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THE EXISTENCE IN EVERY SEPARABLE BANACH SPACE OF A FUNDAMENTAL TOTAL AND BOUNDED BIORTHOGONAL SEQUENCE AND RELATED CONSTRUCTIONS OF UNIFORMLY BOUNDED ORTHONORMAL SYSTEMS IN L^2

by

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Abstract.

1) In every separable Banach space X a biorthogonal sequence (x_n, x_n^*) is constructed such that linear combinations of the x_n's are dense in X, for every x in X if $x_n^*(x) = 0$ for all n then $x = 0$ and $\sup_n \|x_n\| \|x_n^*\| < \infty$.

2) Linear subspaces of $L^2[0,1]$ which admit an orthonormal basis consisting of uniformly bounded functions are characterized.

The present paper consists of three sections. In the first one using a trick invented by Olevskiĭ ([9] Lemmas 3 and 4) we prove

<u>Theorem 1</u> : In every separable Banach space X there exists a fundamental and total biorthogonal sequence (x_n, x_n^*) such that

$$\sup_n \|x_n\| \, \|x_n^*\| < \infty .$$

Recall that a sequence (x_n, x_n^*) of pairs consisting of elements of a Banach space X and bounded linear functionals on X, i.e. elements of X^* - the dual of X is said to be biorthogonal if $x_n^*(x_m) = \delta_n^m$ for $n,m = 1,2,\ldots$ A biorthogonal sequence (x_n, x_n^*) is fundamental if linear combinations of the x_n's are dense in X, and is total if the condition $x_n^*(x) = 0$ for $n = 1,2,\ldots$ implies that $x = 0$.

Theorem 1 answers a question of Banach ([1], p.238). A slightly weaker result has been previously obtained by Davis and Johnson [4].

The main result of the second section is

<u>Theorem 2</u> : Let E be a separable linear subspace of a Hilbert space $L^2(\mu)$ where μ is a probability measure on a sigme field of subsets of a set S. Then E admits an orthonormal basis consisting of uniformly bounded functions if and only if

(i) $E \cap L^\infty(\mu)$ is dense in E in the $L^2(\mu)$ norm,
(ii) $E \cap \{f \in L^\infty(\mu) : \|f\|_\infty \le 1\}$ is not a totally bounded subset of $L^2(\mu)$.

Moreover if $E \cap L^\infty(\mu)$ is a separable subspace of $L^\infty(\mu)$ then the orthonormal basis can be constructed so that it spans a linear subspace which is dense in the norm $\|\cdot\|_\infty$ in $E \cap L^\infty(\mu)$.

As a corollary we obtain that every subspace of $L^2(0,1)$ of finite codimension admits an orthonormal basis consisting of uniformly bounded infinitely many times differentiable functions. This answers a question of H. Shapiro [14].

In the third section we consider the class of such Banach spaces X which admit an isometric embedding, say j, into a space C(S) of all scalar-valued continuous functions on a compact Hausdorff space S such that there exists a Borel probability measure μ on S such that the unit ball of j(X) is not a totally bounded subset of $L^2(\mu)$, i.e. j(X) regarded as a subspace of $L^2(\mu)$ satisfies the condition (ii) of Theorem 2. Using a recent profound result of Rosenthal [13] we show that a Banach space X has the above property if and only if it contains a closed linear subspace isomorphic to the space l^1 of all absolutely convergent series of scalars.

1. Proof of Theorem 1 . We begin with a lemma which is a modification of Olevskiĭ's Lemma 3 of [9]. If A is a non-empty subset of a Banach space X, then [A] denotes the closed linear subspace of X generated by A and lin A - the linear subspace of X generated by A.

