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SUMS OF INDEPENDENT BANACH SPACE VALUED RANDOM VARIABLES

(AFTER J. HOFFMANN-JØRGENSEN)

by S. KWAPIEN

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Let E be a vector space and $\mathfrak{B}$ a σ-field of subsets of E compatible with the linear structure on E.

Let $\rho : E \to R^+$ be a measurable map such that

1) $\rho(x+y) \leq C[\rho(x)+\rho(y)]$ for $x, y \in E$

2) $\rho(\alpha x) \leq \rho(x)$ for $|\alpha| \leq 1$, $x \in E$.

Let $(\Omega, \mathfrak{M}, \mathcal{P})$ be a probability space.

By E-valued random variables we shall mean measurable mappings $X : (\Omega, \mathfrak{M}) \to (E, \mathfrak{B})$.

The definitions of symetricity and of independence for E-valued random variables are the same as usually.

<u>Lemma 1</u> : Let X be an E-valued symetric ramdom variable and $e \in E$ then

$$\mathcal{P}(\rho(e + X) < \frac{\rho(2e)}{2C}) \leq \frac{1}{2}$$

<u>Proof</u> : The subsets of E : $A = \{z | \rho(e+z) < \frac{\rho(2e)}{2C}\}$ and $B = \{x | \rho(e-z) < \frac{\rho(2e)}{2C}\}$ are disjoint and measurable. Since X is symetric $\mathcal{P}(X \in A) = \mathcal{P}(X \in B)$ and $\mathcal{P}(X \in A) + \mathcal{P}(X \in B) \leq 1$. Hence $\mathcal{P}(X \in A) \leq \frac{1}{2}$.

Q. E. D.

In the sequel $X_1, X_2, \ldots X_n$ will denote a fixed sequence of E-valued, independent, symetric random variables.

Let us denote

$$S_k = X_1 + X_2 + \ldots X_k \quad \text{for } k = 1, \ldots n.$$

<u>Lemma 2</u> : $\mathcal{P}(\max_{1 \leq k \leq n} \rho(S_k) \geq t) \leq 2\mathcal{P}(\rho(S_n) \geq \frac{t}{2C})$

<u>Proof</u> : Let $\tau = \min \{k | \rho(S_k) \geq t\}$ ($\tau = +\infty$ if the set is empty). The sets $(\tau = k)$ $k = 1 \ldots n$ are disjoint and

$$\bigcup_{k=1}^{n} (\tau = k) = (\max_{1 \leq k \leq n} \rho(S_k) \geq t)$$

For k = 1, 2...n we have

$$\mathcal{P}(\rho(S_n) < \frac{t}{2C}) \cap (\tau = k)) = \int_{\tau=k} \chi_{(\rho(S_n)<\frac{t}{2C})} \, d\mathcal{P}$$

Let $Q_k(e) = \mathcal{P}(\rho(e + X_{k+1} + \dots X_n) < \frac{t}{2C})$. By lemma 1 if $\rho(e) \geq t$ then $Q_k(e) \leq \frac{1}{2}$. By Fubini theorem and independence arguments :

$$\int_{\tau=k} \chi_{(\rho(S_n)<\frac{t}{2C})} \, d\mathcal{P} = \int_{\tau=k} Q_k(S_k) \, d\mathcal{P} \leq \frac{1}{2}\mathcal{P}(\tau=k)$$

(because $\rho(S_k) \geq t$ on $(\tau = k)$).

Thus $\mathcal{P}((\rho(S_n) < \frac{t}{2C}) \cap (\tau=k)) \leq \frac{1}{2}\mathcal{P}(\tau=k)$ for $k = 1\dots n$.
Hence, adding these inequalities we get

$$\mathcal{P}((\rho(S_n) < \frac{t}{2C}) \cap (\max_k \rho(S_k) \geq t)) \leq \frac{1}{2} \mathcal{P} (\max_k \rho(S_k) \geq t)$$

Because $\mathcal{P}((\rho(S_n) < \frac{t}{2C}) \cap (\max_k \rho(S_k) \geq t)) \geq -\mathcal{P}(\rho(S_n) \geq \frac{t}{2C}) + \mathcal{P}(\max_k \rho(S_k) \geq t)$ we get

$$\mathcal{P}(\max_k \rho(S_k) \geq t) \leq 2\mathcal{P}(\rho(S_n) \geq \frac{t}{2C}) \qquad \text{Q. E. D.}$$

Let us denote by $S = \rho(S_n)$ and by $H = \max\limits_{1\leq k\leq n} \rho(X_k)$

<u>Lemma 3</u> : $\mathcal{P}(H \geq t) \leq 2\mathcal{P}(S \geq \frac{t}{4C^2})$

