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ISOMORPHIC CHARACTERIZATIONS OF HILBERT SPACES BY ORTHOGONAL SERIES WITH VECTOR VALUED COEFFICIENTS

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§ 1. Let $(\varepsilon_i)_{i\in N}$ be the Bernouilli sequence of independent random variables on a probability space $(\Omega,\mathfrak{M},P)$ (e.g. each ε_i is distributed by the law $P(\varepsilon_i = +1) = P(\varepsilon_i = -1) = \frac{1}{2}$) and let $(\gamma_i)_{i\in N}$ be a sequence of independent Gaussian random variables on $(\Omega,\mathfrak{M},P)$ (each of γ_i is distributed by the law :

$$P(\gamma_i < t) = \frac{1}{2\pi}\int_{-\infty}^{t} e^{-\frac{s^2}{2}}\,ds)$$

<u>Theorem 1</u> : Let X be a Banach space. The following conditions are equivalent :

1) X is isomorphic with a Hilbert space.

2) $\exists_C \ \forall\ x_1,x_2,\dots x_n \in X$

$$\frac{1}{C}\sum_{i=1}^{n}\|x_i\|^2 \le E(\|\sum_{i=1}^{n} x_i\varepsilon_i\|^2) \le C\sum_{i=1}^{n}\|x_i\|^2$$

3) $\exists_C \ \forall x_1,x_2,\dots x_n \in X$

$$\frac{1}{C}\sum_{i=1}^{n}\|x_i\|^2 \le E(\|\sum_{i=1}^{n} x_i\gamma_i\|^2) \le C\sum_{i=1}^{n}\|x_i\|^2$$

4) $\exists_C \ \forall\ x_1,x_2,\dots x_n \in X\ ,\ \ \forall (a_{i,j})_{n\times n}$

$$\sum_{i=1}^{n}\ \|\sum_{j=1}^{n} a_{ij}x_j\|^2 \le C^2\ \|a\|^2\ \sum_{i=1}^{n}\|x_i\|^2\ ,$$

where $\|a\|$ is the norm of the operator $a: l_2^n \to l_2^n$ given by the matrix $(a_{i,j})_{n\times n}$.

<u>Proof</u> : 1) ⇒ 2). It follows from the fact that if X is a Hilbert space then

$$E\ \|\sum_{i=1}^{n} x_i\ \varepsilon_i\|^2 = \sum_{i=1}^{n}\|x_i\|^2$$

2) ⇒ 3). Let us fix a positive integer n and let us put

$$\delta_i^m = \frac{1}{\sqrt{m}}\sum_{k=0}^{m-1}\varepsilon_{im+k} \qquad \text{for } i = 1,2,\dots n\ ;\ m = 1,2,\dots$$

By the Moivre-Laplace theorem the common distribution of $(\delta_1^m, \delta_2^m, \dots \delta_n^m)$ converges to the common distribution of $(\gamma_1, \gamma_2, \dots \gamma_n)$ as $m \to \infty$, from which we deduce easily that if $h : R^n \to R$ is a continuous function such that $h(s_1, s_2 \dots s_n) e^{-(|s_1|+|s_2|+\dots|s_n|)}$ is bounded on R^n then

$$E\, h(\gamma_1, \gamma_2, \dots \gamma_n) = \lim_{m\to\infty} Eh(\delta_1^m, \delta_2^m, \dots \delta_n^m)$$

Let $x_1, \dots, x_n \in X$ and let $h : R^n \to R$ be given by $h(s_1, s_2, \dots s_n) = \|s_1 x_1 + s_2 x_2 + \dots + s_n x_n\|^2$.

By 2) we have that

$$\frac{1}{C} \sum_{i=1}^{n} \|x_i\|^2 \le E\, \|x_1\delta_1^m + x_2\delta_2^m + \dots x_n\delta_n^m\|^2 \le C \sum_{i=1}^{n} \|x_i\|^2$$

Now passing to the limit with m we obtain 3).

