilon \int_0^{\pi/2} \cos^{n/2-2} t \, dt$$
$$\geq 1 - 2\gamma_n \, e^{-n\varepsilon^2/4} \cdot \frac{1}{2} (\gamma_{n/2})^{-1} \geq 1 - 2e^{-n\varepsilon^2/4} \quad .$$

Let now X be a Banach space of dimension n with norm $\|.\|$ and let $|||.|||$ be an inner product norm on X so that

$$a|||x||| \leq \|x\| \leq b|||x||| \qquad x \in X \quad . \tag{2}$$

By applying Proposition 2 to the function $r(x) = \|x\|$ on $\{x \,;\, |||x||| = 1\}$ we get the following

<u>Proposition 3</u> : Let M_r be the median of $\|x\|$ on $\{x \,;\, |||x||| = 1\}$, let $\varepsilon > 0$ and let $\{y_i\}_{i=1}^m$ be any m points of norm 1 in ℓ_2^n, where

$$m < \frac{1}{2} e^{n\varepsilon^2/4} \quad . \tag{3}$$

Then there is an isometry U from ℓ_2^n onto $(X, |||.|||)$ so that

$$M_r - b\varepsilon \leq \|Uy_i\| \leq M_r + b\varepsilon \quad , \qquad 1 \leq i \leq m \quad . \tag{4}$$

Proof : Let $A^r = \{x ; |||x||| = 1 , \|x\| = M_r\}$. Since the geodesic distance d(x,y) on the surface of the unit ball dominates $|||x-y|||$ we have for every $y \in (A^r)_\varepsilon$ an $x \in A^r$ such that $|||x-y||| \le \varepsilon$ and thus $\|x-y\| \le b\varepsilon$ i.e. $M_r - b\varepsilon \le \|y\| \le M_r + b\varepsilon$. Let now U_o be any isometry from ℓ_2^n onto $(X, |||.|||)$ and let σ be the normalized Haar measure on the space of all orthogonal transformations V on $(X, |||.|||)$. By (1) and (3) we have that for every $y \in \ell_2^n$ with $\|y\| = 1$

$$\sigma\{V ; M_r - b\varepsilon \le \|V U_o y\| \le M_r + b\varepsilon\} > 1 - \frac{1}{m} .$$

Hence, there is at least one orthogonal transformation V_o of $(X, |||.|||)$ so that $U = V_o U_o$ has the desired property.

Remark : If we take as m a number smaller than $e^{n\varepsilon^2/4}/2$ say $\delta e^{n\varepsilon^2/4}$ with a small δ, then the proof of Proposition 3 shows that "most" isometries U from ℓ_2^n to $(X, |||\ |||)$ have the desired property ("most" means a set of large measure with respect to σ).

We apply Proposition 3 by choosing the points $\{y_i\}_{i=1}^m$ to be a "δ net" in a subspace of ℓ_2^n of a suitable dimension. (A δ net S_o in a metric space (S,ρ) is a set so that for every $x \in S$ there is a $y \in S_o$ with $\rho(x,y) \le \delta$.) We need first two elementary lemmas.

Lemma 4 : Let X be a k dimensional Banach space and let $\delta > 0$. Then $\{x \in x ; \|x\| = 1\}$ has a δ net of cardinality $(1 + 2/\delta)^k$.

Proof : Let $\{x_i\}_{i=1}^m$ be a maximal subset of $S = \{x ; \|x\| = 1\}$ consisting of points whose mutual distances are $> \delta$. The maximality of $\{x_i\}_{i=1}^m$ implies that this set is a δ net of S. The sets $\{x_i + \frac{\delta}{2} S\}_{i=1}^m$ are disjoint and contained in $(1 + \delta/2)S$. By comparing volumes we get that $m(\delta/2)^k \le (1 + \delta/2)^k$ and this proves the lemma.