Lemma 1 : Let X be a Banach space and let n be a positive integer. Let $x_o, x_1, \dots, x_{2^n-1}$ be elements of X and let $x_o^*, x_1^*, \dots, x_{2^n-1}^*$ be elements of X^* such that $x_p^*(x_q) = \delta_p^q$ for $p = 0,1,\dots,2^n-1$. Then there exists a unitary real matrix $(a_{k,j}^n)_{0 \le k,j < 2^n}$ such that if

$$e_k = \sum_{j=0}^{2^n-1} a_{k,j}^n x_j \qquad \text{for } k = 0,1,\dots,2^n-1,$$

and

$$e_k = \sum_{j=0}^{2^n-1} a_{k,j}^n x_j^*$$

then

$$(1) \qquad \max_{0 \le p < 2^n} \|e_p\| < (1+\sqrt{2}) \max_{1 \le j < 2^n} \|x_j\| + 2^{-\frac{n}{2}} \|x_o\|$$

(2) $$\max_{0 \le p < 2^n} \|e_p^*\| < (1+\sqrt{2}) \max_{1 \le j < 2^n} \|x_j^*\| + 2^{-\frac{n}{2}} \|x_o^*\|$$

(3) $$e_p^*(e_q) = \delta_p^q \quad \text{for } p,q = 0,1,\dots,2^n-1$$

(4) $$[\{e_p\}_{0 \le p \le 2^n}] = [\{x_p\}_{0 \le p < 2^n}] \;;\; [\{e_p^*\}_{0 \le p < 2^n}] = [\{x_p^*\}_{0 \le p < 2^n}].$$

<u>Proof</u> : The conditions (3) and (4) are satisfied for every unitary $2^n \times 2^n$ - matrix. The specific unitary matrix for which (1) and (2) hold is defined to be the matrix which transform the unit vector basis of the 2^n-dimensional Hilbert space $l^2_{2^n}$ onto the Haar basis of this space. We put

$$a^n_{k,0} = 2^{-\frac{n}{2}} \quad \text{for } 0 \le k < 2^n ,$$

$$a^n_{k,2^s+r} = \begin{cases} 2^{\frac{s-n}{2}} & \text{for } 2^{n-s-1}2r \le k < 2^{n-s-1}(2r+1) \\ -2^{\frac{s-n}{2}} & \text{for } 2^{n-s-1}(2r+1) \le k < 2^{n-s-1}(2r+2) \\ 0 & \text{for } k < 2^{n-s-1}2r \text{ and for } k \ge 2^{n-s-1}(2r+2). \end{cases}$$

$$(s = 0,1,\dots,n-1 \;;\; r = 0,1,\dots,2^s-1) \;.$$

We have

(5) $$\sum_{j=1}^{2^n-1} |a^n_{k,j}| = \sum_{s=0}^{n-1} 2^{-\frac{n-s}{2}} < 1+\sqrt{2} \quad \text{for } 0 \le k < 2^n \;.$$

Clearly (5) implies (1) and (2).

<u>Proposition 1</u> : Let (x_n, x_n^*) be a fundamental and total biorthogonal sequence in a Banach space X such that there exists an increasing infinite sequences (n_k) such that $\sup_n \|x_{n_k}\| \, \|x^*_{n_k}\| = M < \infty$. Then there exists

a fundamental and total biorthogonal sequence (e_n, e_n^*) in X such that

$$\sup_n \|e_n\| \, \|e_n^*\| \leq M(1+\sqrt{2})^2 + 1$$

$$\text{and} \qquad \operatorname{lin}\{e_n\}_{n=1}^{\infty} = \operatorname{lin}\{x_n\}_{n=1}^{\infty}$$

$$\text{and} \qquad \operatorname{lin}\{e_n^*\}_{n=1}^{\infty} = \operatorname{lin}\{x_n^*\}_{n=1}^{\infty} \quad .$$

<u>Proof</u> : Without loss of generality one may assume that $\|x_n\| = 1$ for all n. Pick a permutation p(.) of the indices and an increasing sequence (m_r) of the indices so that if $\tilde{x}_n = x_{p(n)}$ and $\tilde{x}_n^* = x_{p(n)}^*$ for all n and $q_r = \sum_{p=0}^{r} 2^{m_p}$ for all r then

if $n \neq q_r$ for all r, then $\|\tilde{x}_n\| \, \|\tilde{x}_n^*\| \leq M$,

if $n = q_r$ for some $r = 0,1\ldots,$ then

$$(1+\sqrt{2})^2 M+1 > [(1+\sqrt{2})M + \|\tilde{x}_n^*\| \, 2^{-\frac{m_r}{2}}][(1+\sqrt{2}) + \|\tilde{x}_n\| \, 2^{-\frac{m_r}{2}}].$$

