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On a result of Olevskiĭ : a uniformly bounded orthonormal sequence is not a basis for $C[0,1]$

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ON A RESULT OF OLEVSKIǏ : A UNIFORMLY BOUNDED ORTHONORMAL SEQUENCE IS NOT A BASIS FOR C[0,1]

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The purpose of this lecture is to present a result of Olevskiĭ (Izvestia Akad Nauk SSSR, volume 30 (1966), 387 - 432) that there is no uniformly bounded orthonormal system which is a basis for the space C of all continuous functions on the interval [0,1] and to explain the relation of this result with the conjecture of the non existence of normalized Besselian bases in C.

<u>Definition</u> : A biorthogonal system (e_n, μ_n) in a Banach space E, in particular a basis, is said to be Besselian (Hilbertian) if there exists a constant $K > 0$ such that for each $x \in E$

$$\|x\| \geq K \left(\sum_{n=1}^{\infty} (\mu_n(x) \|e_n\|)^2 \right)^{1/2}$$

$$(\text{resp. } \|x\| \leq K \left(\sum_{n=1}^{\infty} (\mu_n(x) \|e_n\|^2 \right)^{1/2} \quad .$$

<u>Example</u> : A uniformly bounded in L^∞-norm orthonormal system in L^2 is a Besselian biorthogonal system in L^∞.
Indeed if (φ_n) is an orthonormal system in L^2 such that $\|\varphi_n\|_\infty \leq M$ for all n then for $x \in L^\infty$ we have

$$\|x\|_\infty \geq \|x\|_2 \geq \sqrt{\sum_n [\int x(t)\, \varphi_n(t)\, dt]^2} \geq \frac{1}{M} \sqrt{\sum_n ([\int x(t)\, \varphi_n(t)\, dt] \|\varphi_n\|_\infty)^2}$$

<u>Conjecture</u> : There is no Besselian basis in C[0,1] as well as there is no Hilbertian basis in L^1.

The following result of Olevskiĭ (1966) strongly support this conjecture.

Theorem 1 : Let ν be a probability measure on $[0,1]$. Let (φ_n) ba a uniformly bounded orthonormal (with respect to ν) system in $C[0,1]$. Then (φ_n) is not a basis for $C[0,1]$.

Before passing to the proof of Theorem 1 we shall establish a general fact on Besselian biorthogonal systems in $C[0,1]$ which is trivial for orthonormal systems but which shows some relation between the theorem and the conjecture.

Proposition 1 : Let (e_n, μ_n) be a Besselian biorthogonal system in C. Assume that (μ_n) is total, i.e. $\mu_n(x) = 0$ for all n implies $x = 0$, and let for some $M > 0$, $M^{-1} \le \|e_n\| \le M$ for all n.

Then there exists a probability measure ν on $[0,1]$ and $g_n \in L^2(\nu)$ such that $\mu_n(x) = \int_0^1 x(t)\, g_n(t)\, d\nu$ for all $x \in C$ and for all n and

$$M^{-1} \le \int_0^1 |g_n(t)|\, d\nu \le \sqrt{\int_0^1 |g_n(t)|^2\, d\nu} \le K_G\, K\, M$$

and $\left(\int_0^1 |e_n(t)|^2 d\nu\right)^{1/2} \ge \frac{1}{K_G\, K\, M}$ for all n where K is the constant appearing in the definition of a Besselian basis and K_G is a universal (Grothendieck) constant.

Proof : Define $T : C \to l^2$ by $T(x) = (\mu_n(x))$. Since (e_n, μ_n) is a Besselian biorthogonal system and $\|e_n\| \ge M^{-1}$ for all n, T is continuous and $\|T\| \le K M$. Thus, by a result of Grothendieck, T is 2-integral. Hence, by Grothendieck Pietsch factorization theorem, there exists a probability measure ν on $[0,1]$ such that $T = A\, I_\nu$ where $I_\nu : C \to L^2(\nu)$ is the natural embedding and $A : L^2(\nu) \to l^2$ is a bounded linear operator with $\|A\| \le K_G\, K\, M$.

Let δ_n denote the n-th unit vector in l^2. We have for $x \in C[0,1]$

$$\mu_n(x) = \langle Tx, \delta_n \rangle_{l^2} = \langle I_\nu x, A^* \delta_n \rangle_{L^2(\nu)} = \int_0^1 x(t)\, y_n(t)\, d\nu$$

where $g_n = A^* \delta_n \in L^2(\nu)$. Hence $\mu_n = g_n\, d\nu$.

