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The existence in every separable Banach space of a fundamental total and bounded biorthogonal sequence and related constructions of uniformly bounded orthonormal systems in L^2

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THE EXISTENCE IN EVERY SEPARABLE BANACH SPACE OF A FUNDAMENTAL TOTAL
AND BOUNDED BIORTHOGONAL SEQUENCE AND RELATED CONSTRUCTIONS
OF UNIFORMLY BOUNDED ORTHONORMAL SYSTEMS IN L^2

by

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Abstract.

1) In every separable Banach space X a biorthogonal sequence (x_n, x_n^*) is constructed such that linear combinations of the x_n 's are dense in X , for every x in X if $x_n^*(x) = 0$ for all n then $x = 0$ and $\sup_n \|x_n\| \|x_n^*\| < \infty$.

2) Linear subspaces of $L^2[0,1]$ which admit an orthonormal basis consisting of uniformly bounded functions are characterized.

The present paper consists of three sections.
In the first one using a trick invented by Olevskii ([9] Lemmas 3 and 4) we prove

Theorem 1 : In every separable Banach space X there exists a fundamental and total biorthogonal sequence (x_n, x_n^*) such that

$$\sup_n \|x_n\| \|x_n^*\| < \infty$$

Recall that a sequence (x_n, x_n^*) of pairs consisting of elements of a Banach space X and bounded linear functionals on X , i.e. elements of X^* - the dual of X is said to be biorthogonal if $x_n^*(x_m) = \delta_n^m$ for $n, m = 1, 2, \dots$. A biorthogonal sequence (x_n, x_n^*) is fundamental if linear combinations of the x_n 's are dense in X , and is total if the condition $x_n^*(x) = 0$ for $n = 1, 2, \dots$ implies that $x = 0$.

Theorem 1 answers a question of Banach ([1], p.238). A slightly weaker result has been previously obtained by Davis and Johnson [4].

The main result of the second section is

Theorem 2 : Let E be a separable linear subspace of a Hilbert space $L^2(\mu)$ where μ is a probability measure on a sigma field of subsets of a set S . Then E admits an orthonormal basis consisting of uniformly bounded functions if and only if

- (i) $E \cap L^\infty(\mu)$ is dense in E in the $L^2(\mu)$ norm,
- (ii) $E \cap \{f \in L^\infty(\mu) : \|f\|_\infty \leq 1\}$ is not a totally bounded subset of $L^2(\mu)$.

Moreover if $E \cap L^\infty(\mu)$ is a separable subspace of $L^\infty(\mu)$ then the orthonormal basis can be constructed so that it spans a linear subspace which is dense in the norm $\|\cdot\|_\infty$ in $E \cap L^\infty(\mu)$.

As a corollary we obtain that every subspace of $L^2(0,1)$ of finite codimension admits an orthonormal basis consisting of uniformly bounded infinitely many times differentiable functions. This answers a question of H. Shapiro [14].

In the third section we consider the class of such Banach spaces X which admit an isometric embedding, say j , into a space $C(S)$ of all scalar-valued continuous functions on a compact Hausdorff space S such that there exists a Borel probability measure μ on S such that the unit ball of $j(X)$ is not a totally bounded subset of $L^2(\mu)$, i.e. $j(X)$ regarded as a subspace of $L^2(\mu)$ satisfies the condition (ii) of Theorem 2. Using a recent profound result of Rosenthal [13] we show that a Banach space X has the above property if and only if it contains a closed linear subspace isomorphic to the space l^1 of all absolutely convergent series of scalars.

1. Proof of Theorem 1 . We begin with a lemma which is a modification of Olevskii's Lemma 3 of [9]. If A is a non-empty subset of a Banach space X , then $[A]$ denotes the closed linear subspace of X generated by A and $\text{lin } A$ - the linear subspace of X generated by A .