Next we put

$$e_n = \tilde{x}_n \quad \text{and} \quad e_n^* = \tilde{x}_n^* \qquad \text{for } n < 2^{m_0} ,$$

$$e_{k+q_{r-1}} = \sum_{j=0}^{2^{m_r}-1} a_{k,j}^{m_r} \tilde{x}_{j+q_{r-1}} \qquad ; \qquad e_{k+q_{r-1}}^* = \sum_{j=0}^{2^{m_r}-1} a_{k,j}^{m_r} \tilde{x}_{j+q_{r-1}}^*$$

$$\text{for } 0 \leq k < 2^{m_r} \ ; \ r = 1,2,\ldots$$

where $a_{k,j}^{m_r}$ are defined as in Lemma 1 for $n = m_r$. Using Lemma 1 we easily verify that such defined sequence (e_n, e_n^*) has the desired properties.

<u>Proof of Theorem 1</u> : We shall assume that $\dim X = \infty$. Then the separability of X implies that there exist sequences $E_1 \subset E_2 \subset \dots$ of subspaces of X and $F_1 \subset F_2 \subset \dots$ of subspaces of X^* such that $\dim E_i = \dim F_i = i$ for $i = 1,2,\dots$, $\bigcup_{i=1}^{\infty} E_i$ is dense in X and if $f^*(x) = 0$ for all $f^* \in \bigcup_{i=1}^{\infty} F_i$ then $x = 0$. In view of Proposition 1 it is enough to construct a biorthogonal sequence (x_n, x_n^*) in X such that if $G = [x_1, x_2, \dots, x_n]$ and $H_n = [x_1^*, x_2^*, \dots, x_n^*]$ then for all s

$$(6) \qquad G_{3s-1} \supset E_s \quad ; \quad H_{3s-1} \supset F_s \quad ; \quad \|x_{3s}\| \, \|x_{3s}^*\| \leq 3 \; .$$

Pick $x_1 \in X$ and $x_1^* \in X^*$ so that $0 \neq x_1 \in E_1$ and $x_1^*(x_1) = 1$. Assume that for some $n-1 \geq 1$ the elements $x_1, x_2, \dots, x_{n-1}$ in X and the functionals $x_1^*, x_2^*, \dots, x_{n-1}^*$ in X^* have been defined to satisfy (6) and so that $x_p^*(x_q) = \delta_p^q$ for $p,q = 1,2,\dots,n-1$. We consider separately three cases.

1^0) $n = 3s-2$. If $G_{n-1} \supset E_s$ we define $x_n \in X$ and $x_n^* \in X^*$ arbitrarily so that $x_n^*(x^q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p,q = 1,2,\dots,n$. If $E_s \setminus G_{n-1}$ is non empty, say $e \in E_s \setminus G_{n-1}$, then we put $x_n = e - \sum_{p=1}^{n-1} x_p^*(e)\, x_p$ and $G_n = [G_{n-1} \cup \{x_n\}]$. Clearly $x_n \neq 0$. Since $\dim E_s = \dim E_{s-1} + 1$ and $e \in G_n \setminus E_{s-1}$ and since the inductive hypothesis implies that $E_{s-1} \subset G_{n-1}$, we infer that $G_n \supset E_s$. Since $x_n \in G_n \setminus G_{n-1}$, there exists a bounded linear functional on G_n, say g^*, such that $g^*(x_n) = 1$ and $g^*(g) = 0$ for $g \in G_{n-1}$. We define x_n^* to be any extension of g^* to a bounded linear functional on X.