Lemma 5 : Let $(X, \|\ \|)$ be a Banach space and let $|||.|||$ be an equivalent inner product norm on X. Let $0 < \delta$, $\rho < 1$ and let S_o be a δ net of $\{x ; |||x||| = 1\}$ so that $|\,\|x\| - 1| < \rho$ for every $x \in S_o$. Then for every $x \in X$ we have

$$F(\rho,\delta)^{-1} |||x||| \le \|x\| \le F(\rho,\delta) |||x||| \tag{5}$$

where $F(\rho,\delta)$ is a function depending only on ρ and δ (but not on X) with $\lim_{\rho,\delta\to 0} F(\rho,\delta) = 1$.

We omit the quite routine proof.

Theorem 6 : For every $\tau > 0$ there is a constant $\eta_\tau > 0$ so that the following holds. Let $(X, \| \|)$ be an n-dimensional Banach space and let $||| \; |||$ be an inner product norm on X so that (2) holds. Let M_r be the median of $\|x\|$ on $\{x \,;\, |||x||| = 1\}$. Then for $k = [\eta_\tau n M_r^2 / b^2]$ there is a subspace $Y \subset X$ so that $d(Y, \ell_2^k) \le 1+\tau$.

Proof : Given τ choose $\delta > 0$ and $\rho > 0$ so that $F^2(\rho,\delta) < 1+\tau$ where $F(\rho,\delta)$ is the function appearing in Lemma 5. We claim that $\eta_\tau = \rho^2/8 \log 3/\delta$ has the desired property. Indeed, by Proposition 3 with $\varepsilon = \rho M_r/b$ we can, for every choice of $m = \frac{1}{2} e^{n\rho^2 M_r^2/b^2}$ points $\{y_i\}_{i=1}^m$ in ℓ_2^n, find an isometry U from ℓ_2^n onto $(X, ||| \; |||)$ so that $M_r(1-\rho) \le |||Uy_i||| \le M_r(1+\rho)$ for every i. By Lemma 4 and the fact that by our choice of η_τ and k $(3/\delta)^k = e^{k\log(3/\delta)} \le m$ we may choose the points $\{y_i\}_{i=1}^m$ so that they are a δ net in a k dimensional subspace Y_o of ℓ_2^n. By Lemma 5, $M_r |||x||| \cdot (1+\tau)^{-1/2} \le \|x\| \le M_r |||x|||(1+\tau)^{1/2}$ for every $x \in UY_o$ and in particular $d(UY_o, \ell_2^k) \le 1+\tau$.

Remark : The probabilistic nature of the proof shows that if we replace η_t by a smaller constant we can ensure that the set of subspaces Y of X of dimension k which have the desired property form a subset of measure close to 1 (with respect to the natural normalized measure on the Grassman manifold of k-dimensional subspaces of X. This measure depends on the choice of $|||\cdot|||$ and is thus not intrinsically well defined on $(X, \| \|)$.

By using duality it is possible to derive from Theorem 6 a theorem which does not involve the term M_r.

Theorem 7 : For every $\tau > 0$ there is a $\delta_\tau > 0$ so that the following holds. For every Banach space X of dimension n

$$k_\tau(X) k_\tau(X^*) \ge \delta_\tau n^2 \|P\|^2 / d^2(X, \ell_2^n) \quad . \tag{6}$$

The terms $k_\tau(X)$, $k_\tau(X^*)$ and P appearing in (6) have the following meaning : there is a subspace $Y \subset X$ and a subspace $Z \subset X^*$ so that $k_1 = \dim Y = k_\tau(X)$, $k_2 = \dim Z = k_\tau(X^*)$, $d(Y, \ell_2^{k_1}) \le 1+\tau$ and in case $k_1 \le k_2$, P is a projection from X onto Y while if $k_2 < k_1$ P is a projection from X^* onto Z.