Lemma 1 : Let X be a Banach space and let n be a positive integer. Let $x_0, x_1, \dots, x_{2^n-1}$ be elements of X and let $x_0^*, x_1^*, \dots, x_{2^n-1}^*$ be elements of X^* such that $x_p^*(x_q) = \delta_p^q$ for $p = 0, 1, \dots, 2^n-1$. Then there exists a unitary real matrix $(a_{k,j}^n)_{0 \leq k, j < 2^n}$ such that if

$$e_k = \sum_{j=0}^{2^n-1} a_{k,j}^n x_j \quad \text{for } k = 0, 1, \dots, 2^n-1,$$

and

$$e_k = \sum_{j=0}^{2^n-1} a_{k,j}^n x_j^*$$

then

$$(1) \quad \max_{0 \leq p < 2^n} \|e_p\| < (1 + \sqrt{2}) \max_{1 \leq j < 2^n} \|x_j\| + 2^{-\frac{n}{2}} \|x_0\|$$

$$(2) \quad \max_{0 \leq p < 2^n} \|e_p^*\| < (1 + \sqrt{2}) \max_{1 \leq j < 2^n} \|x_j^*\| + 2^{-\frac{n}{2}} \|x_0^*\|$$

$$(3) \quad e_p^*(e_q) = \delta_p^q \quad \text{for } p, q = 0, 1, \dots, 2^n - 1$$

$$(4) \quad [\{e_p\}_{0 \leq p < 2^n}] = [\{x_p\}_{0 \leq p < 2^n}] ; \quad [\{e_p^*\}_{0 \leq p < 2^n}] = [\{x_p^*\}_{0 \leq p < 2^n}].$$

Proof : The conditions (3) and (4) are satisfied for every unitary $2^n \times 2^n$ - matrix. The specific unitary matrix for which (1) and (2) hold is defined to be the matrix which transform the unit vector basis of the 2^n -dimensional Hilbert space $l_{2^n}^2$ onto the Haar basis of this space. We put

$$a_{k,0}^n = 2^{-\frac{n}{2}} \quad \text{for } 0 \leq k < 2^n ,$$

$$a_{k,2^s+r}^n = \begin{cases} 2^{\frac{s-n}{2}} & \text{for } 2^{n-s-1}2r \leq k < 2^{n-s-1}(2r+1) \\ -2^{\frac{s-n}{2}} & \text{for } 2^{n-s-1}(2r+1) \leq k < 2^{n-s-1}(2r+2) \\ 0 & \text{for } k < 2^{n-s-1}2r \text{ and for } k \geq 2^{n-s-1}(2r+2) . \end{cases}$$

$$(s = 0, 1, \dots, n-1 ; r = 0, 1, \dots, 2^s - 1) .$$

We have

$$(5) \quad \sum_{j=1}^{2^n-1} |a_{k,j}^n| = \sum_{s=0}^{n-1} 2^{-\frac{n-s}{2}} < 1 + \sqrt{2} \quad \text{for } 0 \leq k < 2^n .$$

Clearly (5) implies (1) and (2).

Proposition 1 : Let (x_n, x_n^*) be a fundamental and total biorthogonal sequence in a Banach space X such that there exists an increasing infinite sequences (n_k) such that $\sup_n \|x_{n_k}\| \|x_{n_k}^*\| = M < \infty$. Then there exists

a fundamental and total biorthogonal sequence (e_n, e_n^*) in X such that

$$\sup_n \|e_n\| \|e_n^*\| \leq M(1 + \sqrt{2})^2 + 1$$

$$\text{and} \quad \text{lin} \{e_n\}_{n=1}^\infty = \text{lin} \{x_n\}_{n=1}^\infty$$

$$\text{and} \quad \text{lin} \{e_n^*\}_{n=1}^\infty = \text{lin} \{x_n^*\}_{n=1}^\infty.$$

Proof : Without loss of generality one may assume that $\|x_n\| = 1$ for all n . Pick a permutation $p(\cdot)$ of the indices and an increasing sequence (m_r) of the indices so that if $\tilde{x}_n = x_{p(n)}$ and $\tilde{x}_n^* = x_{p(n)}^*$ for all n and $q_r = \sum_{p=0}^r 2^p$ for all r then

$$\text{if } n \neq q_r \text{ for all } r, \text{ then } \|\tilde{x}_n\| \|\tilde{x}_n^*\| \leq M,$$

if $n = q_r$ for some $r = 0, 1, \dots$, then

$$(1 + \sqrt{2})^2 M + 1 > [(1 + \sqrt{2}) M + \|\tilde{x}_n^*\| 2^{-\frac{m_r}{2}}] [(1 + \sqrt{2}) + \|\tilde{x}_n\| 2^{-\frac{m_r}{2}}].$$