2^0) $n = 3s-1$. If $H_{n-1} \supset F_s$ we define $x_n \in X$ and $x_n^* \in X^*$ arbitrarily so that $x_n^*(x_q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p,q = 1,2,\dots,n$. If $F_s \setminus H_{n-1}$ is non empty, say $f^* \in F_s \setminus H_{n-1}$ then we put $x_n^* = f^* - \sum_{q=1}^{n-1} f^*(x_q) x_q^*$. Since $f^* \notin H_{n-1}$, there exists an $x \in X$ such that $1 = f^* - \sum_{q=1}^{n-1} f^*(x_q) x_q^*(x)$.

We put $x_n = x - \sum_{p=1}^{n-1} x_p^*(x)x_p$. It is easy to check that $x_n^*(x_q) = \delta_n^q$ and $x_p^*(x_n) = \delta_p^n$ for $p,q = 1,2,\dots,n$. Let $H_n = [H_{n-1} \cup \{x_n\}]$. Since the inductive hypothesis implies that $F_{s-1} \subset H_{n-1}$ and since $\dim F_s = \dim F_{s-1} + 1$ and $f^* \in F_s \setminus F_{s-1}$, we infer that $H_n \supset F_s$.

3^o) $n = 3s$. Using Mazur's technique (cf. [10] Lemma) we pick an $x_n \in X$ with $\|x_n\| = 1$ so that $x^*(x_n) = 0$ for every $x^* \in H_{n-1}$ and for all g in G_{n-1} and for all scalars t, $\|g + t\,x_n\| \geq (1 - 1/3)\,\|g\|$. Define g^* on G_n by $g^*(g + t\,x_n) = t$. Then $|t| = \|t\,x_n\| \leq \|g + t\,x_n\| + \|g\| \leq (1 + 3/2)\ \|g + t\,x_n\|$.

Thus $\|g^*\| \leq 3$. We define x_n^* to be any norm preserving extension of g^* to a linear functional on X.

<u>Remark 1</u> : Using in the case 3^o Day's technique (cf. [3]) which bases on the Borsuk antipodal mapping theorem one can choose (both in the case of real and of complex scalars) x_{3s} and x_{3s}^* so that $\|x_{3s}\| = \|x_{3s}^*\| = x_{3s}^*(x_{3s}) = 1$ for $s = 1,2,\dots$ Now the inspection of the proof of Theorem 1 yields that in every separable Banach space for every $\varepsilon > 0$ there exists a fundamental and bounded biorthogonal sequence (e_n, e_n^*) such that $\|e_n\|\ \|e_n^*\| < (1+\sqrt{2})^2 + \varepsilon$ for all n. We do not know whether for every $\varepsilon > 0$ this bound can be replaced by $1 + \varepsilon$. However, as was observed by C. Bessaga we have

<u>Corollary 1</u> : In every separable Banach space X there exists an equivalent norm $|||\cdot|||$ such that there exists in X a fundamental and total biorthogonal sequence (e_n, e_n^*) with $|||e_n^*|||\ |||e_n^*||| = 1$.

<u>Proof</u> : We admit $|||x||| = \max\left(\|x\|,\ \sup_n |e_n^*(x)|\right)$ for $x \in X$ where (e_n, e_n^*) is any fundamental and total biorthogonal sequence in X such that $\|e_n\| = 1$ for all n and $\sup_n \|e_n^*\| < \infty$.

<u>Remark 2</u> : A similar argument to that which is used in the proof of Theorem 1 allows to prove the following

Theorem 1' : Let X and Y be Banach spaces and let $T : X \dashrightarrow Y$ be one-to-one bounded linear operator. If X is separable, T(X) is dense in Y and T is not compact, then there exists fundamental and total biorthogonal sequences (x_n, x_n^*) in X and (y_n, y_n^*) in Y such that

$$\sup_n \max (\|x_n\| \ \|x_n^*\| \ , \|y_n\| \ \|y_n^*\|) < \infty$$

and
$$T(x_n) = y_n$$

for all n.

