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ON THE INVARIANT SUBSPACE PROBLEM IN BANACH SPACES

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In this seminar we will outline a construction of an operator with only trivial invariant subspaces on a Banach space (for details see [1]). The Banach space in this example will be constructed at the same time as the operator and will be non-reflexive. There are very serious difficulties in carrying out a similar construction in a reflexive Banach space. So we feel that the construction gives some support to the conjecture that every operator on a Hilbert space has a non-trivial invariant subspace. We now turn to the basic considerations behind this approach. It is clear that every operator with a cyclic vector on a Banach space can be represented as multiplication by x on the set of polynomials under some norm. So what we will do is to construct a norm on the space of polynomials and prove that the shift operator under this norm has only trivial invariant subspaces.

Our next basic consideration is based on the fact that one can have an operator with a dense set of cyclic vectors without having all vectors cyclic. In order to be able to make some limit procedure work we will construct the operator so that it has the following property : Let 1 be a cyclic vector of norm 1 in B. Let $\{p_j\}$ be a sequence which is dense on the unit sphere of B. For every j and every m $\exists$ a positive number C_{jm} such that for every p_n with $\|p_j - p_n\| < \frac{1}{2^{m+4}}$ $\exists$ a polynomial $\ell(T)$ in T with $\|\ell(T)\|_{op} \leq C_{jm}$ such that $\|\ell(T)p_n - 1\| \leq \frac{1}{2^m}$. It is easily verified that such a T has only trivial invariant subspaces. ... (1)

If we have the operator T represented as multiplication by x then $\ell(T)$ will just be multiplication by the polynomial ℓ. This leads us to the next basic consideration. Assume that we have a norm $\| \ \|$ on the space of polynomials. Assume that p is a polynomial of norm 1 and assume that $\|\ell p - 1\| \leq \varepsilon$ and $\|\ell\|_{op} \leq K$. This gives that for every polynomial h we have the inequality

$$\|h\| - K\|hp\| \leq \|h\ell p - h.1\| \leq \varepsilon\|h\|_{op} \quad .$$

And this implies that if $\|h\|_{op} \leq \frac{1}{2\varepsilon}\|h\|$, then $\|hp\| \geq \frac{\|h\|}{2K}$ ..... (2).

In order that the operator also satisfied (1) it is of course necessary that the inequality $\|hp\| \geq \frac{\|h\|}{2K}$ holds uniformly in p in every ball of size $\frac{\varepsilon}{16}$ on the unit sphere. (At least if we put $\varepsilon = \frac{1}{2^m}$.) ..... (3)

There is a sense in which the inequality (2) is sufficient for p to be moved close to 1 by a polynomial with small operator norm. This is given by our Lemma 2 below. In order to describe this Lemma we have to tell something about the way that we construct the final norm. It should be pointed out that this sufficiency of (2) depends on the fact that the norm constructed is non-reflexive. We do not know whether anything similar can be done in a reflexive space.

Consider all pairs (q,ε) where q is an arbitrary polynomial whose coefficients have real and parts rational, and ε is of form 2^{-k}. We enumerate all such pairs and call the sequence (q_n,ε_n). We also insist that for a fixed q, if $n_1\ n_2\ \ldots$ are all the integers such that $q_n = q$, then $\varepsilon_{n_1} > \varepsilon_{n_2} > \varepsilon_{n_3}\ \ldots$. Also we assume $\deg q_n \le n$.

Our construction will be completely determined by a sequence of polynomials ℓ_n and constants $C_n > 2$. $\ell_1 \ldots \ell_k$ and $C_1 \ldots C_K$ will determine a number α_{K+1} inductively as explained below and we define a sequence of norms as in the following definitions.

<u>Definition 1</u> : For any polynomial p, consider all representations $p = \Sigma\, a_{i,\beta}\, x^i\, \ell_1^{\beta_1} \ldots \ell_n^{\beta_n}$ and put

$$|P|_{op\,n} = \inf \Sigma\, |a_{i,\beta}|\, 2^i (C_1\, |\ell_1|_1)^{\beta_1} \ldots (C_n\, |\ell_n|_1)^{\beta_n}$$

where $|\ |_1$ denotes the usual ℓ_1 norm equal to the sum of the absolute value of the coefficients.

<u>Remark</u> : In the final norm the operator x will have norm ≤ 2, and multiplication by ℓ_k norm $\le C_k\, |\ell_k|_1$.