Thus

$$\frac{1}{M} = \frac{1}{M}\,\mu_n(e_n) \leq \|\mu_n\| = \int |g_n(t)|\, d\nu \leq \sqrt{\int |g_n(t)|^2\, d\nu} \leq \|A\| \ .$$

Since $I_\nu C$ is dense in $L^2(\nu)$ and T is one to one because (μ_n) is total, A is one to one. Hence $I_\nu\, e_n = A^{-1}\delta_n$. Thus

$$\sqrt{\int_0^1 |e_n(t)|^2 d\nu} \geq \frac{1}{\|A\|}\, T\,(e_n) = \frac{1}{\|A\|} \ .$$

Our next proposition indicates the strategy of the proof of Theorem 1.

<u>Proposition 2</u> : Let $M > 0$. Let (e_n,μ_n) be a Besselian biorthogonal system in C with $M^{-1} \leq \|e_n\| \leq M$ for all n.
Assume that
(*) for every sequence (c_n) of scalars the condition

$$\sup_N \left\| \sum_{n=1}^{N} c_n\, \mu_n \right\| < \infty \quad \text{implies} \quad \lim_N \frac{1}{N} \sum_{n=1}^{N} c_n^2 = 0 \ .$$

Then (e_n) is not a basis for C.

<u>Proof</u> : If (μ_n) is not total then (e_n) is not a basis. Assume now that (μ_n) is total. Let ν and g_n have the same meaning as in Proposition 1. Assume to the contrary that (e_n) is a basis. Then

$$\sup_N \sup_{\|x\|=1} \left\| \sum_{n=1}^{N} \mu_n(x)\, e_n \right\| = L < \infty \ .$$

Hence for all $t \in [0,1]$:

$$\sup_N \sup_{\|x\|=1} \left| \sum_{n=1}^{N} \mu_n(x)\, e_n(t) \right| \leq L \ .$$

Let us note that

$$\sup_{\|x\|=1} \left| \sum_{n=1}^{N} \mu_n(x)\, e_n(t) \right| = \sup_{\|x\|=1} \left| \left(\sum_{n=1}^{N} e_n(t)\, \mu_n \right)(x) \right|$$

$$= \left\| \sum_{n=1}^{N} e_n(t)\, \mu_n \right\| .$$

Thus for all $t \in [0,1]$

$$\sup_N \left\| \sum_{n=1}^{N} e_n(t)\, \mu_n \right\| \le L .$$

Hence, by (*),

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} [e_n(t)]^2 = 0$$

for all $t \in [0,1]$.

Thus, by the Lebesgue theorem,

$$0 = \lim_N \int_0^1 \frac{1}{N} \sum_{n=1}^{N} e_n^2(t)\, d\nu = \lim_N \sum_{n=1}^{N} \int_0^1 e_n^2(t)\, d\nu .$$

On the other hand, by Proposition 1, there exists $\delta > 0$ such that $\int_0^1 e_n^2(t)\, d\nu \ge \delta$ for all n. Thus

$$\underline{\lim}_N \frac{1}{N} \sum_{n=1}^{N} \int_0^1 e_n^2(t)\, dt \ \ge\ \delta,$$

a contradiction.

Note that if (φ_n) is an orthonormal (in $L^2(\nu)$) system then

$$\mu_n(x) = \int_0^1 x(t)\, \overline{\varphi_n(t)}\, d\nu$$

and
$$\Big\| \sum_{n=1}^{N} c_n \mu_n \Big\| = \int_0^1 \Big| \sum_{n=1}^{N} c_n \varphi_n(t) \Big| \, d\nu .$$

Hence Theorem 1 is an immediate consequence of Proposition 1 and the following crucial result.

<u>Theorem 2</u> : Let ν be a probability measure on $[0,1]$. Let (φ_n) be a uniformly bounded (in $L^\infty(\nu)$) orthonormal (in $L^2(\nu)$) system. Then for every sequence of scalars (c_n) the condition

$$\sup_N \int_0^1 \Big| \sum_{n=1}^{N} c_n \varphi_n(t) \Big| \, d\nu = L < \infty$$

implies
$$\lim_N \frac{1}{N} \sum_{n=1}^{N} |c_n|^2 = 0 .$$

For the proof of Theorem 2 we shall need two lemmas.