2. Constructions of uniformly bounded orthonormal sequences.

We employ the following notation. If μ is a probability measure (= a non negative normalized measure) on a sigma field of subsets of a set S then $< x,y > = \int_S x(s)\ y(s)\ \mu(ds)$, $\|x\|_2 = < x,y >^{1/2}$ and

$$\|x\|_\infty = \inf_{\mu(B)=1} \ \sup_{s \in B} |x(s)|$$

for any μ-absolutely square summable scalar valued functions x and y on S. $L^\infty(\mu)$ and $L^2(\mu)$ denote as usually the Banach spaces of those x that $\|x\|_\infty < \infty$ and $\|x\|_2 < \infty$ respectively.

The proof of Theorem 2 is similar to the proof of Theorem 1. Instead of Proposition 1, we apply the following result due to Olevskiǐ ([9], Lemma 4).

Proposition 2 : Let μ be aprobability measure on a sigma field of subsets of a set S. Let (x_n) be an infinite orthonormal (with respect to the inner product $< .,. >$) sequence of functions in $L^\infty(\mu)$ such that $\liminf_n \|x_n\|_\infty < \infty$. Then there exists an orthonormal sequence (e_n) such that

$$\operatorname{lin} \{x_n\}_{n=1}^\infty = \operatorname{lin} \{e_n\}_{n=1}^\infty$$

and $$\sup_n \|e_n\|_\infty \; .$$

The proof of Proposition 2 can be obtained by a non essential modification of the proofs of Lemma 1 and Proposition 1. Actually Olevskiĭ stated Proposition 2 for the Lebesgue measure on [0,1].

To prove Theorem 2 it is convenient to use the following simple fact.

Lemma 2 : Let (g_n) be a normalized sequence in $L^2(\mu)$ which weakly (in $L^2(\mu)$) converges to zero and let $\sup_n \|g_n\|_\infty = M < \infty$. Then for every finite dimensional subspace of $L^\infty(\mu)$, say F, and for $k > 0$ there exist an index $n_o > k$ and a function h in the orthogonal complement of F such that

$$[F \cup \{g_n\}] = [F \cup \{h\}] \; , \quad \|h\|_2 = 1$$

and
$$\|h\|_\infty < M + 2^{-k} \; .$$

Proof : Let $p = \dim F$. Let $e_1, e_2, \ldots, e_p$ be any orthonormal basis for F. Pick $\varepsilon > 0$ so that

$$\frac{M + \varepsilon \sum_{j=1}^{p} \|e_j\|_\infty}{1 - \varepsilon p} < M + 2^{-k} \; .$$

Since (g_n) converges weakly to 0 in $L^2(\mu)$, there exists an index $n_o > k$ such that $|< g_{n_o}, e_j >| < \varepsilon$ for $1 \le j \le p$. Put

$$h = (g_{n_o} - \sum_{j=1}^{p} < g_{n_o}, e_j > e_j) \; \|g_{n_o} - \sum_{j=1}^{p} < g_{n_o}, e_j > e_j\|_2^{-1} \; .$$

Clearly h belongs to the orthogonal complement of F, $\|h\|_2 = 1$ and

$$[F \cup \{g_{n_o}\}] = [F \cup \{h\}].$$

We have

$$\|g_{n_o} - \sum_{j=1}^{p} \langle g_{n_o}, e_j \rangle e_j\|_\infty \geq \|g_{n_o}\|_\infty + \|\sum_{j=1}^{p} \langle g_{n_o}, e_j \rangle e_j\|_\infty$$

$$\leq M + \varepsilon \sum_{j=1}^{p} \|e_j\|_\infty$$

and

$$\|g_{n_o} - \sum_{j=1}^{p} \langle g_{n_o}, e_j \rangle e_j\|_2 \geq \|g_{n_o}\|_2 - \|\sum_{j=1}^{p} \langle g_{n_o}, e_j \rangle e_j\|_2$$

$$\geq 1 - \varepsilon p .$$

Thus $\|h\|_\infty \leq (M + \varepsilon \sum_{j=1}^{p} \|e_j\|_\infty)(1 - \varepsilon p) < M + 2^{-k}$.