3) $\Rightarrow$ 4). It is well known that the extreme points of the unit ball of the n^2 dimensional space of linear operators on l_2^n are exactly linear isometries. Hence, by the Krein-Milman theorem any $n \times n$ real matrix $(a_{ij})_{n\times n}$ such that $\|A\| \le 1$ is a convex combination of matrices of isometries. Hence to estabilish 3) $\Rightarrow$ 4) it is enough to show that 4) holds for any matrix $(a_{ij})_{n\times n}$ which represents an isometry. Then we have by 3)

$$\sum_{i=1}^{n} \|\sum_{j=1}^{n} a_{ij} x_j\|^2 \le C\,E\, \|\sum_{i=1}^{n} (\sum_{j=1}^{n-1} a_{ij}\, x_j)\gamma_i\|^2 = C\,E\, \|\sum_{j=1}^{n} x_j (\sum_{i=1}^{n} a_{ij}\gamma_i)\|^2 =$$

$$C\,E\, \|\sum_{j=1}^{n} x_j \gamma_j'\|^2$$

where $\gamma_j' = \sum_{i=1}^{n} a_{ij}\gamma_i$. Since (a_{ij}) is isometry $(\gamma_1', \gamma_2', \dots \gamma_n')$ are the same distributed as $(\gamma_1, \gamma_2, \dots \gamma_n)$ and thus

$$C\, E\|\sum_{j=1}^{n} x_j \gamma_j'\|^2 = CE\|\sum_{j=1}^{n} x_j \gamma_j\|^2 \le C^2 \sum_{j=1}^{n} \|x_j\|^2.$$

4) $\Rightarrow$ 1). Let $u \in L(l_1^I, X)$ be a linear operator which is a surjection (if I is of sufficiently great cardinality then there exists such operator) and let $u(\bar{e}_i) = x_i$ for $i \in I$ where $\bar{e}_i$ is the i-th unite vector in l_1^I.

We shall prove that 4) implies that u is 2-absolutely summing. For this it is enough to show that if $v \in L(l_2^I, l_1^I)$ then

$$\sum_{i\in I} \|u \circ v(e_i)\|^2 < +\infty$$

(where $(e_i)_{i\in I}$ is the family of unite vectors in l_2^I). By the Grothendieck theorem $v = \Delta \circ a$ where $a \in L(l_2^I, l_2^I)$ and $\Delta \in L(l_2^I, l_1^I)$ is a diagonal operator e.g. $\Delta((\xi_i)_{i\in I}) = (\lambda_i \xi_i)_{i\in I}$ for fixed $(\lambda_i)_{i\in I}$ with $\sum_{i\in I} |\lambda_i|^2 = \|\Delta\|^2 < +\infty$. Let a be given by a matrix $(a_{i,j})_{i,j\in I}$. Now

$$\sum_{i\in I} \|u \circ v(e_i)\|^2 = \sum_{i\in I} \| \sum_{j\in I} a_{j,i} \lambda_j x_j\|^2 .$$

By 4) it follows that

$$\sum_{i\in I} \| \sum_{j\in I} a_{j,i} \lambda_j x_j\|^2 \le C^2\|a\|^2 \sum_{j\in I} \|\lambda_j x_j\|^2 \le C^2\|a\|^2 \|\Delta\|^2 \|u\|^2$$

(because for each $i \in I$ $\|x_i\| \le \|u\|$).
Thus u is 2-absolutely summing. Hence it follows from the Pietsch factorization theorem that u may be factorized through a Hilbert space e.g. $u = w \circ v$ where $v \in L(l_1^I, H)$ and $w \in L(H, X)$ where H is a Hilbert space. Since u is a surjection the same is true for w and this implies that X is isomorphic with a Hilbert space. Q. E. D.

Remark 1 : The proof of the implication 3) ⇒ 4) of theorem 1 is valid only for real Banach spaces. But it is not difficult to improve the arguments to obtain also the complex case.

Remark 2 : If a Banach space X fulfills the condition

$$\exists_C \forall_{x_1, x_2 \ldots x_n \in X} \; E\|\sum_{i=1}^n x_i \varepsilon_i\|^2 \le C \sum_{i=1}^n \|x_i\|^2$$

(resp.

$$\exists_C \forall_{x_1, x_2 \ldots x_n \in X} \; E\|\sum_{i=1}^n x_i \varepsilon_i\|^2 \ge C \sum_{i=1}^n \|x_i\|^2 \;)$$

then it is called to be of type 2 (resp. of cotype 2). From Theorem 1 it follows that if a Banach space is of type 2 and of cotype 2, then it is isomorphic with a Hilbert space. As it was observed by Maurey this may be generalized on operators in the following way : if X is of type 2 and Y of cotype 2 then each operator $u \in L(X,Y)$ may be factorized through a Hilbert space. A simple counter-example shows that it is not the case when X is of cotype 2 and Y of type 2.

§ 2.

