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**A short proof of Dvoretzky's theorem**

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# A SHORT PROOF OF DVORETZKY'S THEOREM

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The theorem of Dvoretzky [2] states that for any $\varepsilon > 0$ and any positive integer K there exists an $N = N(K,\varepsilon) < \infty$ such that every normed space $(X, \| \ \|)$ with $\dim X > N$ contains a K-dimensional subspace E which is $\varepsilon$-Euclidean (i.e. there exists a Euclidean norm $|.|$ on E such that $\|x\| \leq |x| \leq (1+\varepsilon)\|x\|$ for $x \in E$).

Two proofs of the theorem have already been presented on this seminar (cf. [1], [4], [3]). I am going to show another one based on an idea of Szankowski's [6] but simpler in details. Only the obvious modifications (viz. considering the complex Stiefel and Grassmann manifolds) are needed to obtain a proof of the complex version of the theorem.

In the sequel let K be a fixed integer greater than 1.

We shall need the following consequence of the Dvoretzky-Rogers lemma (cf. [2], [1]).

(D-R) For every integer $n \geq K$ and every normed space $(X, \| \ \|)$ with $\dim X \geq 4n^2$ there exists an operator $I : \mathbb{R}^n \to X$ such that

$$\frac{1}{2}\|x\|_{\ell_\infty^n} \leq \|Ix\| \leq \|x\|_{\ell_2^n} \quad \text{for } x \in \mathbb{R}^n .$$

Let $F = I(\mathbb{R}^n)$ and let $\|x\|_2 = \|I^{-1}x\|_{\ell_2^n}$ for $x \in F$. Since any norm on F can be approximated (uniformly on the unit ball) by smooth ones, we may assume that $\|.\|$ is smooth on F. Thus for each $x \in F \setminus \{0\}$ there is a unique $T_x \in F^*$ such that

$$T_x(x) = \|x\| \|T_x\|_{F^*} = 1 .$$

Clearly, $T_x$ depends continuously on x, and $\|x\| T_x$ is simply the Gâteaux derivative of the norm $\|.\|$ at x.

For any linear subspace $E \subseteq F$ with $\dim E \geq 2$, let $S_E$ denote the unit sphere $\{x \in E : \|x\|_2 = 1\}$ and let $\Sigma_E$ denote the Stiefel manifold of all ordered pairs $\langle x,y \rangle \in S_E \times S_E$ such that $y \in x^\perp = \{f \in F : (I^{-1}f, I^{-1}x)_{\ell_2^n} = 0\}$

The normalized $\| \ \|_2$-rotation invariant measures on $S_E$ and $\Sigma_E$ will be denoted by $\lambda_E$ and $\sigma_E$ respectively.

Our basic invariant characterizing the closeness of $\| \ \|$ to $\| \ \|_2$ on E is defined as follows

$$v(E) = \int_{\Sigma_E} T_x(y)^2 d\sigma_E(x,y).$$

We shall check the following facts :

1) There exists a subspace $E \subseteq F$ such that $\dim E = K$ and $v(E) \le v(F)$ ;

2) $d(E) = \int_{S_E \times S_E} (\|x\| - \|z\|)^2 \, d\lambda_E(x)\, d\lambda_E(z) / \sup_{S_E} \|x\|^2 \le (\pi/2)^2 \, v(E)$ ;

3) $b = 1 - \inf_{S_E} \|x\| / \sup_{S_E} \|x\| \le C d(E)^{1/(K+1)}$, where C depends only on K.

It follows from 1, 2, 3 that $b \le C_1 v(F)^{1/(K+1)}$, where $C_1$ is another constant. Since $\varepsilon \le b/(1-b)$, the proof will be complete, if we also establish :

4) There exists a sequence $(C_n)_{n=2}^{\infty}$ tending to zero such that the F yielded by (D-R) satisfies $v(F) \le C_n$.

<u>Proofs</u> : 1) is an immediate consequence of the formula

$$v(F) = \int_{\Sigma_F} T_x(y)^2 d\sigma_F(x,y)$$

$$= \int_{\Gamma} d\gamma(E) \int_{\Sigma_E} T_x(y)^2 d\sigma_E(X,y) = \int_{\Gamma} v(E)\, d\gamma(E),$$

where $\gamma$ is the normalized $\|\cdot\|_2$-rotation invariant measure on the Grassmann manifold $\Gamma$ of all K-dimensional linear subspaces of F. (The second equality is valid when $T_x(y)^2$ is replaced by any function f(x,y) defined and continuous on $\Sigma_F$ ; it follows from the uniqueness of a normalized invariant measure on $\Sigma_F$).