<u>Proof</u> : $\rho(X_1) = \rho(S_1)$, $\rho(X_2) = \rho(S_2 - S_1) \leq C[\rho(S_2) + \rho(S_1)], \dots$ $\rho(X_n) = \rho(S_n - S_{n-1}) \leq C[\rho(S_n) + \rho(S_{n-1})]$. Therefore

$$\max_{1\leq k\leq n} \rho(X_k) \leq 2C \max_{1\leq k\leq n} \rho(S_k).$$

$$\mathcal{P}(H \geq t) = \mathcal{P}(\max_k \rho(X_k) \geq t) \leq \mathcal{P}(\max_k \rho(S_k) \geq \frac{t}{2C}) \leq 2\mathcal{P}(S \geq \frac{t}{4C^2}) \quad \text{Q. E. D.}$$

<u>Lemma 4</u> : $\mathcal{P}(S \geq s + t + u) \leq \mathcal{P}(H \geq \frac{u}{C}) + 4\mathcal{P}(S > \frac{s}{2C^3}) \, \mathcal{P}(S > \frac{t}{2C^3})$

Proof : Let $\tau = \min \{k \mid \rho(S_k) \geq \frac{t}{C^2}\}$ then

$$(S \geq s + t + u) \subset (H \geq \frac{u}{C}) \cup \bigcup_{k=1}^{n} ((\tau=k) \cap (\rho(S_n - S_k) \geq \frac{s}{C^2})).$$ Thus

$$P(S \geq s + t + u) \leq P(H \geq \frac{u}{C}) + \sum_{k=1}^{n} P((\tau=k) \cap (\rho(S_n - S_k) \geq \frac{s}{C^2}))$$

The events $(\rho(S_n - S_k) \geq \frac{s}{C^2})$ and $(\tau = k)$ are independent therefore

$$P(S \geq s + t + u) \leq P(H \geq \frac{u}{C}) + \sum_{k=1}^{n} P(\tau=k)\, P(\rho(S_n - S_k) \geq \frac{s}{C^2})$$

By lemma 2 $P(\rho(S_n - S_k) \geq \frac{s}{C^2}) = P(\rho(X_{k+1} + \dots X_n) \geq \frac{s}{C^2}) \leqslant 2P(S \geq \frac{s}{2C^3})$

Hence

$$P(S \geq s + t + u) \leq P(H \geq \frac{u}{C}) + 2P(S \geq \frac{s}{2C^3})\, P(\max_k \rho(S_k) \geq \frac{t}{C^2}) \leq$$

$$P(H \geq \frac{u}{C}) + 4P(S \geq \frac{s}{2C^3})\, P(S \geq \frac{t}{2C^3}) .$$ Q. E. D.

Lemma 5 : If ϕ is a real random variable then

$$E|\phi| = \int_0^{+\infty} P(|\phi| \geq t)dt.$$

Proof :

$$\int_0^{+\infty} P(|\phi| \geq t)dt = \int_0^{+\infty} (E\, \chi_{|\phi| \geq t})dt = E(\int_0^{+\infty} \chi_{|\phi| \geq t} dt) = E\int_0^{|\phi|} dt =$$

$E|\phi|$. Q. E. D.

Let us denote by $F(t) = P(S \geq t)$ and by $G(t) = P(H \geq t)$ then it follows from the lemma 4 that

$$(*) \qquad F(3t) \leq G(\frac{t}{C}) + 4\, F^2(\frac{t}{2C^3})$$

Theorem 1 : If $P(S \geq \frac{1}{2C^3}) \leq \frac{1}{48C^3}$ then

$$ES \leq 6\, C\, E\, H + 24$$

Proof : By (*) we obtain

$$\int_0^{+\infty} F(3t)\,dt \le \int_0^{+\infty} G(\tfrac{t}{C})\,dt + 4 + 4\int_1^{+\infty} F(\tfrac{t}{2C^3})\; F(\tfrac{1}{2C^3})\,dt$$. Hence

$$\frac{1}{3}\int_0^{+\infty} F(t)\,dt \le C\int_0^{+\infty} G(t)\,dt + 4 + 4.2C^3F(\frac{1}{2C^3})\int_0^{+\infty} F(t)\,dt$$ or equivalently

$$[\frac{1}{3} - 4.2C^3F(\frac{1}{2}\,C^3)]\int_0^{+\infty} F(t)\,dt \le C\int_0^{+\infty} G(t)\,dt + 4.$$

Since $\frac{1}{3} - 4.2C^3\, F(\frac{1}{2C^3}) \ge \frac{1}{6}$, by lemma 5 we get theorem . Q. E. D.

Remark 1 : From lemma 3, it follows that

$$EH \le 8C^2\, ES$$

In the rest of the paper we shall assume that ρ fulfils : 3') $\rho(\alpha\mathbf{x}) = |\alpha|^p\rho(\mathbf{x})$ for some $0 < p < \infty$ and for all $\mathbf{x} \in E$.

Let $f_1, f_2, \dots f_n$ be a sequence of independent, symetric equi-distributed real random variables.

Let $\mathcal{N}(t) = P(|f_i|^p \ge t)$. We shall assume that

a) $\int_t^{+\infty} \mathcal{N}(s)\,ds \le Kt\mathcal{N}(t)$ for $t \ge 1$

b) $\mathcal{N}(1) > 0$

Let $X_1 = x_1f_1$, $X_2 = x_2f_2, \dots X_n = x_nf_n$ where $x_1, x_2 \dots x_n \in E$. Then $X_1, X_2 \dots X_n$ is a sequence of independent, symetric E-valued random variables.