<u>Proof</u> : By the definition of $d = d(X,\ell_2^n)$ there is an inner product norm $|||.|||$ so that

(7) $$|||x||| \le \|x\| \le d|||x||| \qquad x \in X \quad .$$

By theorem 6 there is a subspace Y of dimension $k_1 \ge \eta_\tau n M_r^2/d^2$ with $d(Y,\ell_2^{k_1}) \le 1+\tau$. In X^* we have clearly $d^{-1}|||x^*||| \le \|x^*\| \le |||x^*|||$. We identify X^* with X (in the canonical manner via the inner product induced by $|||.|||$), and denote by $\|x\|_*$ the norm of x as an element in $(X,\|\ \|)^*$. Thus $d^{-1}|||x||| \le \|x\|_* \le |||x|||$ for every $x \in X$. Let M_{r^*} be the median of $\|x\|_*$ on $\{x ; |||x||| = 1\}$. By using again theorem 6 we deduce that there is a subspace Z of X^* of dimension $k_2 \ge \eta_\tau n M_{r^*}^2$ with $d(Z,\ell_2^{k_2}) \le 1+\tau$. We have $k_1 k_2 \ge \eta_\tau^2 n^2 M_r^2 M_{r^*}^2/d^2$. Thus in order to conclude the proof it is enough to find a suitable projection P so that $\|P\| \le 4 M_r M_{r^*}$.

To this end we recall first the remark that if η_τ is chosen small enough we can ensure that not only $d(Y,\ell_2^{k_1}) \le 1+\tau$ but for a set of isometries U of $(X,|||\ |||)$ of σ measure $> 1/2$, $d(UY,\ell_2^{k_1}) \le 1+\tau$. Thus if e.g $k_1 \le k_2$ there is no loss of generality to assume that Y and Z are chosen so that $Y \subset Z$ (if X^* is canonically identified with X). Thus on Y we have

$$\|y\| \le (1+\tau)M_r|||y||| \quad , \quad \|y\|_* \le (1+\tau)M_{r^*}|||y||| \qquad y \in Y \quad .$$

Let P be the orthogonal (with respect to $|||.|||$) projection from X onto Y. Then for every $x \in X$

$$|||Px|||^2 = (Px,Px) = (Px,x) \le \|Px\|_* \|x\| \le M_{r^*}(1+\tau)|||Px||| \, \|x\| \quad ,$$

and thus

$$\|Px\| \le (1+\tau)M_r \, |||Px||| \le (1+\tau)^2 M_r M_{r^*} \|x\|$$

and this concludes the proof.

Since $d(X,\ell_2^n) \le \sqrt{n}$ for every X (cf. [3]) and since $\|P\| \ge 1$ we get in particular from (6)

$$(8) \qquad k_\tau(X)k_\tau(X^*) \geq \delta_\tau n$$

$$(9) \qquad \max(k_\tau(X),k_\tau(X^*)) \geq (\delta_\tau n)^{1/2} \ .$$

For the application of Theorem 6 it is worthwhile to remark that if $|||.|||$ is such that $b/a \leq \sqrt{n}$ in (2) (this is always possible by [3]) then the median M_r can be replaced by the easier to compute average $\int_{|||x|||=1} \|x\| d\mu_{n-1}$. More precisely we have the following

<u>Lemma 8</u> : There is an absolute constant C so that whenever (2) holds with $b/a \leq \sqrt{n}$ then

$$(10) \qquad C^{-1} \leq \int_{|||x|||=1} \|x\| d\mu_{n-1} / M_r \leq C \quad .$$

<u>Proof</u> : We may assume $a = 1$ and $b \leq \sqrt{n}$. We prove the stronger assertion that with this normalization $|M_r - \int_{|||x|||=1} \|x\| d\mu_{n-1}| < K$ where K is an absolute constant. By the proof of Proposition 3 we have for every integer j

$$\mu_{n-1}\{x \,;\; |||x||| = 1 \;;\; j \leq |\,\|x\| - M_r|\} \leq 2e^{-j^2/4}$$

and thus $|M_r - \int_{|||x|||=1} \|x\| d\mu_{n-1}| \leq \sum_{j=0}^{\infty} 2(j+1)e^{-j^2/4}$.