Proof of Theorem 2 : It follows from (i) that there exists in E an increasing sequence of finite dimensional subspaces $F_1 \subset F_2 \subset \ldots$ such that $\dim F_p = p$ and $\bigcup_{p=1}^{\infty} F_p$ is dense in E. Clearly if $E \cap L^\infty(\mu)$ is a separable subset of $L^\infty(\mu)$ one can choose the sequence (F_p) so that the union $\bigcup_{p=1}^{\infty} F_p$ is dense in $E \cap L^\infty(\mu)$ in the $L^\infty(\mu)$ norm. The condition (ii) yields that there exists in E a sequence (g_n) satisfying the assumption of Lemma 2. In view of Proposition 2, it is enough to define inductively an orthonormal sequence (h_n) in $L^\infty(\mu) \cap E$ so that for $s = 1, 2, \ldots$

(7) $$[\{h_1, h_2, \ldots, h_{2s-1}\}] \supset F_s ,$$

(8) $$\|h_{2s}\|_\infty < M + 2^{-s}$$

where $M = \sup_n \|g_n\|_\infty$.

We define h_1 as any element of F_1 with $\|h_1\|_2 = 1$. Suppose that for some $n-1 \geq 1$ the functions $h_1, h_2, \ldots, h_{n-1}$ have been defined to satisfy the conditions (7) and (8) and so that $\langle h_p, h_q \rangle = \delta_p^q$ for $p, q = 1, 2, \ldots, n-1$. Let us consider separately two cases.

1) $n = 2s$ for some $s = 1,2,\ldots$ We put $h_n = h$ where h is that of Lemma 2 applied for $F = [\{h_1,h_2,\ldots,h_{n-1}\}]$ for (g_p) and for $k = s$.

2) $n = 2s-1$ for some $s = 2,3,\ldots$ If $F_s \subset [\{h_1,h_2,\ldots,h_{n-1}\}]$ we again define $h_n = h$ where h is that of Lemma 2 applied for $F = [\{h_1,h_2,\ldots,h_{n-1}\}]$ for (g_p) and for $k=1$. If $F_m \not\subset [\{h_1,h_2,\ldots,h_{n-1}\}]$ then there exists an f which belongs to $F_s \setminus [\{h_1,h_2,\ldots,h_{n-1}\}]$. Let $\tilde{f}$ be the orthogonal projection of f onto $[\{h_1,h_2,\ldots,h_{n-1}\}]$. We put $h_n = (f-\tilde{f})\,\|f-\tilde{f}\|_2^{-1}$. Clearly $\|h_n\|_2 = 1$ and h_n belongs to the orthogonal complement of $[\{h_1,h_2,\ldots,h_{n-1}\}]$. Obviously we have $f \in [\{h_1,h_2,\ldots,h_1\}] \setminus [\{h_1,h_2,\ldots,h_{n-1}\}]$. By the inductive hypothesis $F_{s-1} \subset \{h_1,h_2,\ldots,h_{n-1}\}]$. Thus $F_s \subset [\{h_1,h_2,\ldots,h_n\}]$ because $\dim F_s = \dim F_{s-1}+1$.

This complete the induction and the proof of the sufficiency of the conditions (i) and (ii). The necessity is trivial.

<u>Remark 1</u> : A similar argument gives

<u>Theorem 2'</u> : Let $T : X \dashrightarrow H$ be a one to one bounded linear operator from a Banach space X into a Hilbert space H. Let $E = T(X)$. If E is separable and T is not compact then there exists a sequence (x_n) in X such that $\sup_n \|x_n\| < \infty$ and $(T(x_n))$ is an orthonormal basis for E. Moreover if X is separable and $x_n^* \in X^*$ is defined by $x_n^*(x) = \langle T(x), x_n \rangle_H$ for $x \in X$ and for $n = 1,2,\ldots$, where $\langle .,. \rangle_H$ denotes the inner product of H, then (x_n) can be chosen so that (x_n, x_n^*) is a fundamental and total biorthogonal sequence in X and $\sup_n \|x_n\|\,\|x_n^*\| < \infty$.