Next we put

$$e_n = \tilde{x}_n \quad \text{and} \quad e_n^* = \tilde{x}_n^* \quad \text{for } n < 2^{m_0},$$

$$e_{k+q_{r-1}} = \sum_{j=0}^{2^{m_r}-1} a_{k,j}^{m_r} \tilde{x}_{j+q_{r-1}}; \quad e_{k+q_{r-1}}^* = \sum_{j=0}^{2^{m_r}-1} a_{k,j}^{m_r} \tilde{x}_{j+q_{r-1}}^*$$

$$\text{for } 0 \leq k < 2^{m_r}; \quad r = 1, 2, \dots$$

where $a_{k,j}^{m_r}$ are defined as in Lemma 1 for $n = m_r$. Using Lemma 1 we easily verify that such defined sequence (e_n, e_n^*) has the desired properties.

Proof of Theorem 1 : We shall assume that $\dim X = \infty$. Then the separability of X implies that there exist sequences $E_1 \subset E_2 \subset \dots$ of subspaces of X and $F_1 \subset F_2 \subset \dots$ of subspaces of X^* such that $\dim E_i = \dim F_i = i$ for $i = 1, 2, \dots$, $\bigcup_{i=1}^{\infty} E_i$ is dense in X and if $f^*(x) = 0$ for all $f^* \in \bigcup_{i=1}^{\infty} F_i$

then $x = 0$. In view of Proposition 1 it is enough to construct a biorthogonal sequence (x_n, x_n^*) in X such that if $G = [x_1, x_2, \dots, x_n]$ and $H_n = [x_1^*, x_2^*, \dots, x_n^*]$ then for all s

$$(6) \quad G_{3s-1} \supset E_s \quad ; \quad H_{3s-1} \supset F_s \quad ; \quad \|x_{3s}\| \quad \|x_{3s}^*\| \leq 3.$$

Pick $x_1 \in X$ and $x_1^* \in X^*$ so that $0 \neq x_1 \in E_1$ and $x_1^*(x_1) = 1$. Assume that for some $n-1 \geq 1$ the elements $x_1, x_2, \dots, x_{n-1}$ in X and the functionals $x_1^*, x_2^*, \dots, x_{n-1}^*$ in X^* have been defined to satisfy (6) and so that $x_p^*(x_q) = \delta_p^q$ for $p, q = 1, 2, \dots, n-1$. We consider separately three cases.

1°) $n = 3s-2$. If $G_{n-1} \supset E_s$ we define $x_n \in X$ and $x_n^* \in X^*$ arbitrarily so that $x_n^*(x^q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p, q = 1, 2, \dots, n$. If $E_s \setminus G_{n-1}$ is non empty, say $e \in E_s \setminus G_{n-1}$, then we put $x_n = e - \sum_{p=1}^{n-1} x_p^*(e) x_p$ and

$G_n = [G_{n-1} \cup \{x_n\}]$. Clearly $x_n \neq 0$. Since $\dim E_s = \dim E_{s-1} + 1$ and $e \in G_n \setminus E_{s-1}$ and since the inductive hypothesis implies that $E_{s-1} \subset G_{n-1}$, we infer that $G_n \supset E_s$. Since $x_n \in G_n \setminus G_{n-1}$, there exists a bounded linear functional on G_n , say g^* , such that $g^*(x_n) = 1$ and $g^*(g) = 0$ for $g \in G_{n-1}$. We define x_n^* to be any extension of g^* to a bounded linear functional on X .

2°) $n = 3s-1$. If $H_{n-1} \supset F_s$ we define $x_n \in X$ and $x_n^* \in X^*$ arbitrarily so that $x_n^*(x_q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p, q = 1, 2, \dots, n$. If $F_s \setminus H_{n-1}$ is non empty, say $f^* \in F_s \setminus H_{n-1}$ then we put $x_n^* = f^* - \sum_{q=1}^{n-1} f^*(x_q) x_q^*$.

Since $f^* \notin H_{n-1}$, there exists an $x \in X$ such that $1 = f^* - \sum_{q=1}^{n-1} f^*(x_q) x_q^*(x)$.