We shall use this observation in order to estimate M_r in terms of the cotype of a space and thus get another useful version of Theorem 6. A Banach space X is said to be of cotype q, $2 \leq q < \infty$, if there is a constant γ_q so that

$$(11) \qquad \int_0^1 \|\sum_{i=1}^m r_i(t)x_i\| dt \geq \gamma_q \left(\sum_{i=1}^m \|x_i\|^q\right)^{1/q}$$

for every choice of $\{x_i\}_{i=1}^m$ in X, where $\{r_i(t)\}_{i=1}^\infty$ are the Rademacher functions on [0,1]. The largest possible number γ_q for which (11) is valid is called the q cotype constant of X.

<u>Theorem 9</u> : There is an absolute constant C so that the following holds. If X is a Banach space of dimension n with q cotype constant γ_q then X has a subspace Y with $\dim Y = k$, $d(Y,\ell_2^k) \leq 2$ and $k \geq C\gamma_q^2 n^{2/q}$.

Proof : Let $(X,\|\ \|)$ be a Banach space of dimension n. Let $|||\ |||$ be the inner product norm on X whose unit ball is the ellipsoid of maximal volume in the unit ball of $(X,\|\ \|)$. Then clearly $\|x\| \le |||x|||$ and by [3] $|||x||| \le \sqrt{n}\,\|x\|$ for every $x \in X$ (thus we can apply Lemma 8). In order to evaluate the mean of $\|\ \|$ we apply the Dvoretzky-Rogers lemma [1]. This lemma shows that there is an orthonormal basis $\{e_i\}$ of $(X,|||\ |||)$ and vectors $\{U_i\}_{i=1}^n \in X$ so that $\|U_i\| = |||U_i||| = 1$ for every i and $U_i = \sum_{j=1}^{i} a_{i,j} e_j$ with $a_{i,i}^2 = 1 - \sum_{j=1}^{i-1} a_{i,j}^2 \ge \frac{n-i+1}{n}$. For $1 \le i \le n/2$ we get therefore

$$(12) \qquad \|e_i\| \ge \|a_{i,i}e_i\| \ge \|U_i\| - |||\sum_{j=1}^{i-1} a_{i,j}e_j||| \ge 1 - \frac{1}{\sqrt{2}} > \frac{1}{4} \ .$$

Hence

$$\int_{|||x|||} \|x\| d\mu_{n-1} = \int_{\Sigma t_i^2=1} \|\Sigma t_i e_i\| d\mu_{n-1} = \int_0^1 \int_{\Sigma t_i^2=1} \|\Sigma r_i(s)\, t_i e_i\| d\mu_{n-1}(t)\, ds$$

$$(13) \quad \ge \int_{\Sigma t_i^2=1} \gamma_q \Big(\sum_{i=1}^{n} (t_i^q \|e_i\|^q)\Big)^{1/q} d\mu_{n-1} \ge \frac{1}{4}\gamma_q\, n^{1/q-1/2} \int_{\Sigma t_i^2=1} \Big(\sum_{i=1}^{n/2} t_i^2\Big)^{1/2} d\mu_{n-1}$$

$$\ge \lambda\, \gamma_q\, n^{1/q-1/2}$$

where λ is some absolute constant. By using this estimate for M_r in Theorem 6 the desired result follows.

Remark : If we insist on having $d(Y,\ell_2^k) \le 1+\tau$ (instead of 2) we get a similar result with C replaced by C_τ. Before we pass to examples and applications of the preceeding results we present one result which gives an upper estimate for the dimension of almost Euclidean subspaces.

Proposition 10 : For every τ there is a constant C_τ so that if $Y \subset \ell_\infty^n$ is such that $\dim Y = k$, $d(Y,\ell_2^k) \le 1+\tau$ then $k \le C_\tau \log n$.