<u>Remark 2</u> : There exists an orthonormal decomposition of $L^2[0,1]$ onto subspaces E_1 and E_2 such that neither E_1 nor E_2 admit uniformly bounded orthonormal bases. It is enough to define $E_1 = [\{x_1\} \cup \{x_{2m}\}_{m=2}^{\infty}]$ and $E_2 = [\{x_2\} \cup \{x_{2m-1}\}]_{m=2}^{\infty}$ where (x_n) is any orthonormal basis for $L^2[0,1]$ such that the functions x_1 and x_2 are unbounded, $x_{2m-1}(t) = 0$ for $0 \le t < \frac{1}{2}$ and $x_{2m}(t) = 0$ for $\frac{1}{2} < t \le 1$ $(m=1,2,\ldots)$. However as was observed earlier by F.G. Arutunian (unpublished) we have

Corollary 2 : If E is a linear subspace of a separable space $L^2(\mu)$ where μ is a non purely atomic probability measure and if the orthogonal complement of E is finite dimensional, then [E] has a uniformly bounded orthonormal basis. Moreover if $E \cap L^\infty(\mu)$ is dense in E then the basis can be chosen from elements of $E \cap L^\infty(\mu)$.

Proof : It is enough to show that [E] satisfies the conditions (i) and (ii) of Theorem 2. To check (i) first observe that the density of $L^\infty(\mu)$ regarded as a subspace of $L^2(\mu)$ in $L^2(\mu)$ implies that for every positive integer p and for every linearly independent $f_1, f_2, \dots, f_{p+1}$ in $L^2(\mu)$ there exist $y_1, y_2, \dots, y_{p+1}$ in $L^\infty(\mu)$ such that the matrix $(y_k, f_j)_{1 \le k, j \le p+1}$ is invertible. Let $(a_{i,k})_{1 \le i, k \le p+1}$ be the inverse matrix and let

$$z_i = \sum_{k=1}^{p+1} a_{i,k}\, y_k \text{ for } i = 1,2,\dots,p+1. \text{ Then } z_i \in L^\infty(\mu) \text{ and } \langle z_i, f_j \rangle = \delta_i^j$$

for $i = 1,2,\dots,p+1$. The above observation applied to any basis of the orthogonal complement of E and any non zero element f of [E] yields the existence of an y in $L^\infty(\mu)$ such that $\langle y, f \rangle = 1$ and $\langle y, g \rangle = 0$ for all g in the orthogonal complement of E. The last condition means that $y \in [E]$. Hence there is no $f \neq 0$ in [E] which is orthogonal to all $y \in [E] \cap L^\infty(\mu)$, equivalently $[E] \cap L^\infty(\mu)$ is dense in [E]. Hence [E] satisfies (i).

The"moreover" part of the Corollary follows from the observation that if [E] satisfies (ii) than E also satisfies (ii).

An immediate consequence of Corollary 2 is

Corollary 3 : Let f be any unbounded function in $L^2[0,1]$. Then the orthogonal complement of f admits a uniformly bounded orthonormal basis consisting of trigonometrical polynomials. This basis has no extension to any uniformly bounded orthonormal basis for $L^2[0,1]$.

Corollary 3 answers a question of Shapiro [14].

3. Fat subspaces of C(S) spaces.

Definition : Let μ be a probability Borel measure on a compact Hausdorff space S. A closed linear subspace Z of C(S) is said to be fat with respect to μ if the unit ball of Z regarded as a subset of the Hilbert space $L^2(\mu)$ is not a totally bounded set.

Let $I_\mu : L^\infty(\mu) \to L^2(\mu)$ denote the natural injection. It is clear that Z is fat with respect to μ iff the restriction of I_μ to Z is not a compact operator or equivalently if $E = I_\mu(Z)$ satisfies the condition (ii) of Theorem 2.