<u>Definition 2</u> : For any p, consider all representations

$$p = r + \sum_1^n S_k(\ell_k \alpha_k q_k - 1) \quad .$$

Put $|p|^n = \inf |r|_1 + \Sigma\, |S_k|_{op\,n}\, \varepsilon_k$. Put $|p|^o = |p|_1$, and let α_k be determined inductively by the condition $|\alpha_k q_k|^{k-1} = 1$.

Remark : $|\ell_k\alpha_k q_k - 1|^n < \varepsilon_k$ and clearly the operator norm of multiplication by g is $\leq |g|_{op\,n}$. We see that $|\ |^n$ is the maximal norm satisfying the following four properties :

1) $|\ |^n \leq |\ |_1$

2) $|\ell_k\alpha_k q_k - 1|^n \leq \varepsilon_k$, $k = 1,2,\dots n$,

3) $|g|_{op} \leq |g|_{op\,n}$

4) $|x|_{op} \leq 2$.

Observe that $|\ |^n$ and $|\ |_{op\,n}$ are decreasing sequences of norms and hence converge to some pseudo-norms. We write $\|\ \| = \lim |\ |^n$.

Lemma 1 : Assume C_n and ℓ_n are given as well as sequences of positive numbers $D_n \nearrow \infty$ and $L_n \nearrow \infty$ satisfying the following :

1) $|p|^m$ is constant for $m \geq (\deg p) - 1$. In particular $|q_n|^m$ is constant for $m \geq n - 1$.

2) For any n, consider all $k \leq n$ such that $\varepsilon_n = \varepsilon_k$, and $|\alpha_k q_k - \alpha_n q_n|^{(n-1)} < \frac{\varepsilon_n}{16}$. Let K be the least such k. Then $|\ell_n|_1 = L_K$, $C_n = D_K$.

Then the resulting limit norm defines a space B, for which multiplication by x has no invariant subspace.

Proof : Let q be an element of B, which we recall is the closure of all polynomials and $\|q\| = 1$. Let ε be a fixed negative power of 2. Choose increasing n_k such that $\varepsilon_{n_k} = \varepsilon$ and $\alpha_{n_k} q_{n_k} \to q$ in B. We can even insist that

$$\|\alpha_{n_k} q_{n_k} - q\| < \frac{\varepsilon}{64} .$$

Hence for $k > 1$, $|\alpha_{n_k} q_{n_k} - \alpha_{n_1} q_{n_1}|^{(n_k-1)} = \|\alpha_{n_k} q_{n_k} - \alpha_{n_1} q_{n_1}\| < \frac{\varepsilon}{32}$, so by 2), $C_{n_k} \leq \operatorname{Max}_{m \leq n_1} D_m$ and $|\ell_{n_k}|_1 \leq \operatorname{Max}_{n \leq n_1} L_m$ so that $|\ell_{n_k}|_{op\,n_k}$ is bounded $\leq A$. Therefore

$$\begin{aligned} \|\ell_{n_k} q - 1\| &\leq \|\ell_{n_k}\alpha_{n_k} q_{n_k} - 1\| + \|\ell_{n_k}(q - \alpha_{n_k} q_{n_k})\| \\ &\leq \varepsilon + |\ell_{n_k}|_{op\,n_k} \|q - \alpha_{n_k} q_{n_k}\| \\ &\leq \varepsilon + A\|q - \alpha_{n_k} q_{n_k}\| . \end{aligned}$$

Letting k tend to infinity, we see that 1 is within distance ε of the space generated by q and hence, letting $\varepsilon \to 0$, we see that 1 is in that space and hence it equals B.

We will now drop α_k in our notation so when it is clear from the context we will denote $\alpha_k q_k$ by q_k and assume $|q_k|^{k-1} = 1$.

<u>Definition 3</u> : <u>deg</u> p = degree of lowest order term of the polynomial p.

<u>Definition 4</u> : Let f be a positive real valued function defined on $[0,\infty)$. We say that $\ell = \sum_{j\geq 0} a_j x^{n_j}$ is more lacunary than f if

$$\underline{\deg}\ \ell = n_o \geq f(0)$$

and

$$n_j \geq f(n_{j-1}) \text{ for every } j \quad .$$

Our next Lemma which we give without proof, shows that, under the assumption of an inequality similar to (2) we can have the first part of Lemma 1 fulfilled.