Proof : It is an obvious consequence of Theorem 6 that it is enough to prove Proposition 10 for some $\tau > 0$, say $\tau = 1/7$. Let $T : \ell_2^k \to \ell_\infty^n$ be such that $7\|x\|/8 \le \|Tx\| \le \|x\|$ for every $x \in \ell_2^k$. Let $x_i = T^* e_i \in \ell_2^k = (\ell_2^k)^*$, $1 \le i \le n$ where $\{e_i\}_{i=1}^n$ is the unit vector basis of ℓ_1^n. Then $\|x_i\| \le 1$ for every i and $\max_i |(y,x_i)| \ge 7/8$ for every $y \in \ell_2^k$ with $\|y\| = 1$. Consequently for every such y there is an i so that $\|y - x_i\| \le 1/2$. The union of the balls in ℓ_2^k with center x_i and radius 3/2 contains the ball of radius 2 with the origin as center. By comparing volumes we get that $2^k \le n(3/2)^k$ and this proves the Proposition.

From Theorem 7 and Proposition 10 we get easily the following result on convex polytopes.

Theorem 11 : There is an absolute constant C so that the following holds. If Q is a symmetric convex polytope in R^n having the origin as interior point and having 2s extreme points and 2t (n-1)-dimensional faces then

$$\log t \ \log s \ge C n \quad . \tag{14}$$

Proof : Let X be the Banach space (namely R^n) which has Q as its unit ball. Then X is isometric to a subspace of ℓ_∞^t and X^* is isometric to a subspace of ℓ_∞^s. Thus with the notation of Theorem 7, $k_2(X) \le \lambda \log t$ and $k_2(X^*) \le \lambda \log s$ for some absolute constant λ. The result follows now by using (8).

The inequalities (8), (9) and (14) are in general the best possible. We illustrate this by one example. Define inductively the Banach spaces $\{X\}_{n=1}^\infty$ as follows. $X_1 = R$ (the one dimensional space), $X_2 = (X_1 \oplus X_1)_1$, $X_3 = (X_2 \oplus X_2)_\infty$, $X_4 = (X_3 \oplus X_3 \oplus X_3)_1$, $X_5 = (X_4 \oplus X_4 \oplus X_4)_\infty, \dots$. Then $\dim X_{2n+1} = (n!)^2$. Let s_n (resp. t_n) denote the number of extreme points in the unit ball of X_n (resp. X_n^*). Then

$$s_{2n+1} = (..(((2.2)^2.3)^3.4)^4.....n)^n$$

$$t_{2n+1} = (..((2^2.2)^3.3)^4.4....)^n.n \quad .$$

We have $\log s_{n+1} \approx C_1 n!$ and $\log t_{2n+1} \approx C_2 n!$ for some constants C_1 and C_2. Note that since (8) is reduced to equality in this case (up to an absolute constant) and since (8) was obtained from (6) by replacing $\|P\|$ by its lower estimate 1 it follows that X_{2n+1} contains a subspace Y_{2n+1} so that $k = \dim Y_{2n+1} \ge C^{-1} n!$, $d(Y_{2n+1}, \ell_2^k) \le 2$ and there is a projection from X_{2n+1} onto Y_{2n+1} of norm $\le C$, where C is a constant independent of n.

We shall now give precise (up to a constant) estimates for the dimension of almost Euclidean subspaces in some concrete space.

Proposition 12 : The space ℓ_p^n has a subspace Y with dim $Y = k$, $d(Y,\ell_2^k) \le 2$ where

$$k = \begin{cases} C \log n & \text{if } p = \infty \\ C.n^{2/p} & \text{if } 2 \le p < \infty \\ C.n & \text{if } 1 \le p \le 2 \end{cases}$$

where C is a suitable constant. These are the best possible estimates. The space C_p^n has a subspace Y with dim $Y = k$, $d(Y,\ell_2^k) \le 2$ where

$$k = \begin{cases} C\, n^{1+2/p} & \text{if } 2 \le p \le \infty \\ C\, n^2 & \text{if } 1 \le p \le 2 \end{cases} .$$

Again these are the best possible estimates.