<u>Lemma 1</u> : Let (a_n) be a sequence of real numbers such that $0 < a_n < K$ for all n and

$$\overline{\lim_n} \frac{1}{n} \sum_{j=1}^{n} a_j > \alpha > 0 .$$

Then for every N there exists indices m and k such that

$$\frac{1}{k} \sum_{j=m+k(r-1)+1}^{m+kr} a_j > \frac{\alpha}{2} \quad \text{for } r = 1,2,\dots,N .$$

<u>Proof</u> : Let ρ be an integer. Pick M so that $K N^\rho < M \frac{\alpha}{4}$ and $\frac{1}{M} \sum_{j=1}^{M} a_j > \alpha$.

Let $M = N^\rho q + r$ with $0 \le r < N^\rho$.

Then

$$\frac{1}{N^\rho q}\sum_{j=1}^{N^\rho q} a_j = \frac{M}{N^\rho q}\,\frac{1}{M}\sum_{j=1}^{M} a_j - \frac{1}{N^\rho q}\sum_{j=N^\rho q+1}^{M} a_j$$

$$\geq \frac{M}{N^\rho q}\,\alpha - \frac{1}{N^\rho q}\,K\,N^\rho$$

$$> \frac{M}{N^\rho q}\,\alpha - \frac{M}{N^\rho q}\,\frac{\alpha}{4}$$

$$\geq \frac{3}{4}\,\alpha\ .$$

Now consider the "blocks" $B^1_\nu = (a_j)_{N^{\rho-1}q(\nu-1)+1\ \leq\ j\ <\ N^{\rho-1}q\,\nu}$ for $1 \leq \nu \leq N$ and let $|B^1_\nu| = \sum_{j=N^{\rho-1}q(\nu-1)+1}^{N^{\rho-1}q\nu} a_j$.

If for all ν with $1 \leq \nu \leq N$, $|B^1_\nu| > N^{\rho-1}q\,\frac{\alpha}{2}$ we put $k = N^{\rho-1}q$ and $m = 1$ and we have N consecutive blocks satisfying the assertion of the lemma. If not the inequality $\frac{1}{N}\sum_{\nu=1}^{N}|B^1_\nu| = \frac{1}{N^\rho q}\sum_{j=1}^{N^\rho q} a_j > \frac{3}{4}\,\alpha$ yields the existence of an index ν_1 with $1 \leq \nu_1 \leq N$ such that

$$|B^1_{\nu_1}| > \frac{3}{4}\,\alpha\;\frac{N-\frac{2}{3}}{N-1}\;N^{\rho-1}q\ .$$

We divide the block $B^1_{\nu_1}$ into N consecutive blocks each of length $N^{\rho-3}q$, say $B^2_1, B^2_2, \ldots, B^2_N$.

If $|B^2_\nu| > \frac{1}{2}\,\alpha\,\frac{N-\frac{2}{3}}{N-1} > \frac{\alpha}{2}$ then we already have the desired division into N consecutive blocks. If not we infer that there exists an index ν_2 such that $|B^2_{\nu_2}| > \frac{3}{4}\,\alpha\,\left(\frac{N-\frac{2}{3}}{N-1}\right)^2 N^{\rho-2}q$ and we repeat the same procedure.

If we repeat the procedure ρ times we finally get a block $B^{\rho}_{\nu_\rho}$ of length q such that $|B^{\rho}_{\nu_\rho}| > \frac{3}{4}\,\alpha\,\left(\frac{N-\frac{2}{3}}{N-1}\right)^{\rho} q$. Since (for any block of length q) we have $|B^{\rho}_{\nu_\rho}| \le Kq$ we infer that $K > \frac{3}{4}\,\alpha\,\left(\frac{N-\frac{2}{3}}{N-1}\right)^{\rho}$ which for ρ large enough is impossible. That means that in some earlier step we must get the desired division into N consecutive blocks.

<u>Lemma 2</u> : Let a measurable function f satisfies the conditions

$$|f(t)| \le cn \quad \text{for} \quad t \in [0,1]$$

$$\int_0^1 |f(t)|^2\, d\nu \ \ge\ \frac{n}{c}$$

$$\int_0^1 |f(t)|\, d\nu \ \le\ c$$

Then
$$\nu\left\{ |f| > \frac{n}{c^3} \right\} \ge \frac{1}{n}\,\frac{c-1}{c^4} .$$

<u>**Proof**</u> :

$$\int_{\{|f| \ge \frac{n}{c^3}\}} |f|^2\, d\nu + \int_{\{|f| < \frac{n}{c^3}\}} |f|^2\, d\nu$$

$$\le n^2 c^2\, \nu\left\{ |f| > \frac{n}{c^3} \right\} + \frac{n}{c^3} \int_{\{|f| < \frac{n}{c^3}\}} |f|\, d\nu$$

$$\le n\, c^2\, \nu\left\{ |f| \ge \frac{n}{c^3} \right\} + \frac{n}{c^3}\, c .$$