We put $x_n = x - \sum_{p=1}^{n-1} x_p^*(x)x_p$. It is easy to check that $x_n^*(x_q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p, q = 1, 2, \dots, n$. Let $H_n = [H_{n-1} \cup \{x_n\}]$. Since the inductive hypothesis implies that $F_{s-1} \subset H_{n-1}$ and since $\dim F_s = \dim F_{s-1} + 1$ and $f^* \in F_s \setminus F_{s-1}$, we infer that $H_n \supset F_s$.

3°) $n = 3s$. Using Mazur's technique (cf. [10] Lemma) we pick an $x_n \in X$ with $\|x_n\| = 1$ so that $x_n^*(x_n) = 0$ for every $x_n^* \in H_{n-1}$ and for all g in G_{n-1} and for all scalars t , $\|g + t x_n\| \geq (1 - 1/3) \|g\|$. Define g^* on G_n by $g^*(g + t x_n) = t$. Then $|t| = \|t x_n\| \leq \|g + t x_n\| + \|g\| \leq (1 + 3/2) \|g + t x_n\|$.

Thus $\|g^*\| \leq 3$. We define x_n^* to be any norm preserving extension of g^* to a linear functional on X .

Remark 1 : Using in the case 3° Day's technique (cf. [3]) which bases on the Borsuk antipodal mapping theorem one can choose (both in the case of real and of complex scalars) x_{3s} and x_{3s}^* so that $\|x_{3s}\| = \|x_{3s}^*\| = x_{3s}^*(x_{3s}) = 1$ for $s = 1, 2, \dots$. Now the inspection of the proof of Theorem 1 yields that in every separable Banach space for every $\varepsilon > 0$ there exists a fundamental and bounded biorthogonal sequence (e_n, e_n^*) such that $\|e_n\| \|e_n^*\| < (1 + \sqrt{2})^2 + \varepsilon$ for all n . We do not know whether for every $\varepsilon > 0$ this bound can be replaced by $1 + \varepsilon$. However, as was observed by C. Bessaga we have

Corollary 1 : In every separable Banach space X there exists an equivalent norm $||| \cdot |||$ such that there exists in X a fundamental and total biorthogonal sequence (e_n, e_n^*) with $|||e_n^*||| |||e_n^*||| = 1$.

Proof : We admit $|||x||| = \max(\|x\|, \sup_n |e_n^*(x)|)$ for $x \in X$ where (e_n, e_n^*) is any fundamental and total biorthogonal sequence in X such that $\|e_n\| = 1$ for all n and $\sup_n \|e_n^*\| < \infty$.

Remark 2 : A similar argument to that which is used in the proof of Theorem 1 allows to prove the following

Theorem 1' : Let X and Y be Banach spaces and let $T : X \rightarrow Y$ be one-to-one bounded linear operator. If X is separable, $T(X)$ is dense in Y and T is not compact, then there exists fundamental and total biorthogonal sequences (x_n, x_n^*) in X and (y_n, y_n^*) in Y such that

$$\sup_n \max (\|x_n\| \|x_n^*\|, \|y_n\| \|y_n^*\|) < \infty$$

and

$$T(x_n) = y_n$$

for all n .

2. Constructions of uniformly bounded orthonormal sequences.

We employ the following notation. If μ is a probability measure (= a non negative normalized measure) on a sigma field of subsets of a set S then $\langle x, y \rangle = \int_S x(s) y(s) \mu(ds)$, $\|x\|_2 = \langle x, x \rangle^{1/2}$ and

$$\|x\|_\infty = \inf_{\mu(B)=1} \sup_{s \in B} |x(s)|$$

for any μ -absolutely square summable scalar valued functions x and y on S . $L^\infty(\mu)$ and $L^2(\mu)$ denote as usually the Banach spaces of those x that $\|x\|_\infty < \infty$ and $\|x\|_2 < \infty$ respectively.

The proof of Theorem 2 is similar to the proof of Theorem 1. Instead of Proposition 1, we apply the following result due to Olevskii ([9], Lemma 4).

Proposition 2 : Let μ be a probability measure on a sigma field of subsets of a set S . Let (x_n) be an infinite orthonormal (with respect to the inner product $\langle \cdot, \cdot \rangle$) sequence of functions in $L^\infty(\mu)$ such that $\liminf_n \|x_n\|_\infty < \infty$. Then there exists an orthonormal sequence (e_n) such that

$$\lim_{n \rightarrow \infty} \|x_n - e_n\|_2 = 0$$

and

$$\sup_n \|e_n\|_\infty.$$

The proof of Proposition 2 can be obtained by a non essential modification of the proofs of Lemma 1 and Proposition 1. Actually Olevskiĭ stated Proposition 2 for the Lebesgue measure on $[0,1]$.