Let us recall that C_p^n is the space of all operators T on ℓ_2^n with $\|T\|_p = (\text{trace}(TT^*)^{p/2})^{1/p}$.

Proposition 12 can be deduced in several ways from the preceeding results. The simplest way is to deduce it from Theorem 7. We shall illustrate this in the case of C_p^n. Let $X = C_p^n$, $2 \le p \le \infty$. Since $d(C_p^n, C_2^n) = n^{1/2 - 1/p}$ and $C_2^n = \ell_2^{n^2}$ we get from (6) (we take $\tau = 1$) that $k.(X)k(X^*) \ge \delta\, n^2.n^{1+2/p}$. Thus in order to prove the proposition it is enough to prove that $k(X) \le C.n^{1+2/p}$ for some C. We do this first in the case $p = \infty$. Since $\|T\|_\infty = \sup|(Tx,y)|$, and in this supremum it is approximatively enough to let x and y range over a δ net of the boundary of the unit ball of ℓ_2^n it follows from Lemma 4 that for some constant λ, there is a subspace $Z \subset \ell^{2^{\lambda n}}$ so that $d(Z,C_\infty^n) \le 2$. An application of Proposition 10 concludes the proof for $p = \infty$. Assume now that $2 \le p < \infty$, and that $Y \subset C_p^n$ with $d(Y,\ell_2^k) \le 2$. Since $d(C_p^n, C_\infty^n) = n^{1/p}$ there is a subspace $Y_o \subset C_\infty^n$ so that $d(Y_o,Y) \le n^{1/p}$ and thus $d(Y_o,\ell_2^k) \le 2n^{1/p}$. By Theorem 6 (observe the $|||\ \ |||$ can always be chosen so that b/M_r is less than the distance from the Euclidean space) we get that $Y_o \supset Y_1$ with dim $Y_1 = k_1$, $d(Y_2,\ell_2^{k_2}) \le 2$ and $k_1 \ge \eta k n^{-2/p}$. By the case $p = \infty$ we get that for some constant C, $\eta k n^{-2/p} \le k_1 \le Cn$ and this concludes the proof.

Let us observe that the assertion of Proposition 12 for ℓ_p^n could be deduced by considering only the cotype and using Theorem 9. In the case of C_p^n the use of cotype is not sufficient for proving Proposition 12 if $2 \le p$. Theorem 9 however gives interesting information for arbitrary finite dimensional subspaces of $L_p(0,1)$ and C_p (= the operators on ℓ_2 with $\|T\|_p = (\mathrm{trace}(TT^*)^{p/2})^{1/p}$) and not only for spaces of the form ℓ_p^n or C_p^n. (recall that the cotype of $L_p(0,1)$ and C_p is p if $p \ge 2$ and is 2 if $1 \le p \le 2$ cf [10]). Let us also remark that there is a partial converse to Theorem 9. If X is an infinite-dimensional Banach space so that for some q and C every subspace $Y \subset X$ contains a subspace Z with $d(Z,\ell_2^{\dim Z}) \le 2$ and $\dim Z = C(\dim Y)^{2/q}$ then X is of cotype $q+\varepsilon$ for every $\varepsilon > 0$. This follows from Proposition 12 and a result of Krivine, Maurey and Pisier (cf. [7]) which asserts that if $q_o = \inf\{q\ ;\ X \text{ of cotype } q\}$ then X has almost isometric copies of $\ell_{q_o}^n$ for every integer n.