Hence

$$\left(\frac{n}{c} - \frac{n}{c^2}\right) \frac{1}{n^2 c^2} = \frac{c-1}{n\, c^4} \ \le\ \nu\left\{ |f| \ge \frac{n}{c^3} \right\} .$$

Proof of Theorem 2 : Assume to the contrary that there exists a sequence of scalars (c_j) such that

$$\sup_n \int_0^1 \left| \sum_{j=1}^{n} c_j \varphi_j(t) \right| d\nu = M_1 < +\infty \quad \text{and} \quad \overline{\lim_n} \frac{1}{n} \sum_{j=1}^{n} |c_j|^2 = \alpha > 0 .$$

Let $\sup_j \|\varphi_j\|_\infty = M_2$. Then $1 = \int_0^1 |\varphi_j(t)|^2 d\nu \le M_2 \int_0^1 |\varphi_j(t)| d\nu$.

Hence for all j $|c_j| \int_0^1 |\varphi_j(t)| d\nu \le 2M_1$ and $\sup_j |c_j| \le 2M_1 M_2$.

Now fix a constant C so large that

$$C\alpha > 2 , \quad C > 2 , \quad C > 2M , \quad C > (2M_1 M_2)^2 .$$

Take v large enough and let $N = v^v$. By lemma 2, there exist m and k such that

$$\sum_{j=m+k(r-1)+1}^{m+kr} |c_j|^2 \ge \frac{\alpha}{2} k \qquad \text{for } 1 \le j \le v^v .$$

We shall define by induction the sequence $(i_s)_{1 \le s \le v}$ of the indices such that if

$$f_s = \sum_{j=m+1}^{m+k i_s} c_j \varphi_j \quad , \quad E_s = \left\{ |f_s| \ge \frac{k\, v^{v-s}}{2\, C^3} \right\}$$

then the following conditions are satisfied

(1) $\quad 1 \le i_s \le \dfrac{v^v - v^{v-s}}{v-1}$

(2) $\quad \displaystyle\int_{E_s} |f_s(t)| \, d\nu \ge s^{1/2}\beta$ where $\beta = \dfrac{C-1}{16C^7}$ for $s = 1,2,\ldots, \nu$.

Clearly having done this we get a contradiction because (2) in particular implies that $\int_0^1 |f_v(t)|\, d\nu \geq v^{1/2}\beta$ while

$$\int_0^1 |f_v(t)|\, d\nu \leq \int_0^1 \left|\sum_{j=1}^{m} c_j\, \varphi_j(t)\right| d\nu + \int_0^1 \left|\sum_{j=1}^{m+k\, i_v} c_j\, \varphi_j(t)\right| d\nu < C\ .$$

Hence $v < (\frac{2C}{\beta})^2$ which for v large enough is impossible.

The construction of $(i_s)_{1 \leq s \leq v}$: Let us set $i_1 = v^{v-1}$.

Then

$$\int_0^1 |f_1(t)|^2 d\nu = \sum_{j=m+1}^{m+k\, i_1} |c_j|^2 = \sum_{j=1}^{i_1} \sum_{j=m+k(r-1)+1}^{m+kr} |c_j|^2$$

$$\geq \frac{\alpha}{2}\, k\, i_1 = \frac{\alpha}{2}\, k\, v^{v-1} > \frac{1}{C}\, k\, v^{v-1}\ .$$

We also have

$$\int_0^1 |f_1(t)|\, d\nu < C\ ,$$

and

$$\sup_{t \in [0,1]} |f_1(t)| \leq k\, v^{v-1}(2 M_1 M_2)^2 \leq C\, k\, v^{v-1}\ .$$

Thus, by lemma 2,

$$\nu(|f_1| > \frac{k\, v^{v-1}}{C^3} \geq \frac{C-1}{kv^{v-1}C^4}\ .$$

Thus

$$\int_{\{|f_1| > \frac{kv^{v-1}}{C^3}\}} |f_1|\, d\nu \geq \frac{kv^{v-1}}{C^3}\ \frac{C-1}{kv^{v-1}C^4} = \frac{C-1}{C^7} > \beta\ .$$

Since $E_1 \subset \{|f_1| > \frac{kv^{v-1}}{C^3}\}$, we get $\int_{E_1} |f_1|\, d\nu > \beta$.