Theorem 2 : Let $(\Phi_i)_{i\in N}$ be an orthonormal complete system in $L_2[0,1]$ and let X be a Banach space. The following two conditions are equivalent :

1) X is isomorphic with a Hilbert space,

2) $\exists\ C\ \forall x_1, x_2, \ldots x_n$

$$\frac{1}{C}\sum_{i=1}^{n}\|x_i\|^2 \le \int_0^1 \|\sum_{i=1}^{n} x_i\,\Phi_i(t)\|^2\,dt \le C\sum_{i=1}^{n}\|x_i\|^2 .$$

Proof : If X is a Hilbert space then the condition 2) holds with $C = 1$. Therefore 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1). Let us observe that if $(\psi_k)_{k\in N}$ is an orthonormal system in $L_2[0,1]$ such that for each k, $l \in N$ $k \neq l$ and $i \in N$ it is $(\psi_k, \varphi_i)(\psi_l, \varphi_i) = 0$ then 2) implies that $\forall x_1, x_2, \ldots x_n \in X$

$$\frac{1}{C}\sum_{k=1}^{n}\|x_k\|^2 \le \int_0^1 \|\sum_{k=1}^{n} x_k\,\psi_k(t)\|^2\,dt \le C\sum_{k=1}^{n}\|x_k\|^2$$

(with the same C as in 2)).
This follows from the two equalities :

$$\int_0^1 \|\sum_{k=1}^{n} x_k\,\psi_k(t)\|^2 dt = \int_0^1 \|\sum_{k=1}^{n} x_k \sum_{i=1}^{\infty}(\psi_k,\varphi_i)\varphi_i(t)\|^2 dt =$$

$$\int_0^1 \|\sum_{i=1}^{\infty}\Big(\sum_{k=1}^{n}(\psi_k,\varphi_i)x_k\Big)\varphi_i(t)\|^2 dt$$

and

$$\sum_{k=1}^{n}\|x_k\|^2 = \sum_{k=1}^{n}\|x_k\|^2\sum_{i=1}^{\infty}|(\psi_k,\varphi_i)|^2 = \sum_{i=1}^{+\infty}\Big(\|\sum_{k=1}^{n}(\psi_k,\varphi_i)x_k\|^2\Big) .$$

Now let $(\varepsilon_n)_{n\in N}$ be a Bernouilli sequence on $[0,1]$, as in Theorem 1 (for example the Rademacher system).

By the standard " gliding hump" method for a fixed $\varepsilon > 0$, we can find an increasing sequence (n_k) of indicies and an orthonormal system $(\phi_k)_{k\in N}$ which fulfills the above mentioned assumption and such that

$$\int_0^1 |\varepsilon_{n_k}(t) - \phi_k(t)|^2 \, dt \leq \frac{\varepsilon}{2^k} \quad .$$

From this we derive easily that $\exists_C \forall_{x_1 \ldots x_n \in X}$

$$\frac{1}{C} \sum_{k=1}^{n} \|x_k\|^2 \leq \int_0^1 \| \sum_{k=1}^{n} x_k \varepsilon_{n_k}(t)\|^2 dt \leq C \sum_{k=1}^{n} \|x_k\|^2 \quad .$$

Since

$$\int_0^1 \| \sum_{k=1}^{n} x_k \varepsilon_{n_k}(t)\|^2 \, dt = \int_0^1 \| \sum_{k=1}^{n} x_k \varepsilon_k(t)\|^2 \, dt$$

(because $(\varepsilon_{n_k})_{k\in N}$ is distributed the same as $(\varepsilon_k)_{k\in N}$), we obtain that X fulfills the condition 2) of theorem 1. By theorem 1 we obtain that X is isomorphic with a Hilbert space. Q. E. D.

§ 3. Let X be a complex Banach space. Denote by $L_0^2(X)$ the normed linear space of all simple functions $f : R \to X$ under the norm $|f| = (\int_{-\infty}^{+\infty} \|f(t)\|^2 \, dt)^{1/2}$. Here by a simple function we mean any function of the form $\sum_{j=1}^{n} \chi_{A_j} x_j$ where $x_j \in X$; A_j are measurable subsets of R of finite Lebesgue measure and χ_{A_j} denotes the characteristic function of A_j $j = 1, \ldots, n$. The completion of $L_0^2(X)$ in the norm $| \, . \, |$ will be denoted by $L^2(X)$. The Fourier transform

$$\mathfrak{F} : L_0^2(X) \to L^2(X)$$

is defined by

$$\mathfrak{F}(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ist} f(s) \, ds \quad \text{for } t \in R, \ f \in L_0^2(X) \quad .$$

And similarly we define the inverse Fourier transform

$$\tilde{\mathfrak{F}} : L_0^2(X) \to L^2(X)$$

by

$$\tilde{\mathfrak{F}}(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ist} f(s) ds \quad \text{for } t \in R, \ f \in L_0^2(X) \quad .$$

Clearly $\mathfrak{F}$, $\tilde{\mathfrak{F}}$ are linear operators in general unbounded. Our next lemma seems to be known. The proof repeat the classical argument used in the Poisson summation formula.