We pass now to an examination of the question "what happens if all subspaces of a given dimension in a finite dimensional Banach space are close to Euclidean ?". We recall first the definition of some constants associated to a Banach space X. We denote by $\alpha_m(X)$ the smallest number for which

$$(15)\qquad \int \Big\|\sum_{i=1}^m r_i(t)x_i\Big\|^2 dt \le \alpha_m(X) \sum_{i=1}^m \|x_i\|^2 \quad , \qquad m = 1,2,..$$

for every choice of $\{x_i\}_{i=1}^m \in X$. Similarly $\beta_m(X)$ is the smallest number for which

$$(16)\qquad \int \Big\|\sum_{i=1}^m r_i(t)x_i\Big\|^2 dt \ge \beta_m^{-1}(X) \sum_{i=1}^m \|x_i\|^2 \quad , \qquad m = 1,2,...$$

We shall use the following known fact (cf. [7])

$$(17)\qquad \alpha_{mk}(X) \le \alpha_m(X)\alpha_k(X) \quad , \quad \beta_{mk}(X) \le \beta_m(X)\beta_k(X) \quad , \quad 1 \le k,m \ .$$

There is an absolute constant C so that

$$(18)\qquad C^{-1} \le \alpha_m(X)/\tilde{\alpha}_m(X) \le C \quad , \quad C^{-1} \le \beta_m(X)/\tilde{\beta}_m(X) \le C$$

for all m and X where $\tilde{\alpha}_m$ (resp. $\tilde{\beta}_m$) is obtained from the definition of α_m (resp. β_m) by replacing the Rademacher functions $\{r_i(t)\}_{i=1}^\infty$ by a sequence of independent normalized Gaussian random variables $\{g_i(\omega)\}_{i=1}^\infty$.

<u>Theorem 13</u> : Let X be a Banach space of dimension n. Then $\tilde{\alpha}_m(X) = \alpha_{n(n+1)/2}(X)$, $\tilde{\beta}_m(X) = \beta_{n(n+1)/2}(X)$ for every $m \geq n(n+1)/2$.

<u>Proof</u> : Let $m > n(n+1)/2$ and let $\{x_i\}_{i=1}^m$ be any m elements in X. Then $Q(x^*) = \sum_{i=1}^m x^*(x_i)^2$ is a quadratic form on X^*. Since the dimension of the space of quadratic forms on X^* is $n(n+1)/2$ it follows from Caratheodory's theorem that there exist $\alpha_i \geq 0$ with $\sum_{i=1}^m \alpha_i = 1$ so that at least one α_i (say α_1) vanishes and $Q = m \sum_{i=1}^m \alpha_i x^*(x_i)^2$. Let $\alpha = \max_{1\leq i\leq m} \alpha_i$ (assume say that $\alpha = \alpha_2$) and put $y_i = \sqrt{\frac{\alpha_i}{\alpha}}\, x_i$ and $z_i = \sqrt{(1-\frac{\alpha_i}{\alpha})}\, x_i$, $1 \leq i \leq m$. Then

(19) $$y_1 = 0 \quad , \quad z_2 = 0$$

(20) $$\sum_{i=1}^m \|x_i\|^2 = \sum_{i=1}^m \|y_i\|^2 + \sum_{i=1}^m \|z_i\|^2 \quad .$$

Also $\sum_{i=1}^m x^*(y_i)^2 = \frac{1}{\alpha m} Q$ and thus by a basic property of Gaussian variables

(21) $$\int \Big\|\sum_{i=1}^m g_i(\omega) y_i\Big\|^2 d\omega = \frac{1}{\alpha m} \int \Big\|\sum_{i=1}^m g_i(\omega) x_i\Big\|^2 d\omega$$

and similarly

(22) $$\int \Big\|\sum_{i=1}^m g_i(\omega) z_i\Big\|^2 d\omega = \Big(1-\frac{1}{\alpha m}\Big) \int \Big\|\sum_{i=1}^m g_i(\omega) x_i\Big\|^2 d\omega \quad .$$

Since by (19) the sequences $\{z_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ consist of at most m-1 vectors different from 0 we get by (20), (21) and (22) that $\tilde{\alpha}_m(X) \leq \tilde{\alpha}_{m-1}(X)$ and $\tilde{\beta}_m(X) \leq \tilde{\beta}_{m-1}(X)$ and this concludes the proof.