2) For any $x,z \in S_E$ with $z \neq \pm x$, let $a_{x,z}(t)$, $0 \le t \le 2\pi$, be the arc-length parametrization of the great circle of $S_E$ starting at x and passing through z, -x , - z back to x. We have

$$4\big|\|x\| - \|z\|\big| \le \int_0^{2\pi} \Big|\frac{d}{dt} \|a_{x,z}(t)\|\Big| dt \le \sqrt{2\pi}\Big(\int_0^{2\pi} \Big(\frac{d}{dt} \|a_{x,z}(t)\|\Big)^2 dt\Big)^{1/2}$$

hence

$$\int_{S_E\times S_E}(\|x\|-\|z\|)^2 d\lambda(x)\,d\lambda(z) \le (\pi/8)\int_{S\times S}\int_0^{2\pi}\left(\frac{\partial}{\partial t}\|a_{x,z}(t)\|\right)^2 dt\,d\lambda(x)\,d\lambda(z)$$

$$= (\pi/8)\int_0^{2\pi} du \int_{S\times S}\left(\frac{\partial}{\partial t}\|a_{x,z}(t)\|\Big|_{t=u}\right)^2 d\lambda(x)\,d\lambda(z)$$

$$= (\pi/2)^2 \int_{S\times S}\left(\frac{\partial}{\partial t}\|a_{x,z}(t)\|\Big|_{t=0}\right)^2 d\lambda(x)\,d\lambda(z)$$

$$= (\pi/2)^2 \int_{S\times S}\left[(D\|\ \|)(x)\left(\frac{\partial a_{x,z}(t)}{\partial t}\Big|_{t=0}\right)\right]^2 d\lambda(x)\,d\lambda(z$$

$$= (\pi/2)^2\int_S d\lambda_S(x)\int_{S\cap x^\perp}(D\|\ \|)(x)(y)^2 d\lambda_{S\cap x^\perp}(y)$$

$$= (\pi/2)^2\int_{\Sigma_E}\|x\|^2 T_x(y)^2 d\sigma_E(x)$$

$$\le (\pi/2)^2 \sup_{S_E}\|x\|^2\, v(E)\,.$$

3) Let us write $P(\varphi(x))$ instead of $\lambda_E(\{x\in S_E : \varphi(x)\})$. Let $a = \sup_{x\in E}\|x\|$ and let $t\in(0,1)$ be fixed.

If $P(\|x\| \ge a(1-\frac{1}{2}b)) \ge \frac{1}{2}$, we pick an $x_o$ such that $\|x_o\| = a(1-b) = \inf_{S_E}\|x\|$. (Otherwise we would take $x_o$ with $\|x_o\| = a$, and proceed analogously).

Observe that there is an $s>0$, depending only on K, such that $P(\|x-x_o\| \ge s\,b^{K-1})$ for $b\le 1$. Thus we have

$$a^2 d(E) \ge (\tfrac{1}{2}abt)^2 P(\|x\|-\|x_o\| \le \tfrac{1}{2}ab(1-t))P(\|y\|\ge a(1-\tfrac{1}{2}b))$$

$$\ge (\tfrac{1}{2}abt)^2 2P(\|x-x_o\| \le \tfrac{1}{2}b(1-t))\,.\,\tfrac{1}{2}$$

$$\ge \tfrac{1}{4}a^2 b^{K+1} st^2(1-t)^{K-1}\quad ,$$

which implies the desired inequality.

To get 4) observe first that

$$v(F) = \int_{S_F} d\lambda_F(x) \int_{S_{F \cap x^\perp}} (T_x(y))^2 \, d\lambda_{F \cap x^\perp}(y)$$

$$\leq \int_{S_F} d\lambda_F(x) \frac{1}{n-1} (\|T_x\|_2^* \leq \frac{1}{n-1} \int_{S_F} \|x\|^{-2} d\lambda_F(x)$$

(recall that $\|T_x\|_2^* \leq \|T_x\|_{F^*} = \|x\|^{-1}$).

$$= \frac{1}{n-1} \int_{S_{\mathbb{R}^n}} \|Ix\|^{-2} d\lambda_{\mathbb{R}^n}(x) \leq \frac{4}{n-1} \int_{S_{\mathbb{R}^n}} \|x\|_{\ell_\infty^n}^{-2} d\lambda_{\mathbb{R}^n}(x) \; .$$

The following short reasoning was shown to the author by D. Burkholder. Let $X_1, X_2, \ldots$ be independent normalized Gaussian variables on a probability space $(\Omega, \Sigma, P)$. Then one has

$$\frac{1}{n} \int_{S_{\mathbb{R}^n}} (\max_{1 \leq i \leq n} |x_i|)^{-2} d\lambda_{\mathbb{R}^n}(x) = \frac{1}{n} \int_\Omega (\max_{1 \leq i \leq n} \frac{|X_i(\omega)|}{(\sum_{i=1}^n X_i(\omega)^2)^{1/2}})^{-2} \, dP(\omega)$$

$$= \frac{1}{n} \int_\Omega \frac{\sum_{i=1}^n X_i(\omega)^2}{\max_{1 \leq i \leq n} X_i(\omega)^2} dP(\omega) = \int_\Omega \frac{X_1(\omega)^2}{\max_{1 \leq i \leq n} X_i(\omega)^2} dP(\omega) \overset{\text{def}}{=} b_n \, .$$

The fact that the $b_n$'s tend to zero is a well-known consequence of the unboundedness of the $X_i$'s and the Lebesgue dominated convergence theorem. This is sufficient to establish 4) and complete the proof.

It is easy to prove that in fact $b_n = O((\operatorname{Log} n)^{-1})$ which yields an estimate $N(K, \varepsilon) \leq \exp(C_2 \varepsilon^{-K-1})$ for small $\varepsilon > 0$. This bound can be considerably improved by using the p-th powers instead of squares in the definition of $v(E)$ (p being a large number depending on n) and more careful estimates of the appearing integrals. The result seems to be slightly stronger than those found in [2], [5] and [6] .

## REFERENCES

[1] Séminaire Maurey-Schwartz 1972-1973, Exposés XXVI et XXVII .

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