This completes the first step of induction.

Now assume that for some $s \le v-1$ the index i_s has been defined to satisfy the conditions (1) and (2). Let us set

$$\mathcal{U}_s = \{\frac{kv^{v-s-1}}{2C^3} \le |f_s| < \frac{kv^{v-s}}{2C^3}\} \quad , \quad \int_{\mathcal{U}_s} f_s(t)\, d\nu = \delta_s \quad .$$

We put

$$i_{s+1} = \begin{cases} i_s & \text{if } \delta_s \ge \beta \\ i_s + v^{v-s-1} & \text{if } \delta_s < \beta \end{cases} \quad .$$

Clearly $1 \le i_{s+1} \le i_s + v^{v-s-1} \le \frac{v-v^{v-s}}{v-1} + v^{v-s-1} = \frac{v-v^{v-s-1}}{v-1}$.

To complete the proof we have to check (2). Let us consider separately two cases:

1) $\delta_s \ge \beta$. Then $f_{s+1} = f_s$ and $E_{s+1} = E_s \cup \mathcal{U}_s$. Since $\mathcal{U}_s \cap E_s = \emptyset$, we get (by inductive hypothesis)

$$\int_{E_{s+1}} |f_{s+1}|\, d\nu = \int_{E_s} |f_s|\, d\nu + \int_{\mathcal{U}_s} |f_s|\, d\nu = s^{\frac{1}{2}}\beta + \beta > (s+1)^{\frac{1}{2}}\beta \; .$$

2) $\delta_s < \beta$. Let us set

$$F_s = \sum_{j=m+i_s k+1}^{m+i_{s+1}k} c_j\, \varphi_j \; .$$

Then

$$\int_0^1 |F_s|^2\, d\nu = \sum_{j=m+i_s k+1}^{m+i_{s+1}k} |c_j|^2 = \sum_{r=i_s+1}^{i_{s+1}} \sum_{j=m+k(r-1)+1}^{m+kr} |c_j|^2$$

$$\ge \frac{\alpha}{2}\, k\, (i_{s+1} - i_s) = \frac{\alpha}{2}\, k\, v^{v-s-1} > C\, k\, v^{v-s-1} \; .$$

We also have $$\int_0^1 |F_s| \, d\nu < C$$

and
$$\sup_{t \in [0,1]} |F_s(t)| \leq k\, v^{v-s-1}(2 M_1 M_2)^2 \leq C\, k\, v^{v-s-1} .$$

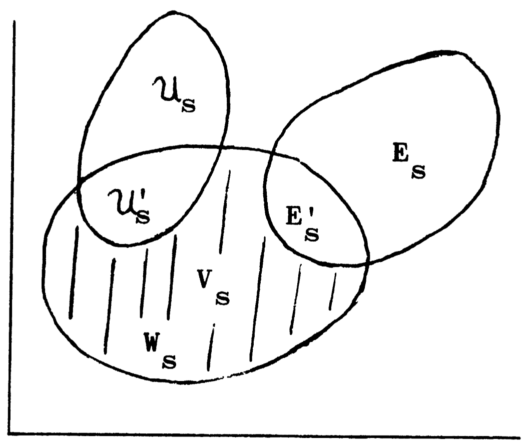
Let $V_s = \{ |F_s| \geq \dfrac{k\, v^{v-s-1}}{C^3} \}$.

Then, by lemma 2,
$$\nu(V_s) \geq \frac{C-1}{C^4\, k\, v^{v-s-1}} .$$

Let us set $E'_s = V_s \cap E_s$

$$\mathcal{U}'_s = V_s \cap \mathcal{U}_s$$

$$W_s = V_s \setminus (E'_s \cup \mathcal{U}'_s)$$



Picture 1

Clearly $\mathcal{U}_s \cap E_s = \emptyset$, $W_s \cap E_s = \emptyset$, $W_s \cap \mathcal{U}_s = \emptyset$.

We first show that $E_s \cup W_s \subset E_{s+1}$.

If a) $t \in E_s$ then

$$|f_{s+1}(t)| \geq |f_s(t)| - |F_s(t)| \geq \frac{k\, v^{v-s}}{2\, C^3} - C\, k\, v^{v-s-1} = \frac{k\, v^{v-s-1}}{2\, C^3} (v - 2\, C^4)$$
$$> \frac{k\, v^{v-s-1}}{2\, C^3}$$

for v large enough $(v > C^5)$.

b) $t \in W_s$ then $t \notin E_s$ and $t \notin \mathcal{U}_s$, that means that $t \in \{ |f_s| < \dfrac{k\, v^{v-s-1}}{2\, C^3} \}$.