__Lemma__ : Let $h = \sum_{k=-n}^{n} \frac{x_k}{\sqrt{a}} \chi_{[k a, (k+1) a)}$, where $a > 0$, $x_k \in X$ $(k = 0, \pm 1, \dots, \pm n)$, n-any positive integer. Then

$$|h|^2 = \sum_{k=-n}^{n} \|x_k\|^2 \ ; \ |\mathfrak{F}(h)|^2 = \int_0^1 \| \sum_{k=-n}^{n} e^{-2\pi kti} x_k \|^2 dt \ .$$

__Proof__ : The computation of the norm $|h|$ is trivial. To establish the second formula we compute directly $\mathfrak{F}(h)$. We have

$$\mathfrak{F}(h)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum_{k=-n}^{n} \frac{x_k}{a} \chi_{[k a, (k+1) a)}(s)\, e^{-ist} ds =$$

$$\frac{1}{\sqrt{2\pi a}} \sum_{k=-n}^{n} x_k \int_{ka}^{(k+1)a} e^{ist} ds = \sqrt{\frac{a}{2\pi}}\ \frac{\sin \frac{at}{2}}{\frac{at}{2}} (-e^{-i\frac{at}{2}}) \sum_{k=-n}^{n} x_k e^{-kati} .$$

Hence, changing the variable $u = \frac{at}{2\pi}$, we get

$$\|\mathfrak{F}(h)\|^2 = \int_{-\infty}^{+\infty} \frac{\sin^2 u\pi}{(u\pi)^2} \| \sum_{k=-n}^{n} x_k e^{-2\pi iku} \|^2 du =$$

$$\sum_{\gamma=-\infty}^{+\infty} \int_{\gamma}^{\gamma+1} \frac{\sin^2 u\pi}{(u\pi)^2} \| \sum_{k=-n}^{n} x_k e^{-2\pi iku} \|^2 du =$$

$$\int_0^1 \sum_{\gamma=-\infty}^{+\infty} \frac{\sin^2 u\pi}{[\pi(u+\gamma)]^2} \| \sum_{k=-n}^{n} x_k e^{-2\pi iku} \|^2 du .$$

Since $\sum_{\gamma=-\infty}^{+\infty} \frac{\sin^2 u\pi}{[\pi(u+\gamma)]^2} = 1$ for all real u we get

$$\|\mathfrak{F}(h)\|^2 = \int_0^1 \| \sum_{k=-n}^{n} x_k e^{-2\pi kui} \|^2 du . \qquad \text{Q. E. D.}$$

__Theorem 3__ : Let X be a complex Banach space. The following conditions are equivalent :

1) X is isomorphic with a Hilbert space,

2) $\exists\, C\ \forall\, x_0, x_1, x_{-1}, \dots\, x_n, x_{-n} \in X$

$$\int_0^1 \| \sum_{k=-n}^{n} x_k e^{2\pi ikt} \|^2 dt \le C \sum_{k=-n}^{n} \|x_k\|^2 \ ,$$

3) $\exists\, C\ \forall\ x_o, x_1, x_{-1}, \ldots, x_n, x_{-n} \in X$

$$\int_0^1 \Big\| \sum_{k=-n}^{n} x_k\, e^{2\pi i k t} \Big\|^2 dt \geq \frac{1}{C} \sum_{k=-n}^{n} \|x_k\|^2$$

4) The Fourier transform $\mathfrak{F} : L_o^2(X) \to L^2(X)$ is bounded.

<u>Proof</u> : 1) ⇒ 2). If X is a Hilbert space, then

$$\int_0^1 \Big\| \sum_{k=-n}^{n} x_k\, e^{2\pi i k t} \Big\|^2 dt = \sum_{k=-n}^{n} \|x_k\|^2$$

and hence 1) ⇒ 2). Next, we prove that 2) ⇔ 4) ⇔ 3).

Since the simple functions h of the form as in Lemma are dense in $L_o^2(X)$ we get by Lemma that 2) ⇔ 4), and also that there exists $C > 0$ such that $|\mathfrak{F} h|^2 \geq \frac{1}{C} |h|^2$. This means exactly that the inverse Fourier transform $\tilde{\mathfrak{F}}$ is bounded. But it is clear that $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ are simultaneously bounded or unbounded. Thus we get that 4) ⇔ 3).
Now, if any of the conditions 2), 3), 4) is satisfied then the conditions 2) and 3) are satisfied and they together , by Theorem 2, imply the condition 1). Q. E. D.