To prove Theorem 2 it is convenient to use the following simple fact.

Lemma 2 : Let (g_n) be a normalized sequence in $L^2(\mu)$ which weakly (in $L^2(\mu)$) converges to zero and let $\sup_n \|g_n\|_\infty = M < \infty$. Then for every finite dimensional subspace of $L^\infty(\mu)$, say F , and for $k > 0$ there exist an index $n_0 > k$ and a function h in the orthogonal complement of F such that

$$[F \cup \{g_n\}] = [F \cup \{h\}] , \quad \|h\|_2 = 1$$

and

$$\|h\|_\infty < M + 2^{-k}.$$

Proof : Let $p = \dim F$. Let $e_1, e_2, \dots, e_p$ be any orthonormal basis for F . Pick $\varepsilon > 0$ so that

$$\frac{M + \varepsilon \sum_{j=1}^p \|e_j\|_\infty}{1 - \varepsilon p} < M + 2^{-k}.$$

Since (g_n) converges weakly to 0 in $L^2(\mu)$, there exists an index $n_0 > k$ such that $|\langle g_{n_0}, e_j \rangle| < \varepsilon$ for $1 \leq j \leq p$. Put

$$h = (g_{n_0} - \sum_{j=1}^p \langle g_{n_0}, e_j \rangle e_j) \left\| g_{n_0} - \sum_{j=1}^p \langle g_{n_0}, e_j \rangle e_j \right\|_2^{-1}.$$

Clearly h belongs to the orthogonal complement of F , $\|h\|_2 = 1$ and

$$[F \cup \{g_{n_0}\}] = [F \cup \{h\}].$$

We have

$$\begin{aligned} \left\| g_{n_0} - \sum_{j=1}^p \langle g_{n_0}, e_j \rangle e_j \right\|_{\infty} &\geq \left\| g_{n_0} \right\|_{\infty} + \left\| \sum_{j=1}^p \langle g_{n_0}, e_j \rangle e_j \right\|_{\infty} \\ &\leq M + \varepsilon \sum_{j=1}^p \|e_j\|_{\infty} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \left\| g_{n_0} - \sum_{j=1}^p \langle g_{n_0}, e_j \rangle e_j \right\|_2 &\geq \left\| g_{n_0} \right\|_2 - \left\| \sum_{j=1}^p \langle g_{n_0}, e_j \rangle e_j \right\|_2 \\ &\geq 1 - \varepsilon p . \end{aligned}$$

$$\text{Thus } \|h\|_{\infty} \leq (M + \varepsilon \sum_{j=1}^p \|e_j\|_{\infty}) (1 - \varepsilon p) < M + 2^{-k} .$$

Proof of Theorem 2 : It follows from (i) that there exists in E an increasing sequence of finite dimensional subspaces $F_1 \subset F_2 \subset \dots$ such that $\dim F_p = p$ and $\bigcup_{p=1}^{\infty} F_p$ is dense in E . Clearly if $E \cap L^{\infty}(\mu)$ is a separable subset of $L^{\infty}(\mu)$ one can choose the sequence (F_p) so that the union $\bigcup_{p=1}^{\infty} F_p$ is dense in $E \cap L^{\infty}(\mu)$ in the $L^{\infty}(\mu)$ norm. The condition (ii) yields that there exists in E a sequence (g_n) satisfying the assumption of Lemma 2. In view of Proposition 2, it is enough to define inductively an orthonormal sequence (h_n) in $L^{\infty}(\mu) \cap E$ so that for $s = 1, 2, \dots$

$$(7) \quad [\{h_1, h_2, \dots, h_{2s-1}\}] \supset F_s ,$$

$$(8) \quad \|h_{2s}\|_{\infty} < M + 2^{-s}$$

$$\text{where } M = \sup_n \|g_n\|_{\infty} .$$

We define h_1 as any element of F_1 with $\|h_1\|_2 = 1$. Suppose that for some $n-1 \geq 1$ the functions $h_1, h_2, \dots, h_{n-1}$ have been defined to satisfy the conditions (7) and (8) and so that $\langle h_p, h_q \rangle = \delta_p^q$ for $p, q = 1, 2, \dots, n-1$. Let us consider separately two cases.