<u>Theorem 14</u> : Let X be a Banach space of type $1 \leq p \leq 2$ and cotype $2 \leq q \leq \infty$. Then there exists a constant C so that for every subspace $Y \subset X$ with $\dim Y = n$, $d(Y, \ell_2^n) \leq C\, n^{2(\frac{1}{p}-\frac{1}{q})}$.

<u>Proof</u> : Since X is of cotype p, $\alpha_m(X) \leq \lambda . m^{\frac{1}{p}-\frac{1}{2}}$ and since X is of cotype q, $\beta_m(X) \leq \lambda\, m^{\frac{1}{2}-\frac{1}{q}}$ for some constant λ and $m = 1,2,\ldots$. By (18) and Theorem 13 we deduce that for some constant γ

$$\sup_m \alpha_m(Y) \leq \gamma\, m^{2/p-1} \quad , \quad \sup_m \beta_m(Y) \leq \gamma\, m^{1-2/q} \quad .$$

The desired result follows now from **Kwapien's** theorem [4].

<u>Theorem 15</u> : Let $f(n)$ be a function of n so that $\log f(n)/\log n > \gamma$ for all n. Let X be a Banach space of dimension n so that every $Y \subset X$ with $k = \dim Y \le f(n)$ satisfies $d(Y, \ell_2^k) \le 2$. Then, for some constant C depending only on γ, $d(X, \ell_2^n) \le C$.

<u>Proof</u> : By our assumption $\alpha_{f(n)}(X)$, $\beta_{f(n)}(X) \le 2$. By (17), (18) and Theorem 13

$$\sup_m \alpha_m(X) \quad , \quad \sup_m \beta_m(X) \le \lambda 2^{\log n^2/\log f(n)}$$

for some constant λ. A use of **Kwapien's** theorem concludes the proof.

<u>Remark</u> : The assumption on $f(n)$ in Theorem 15 cannot be weakened. Indeed, if $X = \ell_{p_n}^n$ with $p_n \ge 2$ then by Theorem 14 for every $Y \subset X$ with $k = \dim Y \le f(n)$ we have $d(Y, \ell_2^k) \le C\left(f(n)^{\frac{1}{p_n} - \frac{1}{2}}\right)^2$. Assume now that p_n is chosen so that $f(n)^{\frac{1}{p_n} - \frac{1}{2}} = 2$. Then if $\log f(n)/\log n \to 0$ we get that $d(X, \ell_2^n) = n^{\frac{1}{p_n} - \frac{1}{2}} \to \infty$.

<u>Proposition 16</u> : There are absolute constants C and γ so that the following holds. If X is a Banach space so that $X \supset Y$ with Y and X/Y both isometric to inner product spaces then for every $Z \subset X$ with $\dim Z = n$, $d(Z, \ell_2^n) \le C \log n^{\gamma}$.

<u>Proof</u> : This follows from the estimates on $\alpha_n(X)$ and $\beta_n(X)$ given in [2] and Theorem 13. Let us observe that the example in [2] shows that (up to an estimate of γ) this is the best possible result.

<u>Theorem 17</u> : There is a function $\lambda \to f(\lambda)$ so that the following is true. If X is a Banach space of dimension n so that on any subspace of it there is a projection with norm $\le \lambda$ then $d(X, \ell_2^n) \le f(\lambda)$.

<u>Proof</u> : By the assumption of the theorem and (9) there is a constant $C(\lambda)$ so that every $Y \subset X$ with $\dim Y \ge n/2$ contains a subspace Z with $\dim Z = k$, $d(Z, \ell_2^k) \le 2$ and $k \ge C(\lambda) n^{1/2}$. The proof of the main result of [5] shows that this implies that for any subspace $U \subset X$ with $\dim U = C(\lambda) n^{1/2}$, $d(U, \ell_2^{\dim U}) \le g(\lambda)$ for some function $g(\lambda)$. The proof is concluded by using Theorem 15.