Our next result characterizes Banach spaces which admit fat isometric embeddings into C(S) spaces. Some of the equivalent conditions are stated in terms of 2-absolutely summing operators, i.e. such bounded linear operators which admit a factorization through a natural injection I_μ for some measure μ (cf. [12] and [8]).

Proposition 3 : For every Banach space X the follwing conditions are equivalent :

(a) there exists a uniformly bounded sequence (x_n) of elements of X such that no subsequence of (x_n) is a weak Cauchy sequence,

(b) X contains a subspace isomorphic to l^1,

(c) there exists a 2-absolutely summing operator from X onto l^2,

(d) there exists a 2-absolutely summing non compact operator from X into l^2,

(e) for every for some isometric embedding j of X into a C(S) space there exists a probability Borel measure μ on S such that j(X) is fat with respect to μ.

Proof : (a) $\Rightarrow$ (b). This is a profound recent result of Rosenthal [13].
(b) $\Rightarrow$ (c). Let T ba a bounded linear operator from l^1 onto l^2

(cf. [2] for the existence of such an operator). Then by a result of Grothendieck [7] (cf. also [8]) T is 2-absolutely summing. Hence, by [12] T admits an extension to a 2-absolutely summing operator from X onto l^2.

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (e). Let $T : X \dashrightarrow l^2$ be a non compact 2-absolutely summing operator and let S be a compact Housdorff space. By a result of Persson and Pietsch [11], for every isometric embedding $j : X \dashrightarrow C(S)$ there exists a Borel probability measure μ on S such that $T = A I_\mu j$ for some bounded linear operator $A : L^2(\mu) \dashrightarrow l^2$. Since T is not compact, the image of the unit ball of $j(X)$ under I_μ is not a totally bounded subset of $L^2(\mu)$. Thus $j(X)$ is a fat subspace of $C(S)$ with respect to μ.

(e) $\Rightarrow$ (a). It follows from (e) that there exists a uniformly bounded sequence (x_n) in X such that $\|I_\mu j(x_n) - I_\mu j(x_m)\|_2 \geq 1$ for $n \neq m$ $(n,m=1,2,\ldots)$. Thus the sequence (x_n) does not contain weak Cauchy sequences because I_μ takes weak Cauchy sequences into strong Cauchy sequences.

A similar result to our Proposition 3 was recently independently discovered by Weis [16].

Our last result is related to Gaposhkin's [6] generalization of a result of Sidon [15].

<u>Corollary 4</u> : Let μ be a probability measure on a sigme field of subsets of S. Let (g_n) be a uniformly bounded sequence in $L^\infty(\mu)$ such that (g_n) tends weakly to zero in $L^2(\mu)$ and $\limsup_n \|g_n\|_2 > 0$. Then there exists an infinite subsequence (g_{n_k}) and $c > 0$ such that

$$\left\| \sum_{k=1}^{p} c_k g_{n_k} \right\|_\infty > c \sum_{k=1}^{p} |c_k|$$

for every finite sequence of scalars $c_1, c_2, \ldots, c_p$ $(p=1,2,\ldots)$.

Proof : Without loss of generality we may assume that $\inf_n \|g_n\|_2 > 0$. Then (g_n) does not have Cauchy (in $L^2(\mu)$) subsequences because (g_n) weakly converges in $L^2(\mu)$ to zero but no subsequence of (g_n) strongly converges to zero. Thus $g_n)$ regarded as a sequence of elements of $L^\infty(\mu)$ does not contain weak (in $L^\infty(\mu)$) Cauchy sequences because the natural injection $I\mu : L^\infty(\mu) \dashrightarrow L^2(\mu)$ takes weak Cauchy sequences in $L^\infty(\mu)$ into strong Cauchy sequences in $L^2(\mu)$. Since $\sup_n \|g_n\|_\infty < \infty$, to complete the proof it is enough to apply Rosenthal's criterion (cf. Rosenthal [13] for the real case and Dor [5] for the complex case).

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