- 1) $n = 2s$ for some $s = 1, 2, \dots$. We put $h_n = h$ where h is that of Lemma 2 applied for $F = [\{h_1, h_2, \dots, h_{n-1}\}]$ for (g_p) and for $k = s$.
- 2) $n = 2s-1$ for some $s = 2, 3, \dots$. If $F_s \subset [\{h_1, h_2, \dots, h_{n-1}\}]$ we again define $h_n = h$ where h is that of Lemma 2 applied for $F = [\{h_1, h_2, \dots, h_{n-1}\}]$ for (g_p) and for $k=1$. If $F_m \not\subset [\{h_1, h_2, \dots, h_{n-1}\}]$ then there exists an f which belongs to $F_s \setminus [\{h_1, h_2, \dots, h_{n-1}\}]$. Let $\tilde{f}$ be the orthogonal projection of f onto $[\{h_1, h_2, \dots, h_{n-1}\}]$. We put $h_n = (f - \tilde{f}) \|f - \tilde{f}\|_2^{-1}$. Clearly $\|h_n\|_2 = 1$ and h_n belongs to the orthogonal complement of $[\{h_1, h_2, \dots, h_{n-1}\}]$. Obviously we have $f \in [\{h_1, h_2, \dots, h_1\}] \setminus [\{h_1, h_2, \dots, h_{n-1}\}]$.
- By the inductive hypothesis $F_{s-1} \subset [\{h_1, h_2, \dots, h_{n-1}\}]$. Thus $F_s \subset [\{h_1, h_2, \dots, h_n\}]$ because $\dim F_s = \dim F_{s-1} + 1$.

This completes the induction and the proof of the sufficiency of the conditions (i) and (ii). The necessity is trivial.

Remark 1 : A similar argument gives

Theorem 2' : Let $T : X \rightarrow H$ be a one to one bounded linear operator from a Banach space X into a Hilbert space H . Let $E = T(X)$. If E is separable and T is not compact then there exists a sequence (x_n) in X such that $\sup_n \|x_n\| < \infty$ and $(T(x_n))$ is an orthonormal basis for E . Moreover if X is separable and $x_n^* \in X^*$ is defined by $x_n^*(x) = \langle T(x), x_n \rangle_H$ for $x \in X$ and for $n = 1, 2, \dots$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product of H , then (x_n) can be chosen so that (x_n, x_n^*) is a fundamental and total biorthogonal sequence in X and $\sup_n \|x_n\| \|x_n^*\| < \infty$.

Remark 2 : There exists an orthonormal decomposition of $L^2[0,1]$ onto subspaces E_1 and E_2 such that neither E_1 nor E_2 admit uniformly bounded orthonormal bases. It is enough to define $E_1 = [\{x_1\} \cup \{x_{2m}\}_{m=2}^\infty]$ and $E_2 = [\{x_2\} \cup \{x_{2m-1}\}_{m=2}^\infty]$ where (x_n) is any orthonormal basis for $L^2[0,1]$ such that the functions x_1 and x_2 are unbounded, $x_{2m-1}(t) = 0$ for $0 \leq t < \frac{1}{2}$ and $x_{2m}(t) = 0$ for $\frac{1}{2} < t \leq 1$ ($m=1, 2, \dots$). However as was observed earlier by F.G. Arutunian (unpublished) we have

Corollary 2 : If E is a linear subspace of a separable space $L^2(\mu)$ where μ is a non purely atomic probability measure and if the orthogonal complement of E is finite dimensional, then $[E]$ has a uniformly bounded orthonormal basis. Moreover if $E \cap L^\infty(\mu)$ is dense in E then the basis can be chosen from elements of $E \cap L^\infty(\mu)$.