Theorem 2 : If $P(H \ge 1) < 1 - e^{-\mathcal{N}(1)}$ then $EH \le K\mathcal{N}(1) + 1$

Proof : Since

$$G(s) = P(\max_k \rho(X_k) \ge s) \le \sum_{k=1}^{n} P(|f_k|^p \ge \frac{s}{\rho(x_k)}) = \sum_{k=1}^{n} \mathcal{N}(\frac{s}{\rho(x_k)})$$ we have

$$EH = \int_0^{+\infty} G(s)\,ds \le 1 + \sum_{k=1}^{n}\int_1^{+\infty} \mathcal{N}(\frac{s}{\rho(x_k)})\,ds = 1 + \sum_{k=1}^{n} \rho(x_k)\int_{\frac{1}{\rho(x_k)}}^{+\infty} \mathcal{N}(s)\,ds$$

But $G(s) = \mathcal{P}(\max_k \rho(X_k) \geq s) = 1 - \prod_{k=1}^{n} (1 - \mathcal{P}(\rho(X_k) \geq s)) \geq 1 - e^{-\sum_{k=1}^{n} \mathcal{N}(\frac{s}{\rho(x_k)})}$

Therefore

$$(**)\ \sum_{k=1}^{n} \mathcal{N}\left(\frac{1}{\rho(x_k)}\right) \leq -\log(1 - G(1)) < \mathcal{N}(1) \quad \text{(by the assumptions)}$$

Hence $\frac{1}{\rho(x_k)} < 1$ and we can apply a) and then

$$EH \leq 1 + \sum_{k=1}^{n} K\mathcal{N}\left(\frac{1}{\rho(x_k)}\right) \leq 1 + K\mathcal{N}(1) \quad \text{(by **)} . \qquad \text{Q. E. D.}$$

Now combining theorem 1, 2 and lemma 3 we arrive at

__Theorem 3__ : Let $\varepsilon = \min\{\frac{1 - \mathcal{N}(1)}{2} , \frac{1}{48 C^3}\}$ $\delta = \min\{\frac{1}{4C^2} , \frac{1}{2C^3}\}$ and $M = 6C[5 + \mathcal{N}(1)]$ then if $\mathcal{P}(S \geq \delta) < \varepsilon$ then $E\, S < M$

__Remark 2__ : If E is a topological vector space and ρ is continuous then it is easy to pass from finite series to infinite ones.

Let $f_1, f_2 \ldots f_n$ be independent realvariables such that they are equi-distributed with characteristic function equal to $e^{-|t|^q}$ $0 < q \leq 2$. Then $\mathcal{N}(t) = \mathcal{P}(|f_i|^p \geq t)$ fulfils the conditions a) , b) for each

$$p < q^* = \begin{cases} +\infty & \text{if } q = 2 \\ q & \text{if } q < 2 \end{cases} .$$

Let E be a quasi-normed space e.g. the norm in E fulfils $\|\lambda x\| = |\lambda|\ \|x\|$ and $\|x + y\| \leq C'(\|x\| + \|y\|)$ for some C'. It we apply theorem 3 to $\rho(x) = \|x\|^p$ then we obtain

__Corollary__ : Let E be a quasi-normed space, and $f_1, f_2 \ldots f_n$ a sequence of independent real random variables each of them with the characteristic function equal to $e^{-|t|^q}$.

For each $p < q^*$, there exists ε, δ and M such that for each $x_1 \ldots x_n \in E$ if

$$\mathcal{P}(\|\sum_{i=1}^{n} x_i f_i\| \geq \delta) < \varepsilon \quad \text{then}$$

$$E\ \|\sum_{i=1}^{n} x_i f_i\|^p \leq M$$

<u>Remark 2</u> : Let $0 < p \leq 2$ and that λ_p be a canonical cylindrical measure on a locally convex vector space E with the dual $\mathcal{S}_p$ (cf exposé V). Assume that F is a quasi-normed space such that F' separates F, and that $u : E \to F$ is a continuous linear operator. Using corollary one can prove the following : if $u(\lambda_p)$ is a Radon measure then $u(\lambda_p)$ is of each order $< p^*$.

<u>Remark 3</u> : Using methods presented here it is possible to prove that if $u(\lambda_2)$ is a Radon measure then

$$\int_F e^{\varepsilon\|x\|} du(\lambda_2)(x) < +\infty \quad \text{for some } \varepsilon.$$

But it is not possible to obtain that

$$\int_F e^{\varepsilon\|x\|^2} du(\lambda_2)(x) < +\infty \quad \text{e.g. the Shepp's result.}$$

BIBLIOGRAPHY

[1] J. Hoffmann-Jørgensen : Sums of independent Banach space valued random variables, Aarhus University, Preprint series 1972-1973 N° 15.

[2] B. Maurey : Séminaire Maurey-Schwartz 1972-1973, Exposé N°V.