<u>Lemma 2</u> : Let $\ell_1,\ldots,\ell_n$, $C_1,\ldots,C_n$ given with $C_k > 2$. Assume for all h and some B that

$$\frac{|h|_{op\,n}}{|h|^o} < \frac{1}{\varepsilon_{n+1}} \Longrightarrow |hq_{n+1}|^n \geq \frac{|h|^o}{B}$$

then, given N, $K > 4B/\varepsilon_{n+1}$, there exists a lacunarity function f such that if

1) $|\ell_{n+1}|_1 = K$

2) The lacunarity of $\ell_{n+1} \geq f$

3) $C_{n+1} > 2$.

Then with this choice of ℓ_{n+1} and C_{n+1} we have

$$|g|^{n+1} = |g|^n \quad \text{for all } g \text{ with } \deg g \leq N \quad .$$

We now assume that we have two sequences $D_n \nearrow \infty$ and $L_n \nearrow \infty$. Assume that $|\ |^{n-1}$ is defined. We will then define $|\ |^n$ according to the following rule : consider all $k \leq n$ such that $\varepsilon_k = \varepsilon_n$ and $|q_k - q_n|^{n-1} < \frac{\varepsilon_n}{16}$. Let K be the least such k. Then $|\ell_n|_1 = L_K$, $C_n = D_K$. If this rule is fulfilled for all $n \leq N$ we say that $|\ |^N$ is defined in a compatible way from the sequences D_n and L_n . If for every N $|\ |^N$ is defined in a compatible way from the sequences D_n and L_n then obviously condition 2) of Lemma 1 is fulfilled. Our next Lemma combined with Lemma 2 will now enable us to get also the condition 1) of Lemma 1 fulfilled. We first make some

Definition 5 : A growth function F is a function that for every n and every 3n-tuple $D_1 \ldots D_n$, $L_1 \ldots L_n$, $\ell_1 \ldots \ell_n$ gives a positive number $F(D_1 \ldots D_n, L_1 \ldots L_n, \ell_1 \ldots \ell_n)$, and for every n and every (3n+2)-tuple $D_1 \ldots D_{n+1}$, $L_1 \ldots L_{n+1}$, $\ell_1 \ldots \ell_n$ gives a lacunarity function f and a positive number δ. We say that the sequence $\{D_n, L_n, \ell_n, C_n\}$ grows faster than F if

1) ℓ_k and C_k are defined in a compatible way from the sequences D_n and L_n for every k.

2) For every n, D_{n+1} and L_{n+1} are $> F(D_1 \ldots, D_n, L_1 \ldots, L_n, \ell_1 \ldots \ell_n)$.

3) For every n the lacunarity of $\ell_{n+1} \geq f$ and the moduli of the coefficients of ℓ_{n+1} are $\leq \delta$ where f and δ are given by the growth function applied to $D_1 \ldots D_{n+1}$, $L_{n+1} \ldots L_{n+1}$, $\ell_1 \ldots \ell_n$.

We now have

Lemma 3 : There is a growth function F such that if $\{D_n, L_n, \ell_n, C_n\}$ grows faster than F, then for every n $\exists$ B_n depending only on $|\ |^{n-1}$, such that for all $N \geq n$

$$|q - q_n|^N \quad \text{and} \quad \frac{|h|_{op\,N}}{|h|^o} < \frac{1}{\varepsilon_n} \quad \text{imply} \quad |hq|^N \geq \frac{|h|^o}{B_n} \quad .$$

As mentioned above this Lemma can be combined with Lemma 2 to give also 1) of Lemma 1. The main difficulty in the construction is to prove Lemma 3. The main tool is the following

Theorem : Let A, B be homogeneous polynomials in many variables of degree d_1 d_2. Then

$$|AB|_1 \geq K(d_1, d_2)\, |A|_1\, |B|_1 \quad .$$

<u>Remark</u> : $|\ |_1$ denotes the usual ℓ_1-norm, the sum of the moduli of the coefficients. The essential point is that $K(d_1\ d_2)$ is independent of the number of variables. For generalizations of this result to other norms the reader is referred to [2].

REFERENCES

[1] P. Enflo, On the invariant subspace problem in Banach spaces, to appear.

[2] P. Enflo and H. Montgomery, Norms and products of polynomials, to appear.