Proof : It is enough to show that $[E]$ satisfies the conditions (i) and (ii) of Theorem 2. To check (i) first observe that the density of $L^\infty(\mu)$ regarded as a subspace of $L^2(\mu)$ in $L^2(\mu)$ implies that for every positive integer p and for every linearly independent $f_1, f_2, \dots, f_{p+1}$ in $L^2(\mu)$ there exist $y_1, y_2, \dots, y_{p+1}$ in $L^\infty(\mu)$ such that the matrix $(y_k, f_j)_{1 \leq k, j \leq p+1}$ is invertible. Let $(a_{i,k})_{1 \leq i, k \leq p+1}$ be the inverse matrix and let

$$z_i = \sum_{k=1}^{p+1} a_{i,k} y_k \text{ for } i = 1, 2, \dots, p+1. \text{ Then } z_i \in L^\infty(\mu) \text{ and } \langle z_i, f_j \rangle = \delta_i^j$$

for $i = 1, 2, \dots, p+1$. The above observation applied to any basis of the orthogonal complement of E and any non zero element f of $[E]$ yields the existence of an y in $L^\infty(\mu)$ such that $\langle y, f \rangle = 1$ and $\langle y, g \rangle = 0$ for all g in the orthogonal complement of E . The last condition means that $y \in [E]$. Hence there is no $f \neq 0$ in $[E]$ which is orthogonal to all $y \in [E] \cap L^\infty(\mu)$, equivalently $[E] \cap L^\infty(\mu)$ is dense in $[E]$. Hence $[E]$ satisfies (i).

The "moreover" part of the Corollary follows from the observation that if $[E]$ satisfies (ii) than E also satisfies (ii).

An immediate consequence of Corollary 2 is

Corollary 3 : Let f be any unbounded function in $L^2[0,1]$. Then the orthogonal complement of f admits a uniformly bounded orthonormal basis consisting of trigonometrical polynomials. This basis has no extension to any uniformly bounded orthonormal basis for $L^2[0,1]$.

Corollary 3 answers a question of Shapiro [14].

3. Fat subspaces of $C(S)$ spaces.

Definition : Let μ be a probability Borel measure on a compact Hausdorff space S . A closed linear subspace Z of $C(S)$ is said to be fat with respect to μ if the unit ball of Z regarded as a subset of the Hilbert space $L^2(\mu)$ is not a totally bounded set.

Let $I_\mu : L^\infty(\mu) \rightarrow L^2(\mu)$ denote the natural injection. It is clear that Z is fat with respect to μ iff the restriction of I_μ to Z is not a compact operator or equivalently if $E = I_\mu(Z)$ satisfies the condition (ii) of Theorem 2.

Our next result characterizes Banach spaces which admit fat isometric embeddings into $C(S)$ spaces. Some of the equivalent conditions are stated in terms of 2-absolutely summing operators, i.e. such bounded linear operators which admit a factorization through a natural injection I_μ for some measure μ (cf. [12] and [8]).

Proposition 3 : For every Banach space X the following conditions are equivalent :

- (a) there exists a uniformly bounded sequence (x_n) of elements of X such that no subsequence of (x_n) is a weak Cauchy sequence,
- (b) X contains a subspace isomorphic to l^1 ,
- (c) there exists a 2-absolutely summing operator from X onto l^2 ,
- (d) there exists a 2-absolutely summing non compact operator from X into l^2 ,
- (e) for every for some isometric embedding j of X into a $C(S)$ space there exists a probability Borel measure μ on S such that $j(X)$ is fat with respect to μ .

Proof : (a) $\Rightarrow$ (b). This is a profound recent result of Rosenthal [13].
 (b) $\Rightarrow$ (c). Let T be a bounded linear operator from l^1 onto l^2

(cf. [2] for the existence of such an operator). Then by a result of Grothendieck [7] (cf. also [8]) T is 2-absolutely summing. Hence, by [12] T admits an extension to a 2-absolutely summing operator from X onto l^2 .

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (e). Let $T : X \rightarrow l^2$ be a non compact 2-absolutely summing operator and let S be a compact Hausdorff space. By a result of Persson and Pietsch [11], for every isometric embedding $j : X \rightarrow C(S)$ there exists a Borel probability measure μ on S such that $T = A I_\mu j$ for some bounded linear operator $A : L^2(\mu) \rightarrow l^2$. Since T is not compact, the image of the unit ball of $j(X)$ under I_μ is not a totally bounded subset of $L^2(\mu)$. Thus $j(X)$ is a fat subspace of $C(S)$ with respect to μ .