On the other hand $t \in V_s = \{ |F_s| \geq \frac{k\, v^{v-s-1}}{C^3} \}$.

Thus
$$|f_{s+1}(t)| \geq |F_s(t)| - |f_s(t)| \geq \frac{k\, v^{v-s-1}}{2\, C^3} .$$

Now we separately estimate from below the integrals $\int_{E_s} |f_{s+1}|\, d\nu$ and $\int_{W_s} |f_{s+1}|\, d\nu$. We have for $t \in E_s$,

$$|f_{s+1}(t)| \geq |f_s(t)| - |F_s(t)| \geq |f_s(t)| - C\, k\, v^{v-s-1}$$

$$= |f_s(t)| - \frac{k\, v^{v-s}}{2\, C^3} \frac{2\, C^4}{v} \geq |f_s(t)| \; (1 - \frac{2\, C^4}{v}) .$$

Thus using the inductive hypothesis we get

$$\int_{E_s} |f_s(t)|\, d\nu \geq [1 - \frac{2\, C^4}{v}] \int_{E_s} |f_s(t)|\, d\nu \geq \beta\, s^{1/2} (1 - \frac{2\, C^4}{v}) .$$

Since $s < v$, for v large enough (precisely for $v > C^{10} > (2\, C^4)^2$) we have

$$\beta\, s^{1/2}\, (1 - \frac{2\, C^4}{v}) \geq \beta\, (s^{1/2} - 1) .$$

Hence
$$\int_{E_s} |f_s(t)|\, d\nu \geq \beta\, (s^{1/2} - 1) .$$

Now we estimate the second integral $\int_{W_s} |f_s(t)|\, d\nu$.

The inclusion $W_s \subset E_{s+1}$ yields
$$\int_{W_s} |f_{s+1}(t)|\, d\nu \geq \frac{k\, v^{v-s-1}}{2\, C^3}\, \nu(W_s) .$$

Thus our last aim is to estimate from below the measure of W_s. We have

$$\begin{aligned}\nu(W_s) &\geq \nu(V_s) - \nu(E'_s) - \nu(\mathcal{U}'_s)\\ &\geq \nu(V_s) - \nu(E_s) - \nu(\mathcal{U}_s)\\ &\geq \frac{C-1}{C^4\ k\ v^{v-s-1}} - \nu(E_s) - \nu(\mathcal{U}_s).\end{aligned}$$

We have

$$C \geq \int_0^1 |f_s(t)|\ d\nu \geq \int_{E_s} |f_s(t)|\ d\nu \geq \nu(E_s)\ \frac{k\ v^{v-s}}{2C^3},$$

thus

$$\nu(E_s) \geq \frac{2C^4}{k\ v^{v-s}}\ .$$

Similarly

$$\delta_s = \int_{\mathcal{U}_s} |f_s(t)|\ d\nu \geq \nu(\mathcal{U}_s)\ \frac{k\ v^{v-s-1}}{2\ C^3}\ ,$$

thus using the assumption that $\delta_s < \beta$, we have

$$\nu(\mathcal{U}_s) \leq \frac{\delta_s}{\frac{k\ v^{v-s-1}}{2C^3}} < \frac{2C^3\beta}{k\ v^{v-s-1}}\ .$$

Therefore

$$\nu(W_s) \geq \frac{1}{k\ v^{v-s-1}}\left(\frac{C-1}{C^4} - \frac{2\ C^4}{v} - 2\beta\ C^3\right).$$

Hence

$$\int_{W_s} |f_{s+1}(t)|\ d\nu \geq \frac{1}{2C^3}\left(\frac{C-1}{C^4} - \frac{2\ C^4}{v} - 2\beta\ C^3\right)$$

Thus for v large enough (remembering that $\beta = \dfrac{C-1}{16\ C^7}$) we get

$$\int_{W_s} |f_{s+1}(t)|\ d\nu \ \geq\ 2\beta\ .$$

Hence

$$\int_{E_{s+1}} |f_{s+1}(t)|\ d\nu \geq \int_{E_s} |f_{s+1}(t)|\ d\nu + \int_{W_s} |f_{s+1}(t)|\ d\nu \geq s^{1/2}\beta + \beta$$

$$\geq (s+1)^{1/2}\beta\ .$$

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