(e) $\Rightarrow$ (a). It follows from (e) that there exists a uniformly bounded sequence (x_n) in X such that $\|I_\mu j(x_n) - I_\mu j(x_m)\|_2 \geq 1$ for $n \neq m$ ($n, m = 1, 2, \dots$). Thus the sequence (x_n) does not contain weak Cauchy sequences because I_μ takes weak Cauchy sequences into strong Cauchy sequences.

A similar result to our Proposition 3 was recently independently discovered by Weis [16].

Our last result is related to Gaposhkin's [6] generalization of a result of Sidon [15].

Corollary 4 : Let μ be a probability measure on a sigma field of subsets of S . Let (g_n) be a uniformly bounded sequence in $L^\infty(\mu)$ such that (g_n) tends weakly to zero in $L^2(\mu)$ and $\limsup_n \|g_n\|_2 > 0$. Then there exists an infinite subsequence (g_{n_k}) and $c > 0$ such that

$$\left\| \sum_{k=1}^p c_k g_{n_k} \right\|_\infty > c \sum_{k=1}^p |c_k|$$

for every finite sequence of scalars $c_1, c_2, \dots, c_p$ ($p = 1, 2, \dots$).

Proof : Without loss of generality we may assume that $\inf_n \|g_n\|_2 > 0$.

Then (g_n) does not have Cauchy (in $L^2(\mu)$) subsequences because (g_n) weakly converges in $L^2(\mu)$ to zero but no subsequence of (g_n) strongly converges to zero. Thus (g_n) regarded as a sequence of elements of $L^\infty(\mu)$ does not contain weak (in $L^\infty(\mu)$) Cauchy sequences because the natural injection $I_\mu : L^\infty(\mu) \rightarrow L^2(\mu)$ takes weak Cauchy sequences in $L^\infty(\mu)$ into strong Cauchy sequences in $L^2(\mu)$. Since $\sup_n \|g_n\|_\infty < \infty$, to complete

the proof it is enough to apply Rosenthal's criterion (cf. Rosenthal [13] for the real case and Dor [5] for the complex case).

REFERENCES

- [1] S. BANACH : Théorie des opérations linéaires, Monografie Mat., Warszawa 1932.
- [2] S. BANACH und S. MAZUR : Zur Theorie der linearen Dimension, Studia Math. 4 (1933), 100 - 112.
- [3] M.M. DAY : On the basis problem in normed linear spaces, Proc. Amer. Math. Soc. 13 (1962), 655 - 658.
- [4] W.J. DAVIS and W.B. JOHNSON : On the existence of fundamental and total bounded biorthogonal systems in Banach spaces, Studia Math. 45 (1973), 173 - 179.
- [5] L. DOR : On sequences spanning a complex l_1 space, to appear.
- [6] V.F. GAPOSHKIN : Lacunar series and independent functions, Usp. Mat. Nauk, 21 (132) (1966), 3 - 82 (Russian).
- [7] A. GROTHENDIECK : Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Matem. Sao Paulo 8 (1956), 1 - 79.
- [8] J. LINDENSTRAUSS and A. PEŁCZYŃSKI : Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications, Studia Math. 29 (1968), 275 - 326.
- [9] A. M. OLEVSKIĬ : Fourier series of continuous functions with respect to bounded orthonormal systems, Izv. Akad. Nauk SSSR, Ser. Mat. 30 (1966), 387 - 432 (Russian).

- [10] A. PEŁCZYŃSKI : A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces", *Studia Math.* 21 (1962), 371 - 374.
- [11] A. PERSSON und A. PIETSCH : p-nukleare und p-integrale Abbildungen in Banachräumen, *Studia Math.* 33 (1969), 19-62.
- [12] A. PIETSCH : Absolutely p-summierende Abbildungen in normierten Räumen, *Studia Math.* 28 (1967), 333 - 353.
- [13] H.P. ROSENTHAL : A characterization of Banach spaces containing l^1 , *Proc. Nat. Acad. Sci. USA* to appear.
- [14] H.S. SHAPIRO : Incomplete orthogonal families and related question on orthogonal matrices, *Michigan J. Math.* 11 (1964), 15 - 18.
- [15] S. SIDON : Über orthogonalen Entwicklungen, *Acta Math. Szeged* 10 (1943), 206 - 253.
- [16] L. WEISS : On strictly singular and strictly cosingular operators in preparation.
