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On the Lie transformation algebra of monoids in symmetric monoidal categories

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ABSTRACT - We define the Lie transformation algebra of a (not necessarily associative) monoid object A in a K-linear symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$, where K is a field. When A is associative and satisfies certain conditions, we describe explicity the Lie transformation algebra and inner derivations of A. Additionally, we also show that derivations preserve the nucleus of the monoid A

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1. Introduction

Given an associative $\mathbb{Z}$-algebra A and an element $a \in A$, the morphism $D_a : A \longrightarrow A$ defined by $D_a(x) := ax - xa$ defines an inner derivation on A. However, if A is not associative, the morphism D_a is not necessarily a derivation. For nonassociative algebras, a theory of inner derivations has been developed by Schafer [4]. The purpose of this paper is to extend this theory to monoids over a K-linear symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$, where K is a field.

More precisely, let A be a (not necessarily associative) unital monoid object in a K-linear symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$. A morphism $f : 1 \longrightarrow A$ induces a morphism $L_A(f) : A \longrightarrow A$ (resp. $R_A(f) : A \longrightarrow A$) by left multiplication (resp. right multiplication) on the monoid A (in the sense of (2.9)). We consider the subspace $\mathcal{L}(A)$ (resp. $\mathcal{R}(A)$) of $Hom(A, A)$ generated by morphisms of the form $L_A(f)$ (resp. $R_A(f)$). Then, we start by defining the Lie transformation algebra $\mathscr{L}(A)$ of A to be the smallest Lie algebra con-

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taining the subspace $\mathcal{L}(A) + \mathcal{R}(A)$ of $Hom(A, A)$. If D is a derivation on A (see Definition 2.10), we will say that D is an inner derivation if $D \in \mathscr{L}(A)$.

When A is an associative monoid object, we show that the Lie transformation algebra $\mathscr{L}(A)$ of A is actually equal to $\mathcal{L}(A) + \mathcal{R}(A)$. In particular, if A is associative and has no left (or right) "absolute divisors of zero", we show that a derivation D on A is inner if and only if D is of the form $D = L_A(f) - R_A(f)$ for some morphism $f : 1 \longrightarrow A$. Moreover, we verify that for any (not necessarily associative) monoid A, the collection of inner derivations is always an ideal in the Lie algebra $Der(A)$ of derivations on A. Finally, we also show that if $f : 1 \longrightarrow A$ is a morphism in the nucleus of A (see (2.13)), for any derivation $D \in Der(A)$, $D \circ f$ is also in the nucleus of A.

We mention here that the notion of derivations on monoid objects appears elsewhere in the literature (see, for instance, Baues, Jibladze and Tonks [1]). For more on derivations and nonassociative algebras, we refer the reader to Jacobson [2] and Schafer [4], [5].

2. Derivations on monoids

Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a K-linear symmetric monoidal category. Since $\mathbf{C}$ is symmetric, for every pair X, Y of objects in $\mathbf{C}$, we have an isomorphism: $t_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ such that $t_{X,Y} \circ t_{Y,X} = 1_{Y\otimes X}$ and $t_{Y,X} \circ t_{X,Y} = 1_{X\otimes Y}$. When there is no danger of confusion, we shall omit the subscripts and simply write $t : X \otimes Y \xrightarrow{\cong} Y \otimes X$. Further, for any object X in $\mathbf{C}$, we have two isomorphisms $l_X : X \xrightarrow{\cong} 1 \otimes X$ and $r_X : X \xrightarrow{\cong} X \otimes 1$ satisfying $r_X = tl_X$.

Given $(\mathbf{C}, \otimes, \mathbf{1})$, we shall let $Mon(\mathbf{C})$ denote the category of unital, not necessarily associative, monoids object in $\mathbf{C}$. For any monoid A in $\mathbf{C}$, we will denote by $m_A : A \otimes A \longrightarrow A$ and $e_A : 1 \longrightarrow A$ resp. the "multiplication map" and the "unit map" on the monoid A. We start by defining the notion of a derivation on A.

DEFINITION 2.1. *Let A be an object of $Mon(\mathbf{C})$. A morphism $D : A \longrightarrow A$ is referred to as a derivation on A if it satisfies the following condition:*

$$m_A \circ (D \otimes 1 + 1 \otimes D) = D \circ m_A : A \otimes A \longrightarrow A \tag{2.1}$$

Given derivations D, D' of a monoid A, it may be easily verified that the commutator $[D, D'] := D \circ D' - D' \circ D$ is also a derivation on A. Then, since the category $\mathbf{C}$ is K-linear, the space $Der(A)$ of derivations on A is a Lie algebra.

Further, for any monoid A, given a morphism $f : 1 \longrightarrow A$, we define:

$$(2.2)\qquad \begin{aligned} L_A(f) &:= \big(A \xrightarrow[\cong]{l_A} 1 \otimes A \xrightarrow{f \otimes 1} A \otimes A \xrightarrow{m_A} A\big) \\ R_A(f) &:= \big(A \xrightarrow[\cong]{r_A} A \otimes 1 \xrightarrow{1 \otimes f} A \otimes A \xrightarrow{m_A} A\big) \\ &:= \big(A \xrightarrow[\cong]{l_A} 1 \otimes A \xrightarrow{f \otimes 1} A \otimes A \xrightarrow{m_A t} A\big) \end{aligned}$$

We denote by $\mathcal{L}(A)$ (resp. $\mathcal{R}(A)$) the subspace of $Hom(A,A)$ generated by morphisms of the form $L_A(f)$ (resp. $R_A(f)$) where $f \in Hom(1,A)$. We will say that $f : 1 \longrightarrow A$ is a left (resp. right) absolute divisor of zero if $L_A(f) = 0$ (resp. $R_A(f) = 0$).

Suppose that $\mathscr{M} := \mathcal{L}(A) + \mathcal{R}(A) \subseteq Hom(A,A)$. If we define the sequence of spaces $\{\mathscr{M}_i\}_{i\in\mathbb{N}}$ as follows:

$$(2.3)\qquad \mathscr{M}_1 = \mathscr{M} \qquad \mathscr{M}_i := [\mathscr{M}_1, \mathscr{M}_{i-1}], \quad i = 2,3,....$$

then, as in [4, § 2], the space $\mathscr{L}(A) := \mathscr{M}_1 + \mathscr{M}_2 + \dots$ is the smallest Lie algebra containing $\mathscr{M}_1 = \mathscr{M}$. Then $\mathscr{L}(A)$ is referred to as the Lie transformation algebra of A. Following [4], we will say that a derivation $D : A \longrightarrow A$ is inner if $D \in \mathscr{L}(A)$.

In particular, suppose that A is an associative monoid. Then, for any morphism $f : 1 \longrightarrow A$, it is known (see [1, § 4]) that $L_A(f) - R_A(f) \in Hom(A,A)$ is a derivation on A. Further, for any $f, g : 1 \longrightarrow A$, it follows from associativity of A that either of the compositions $L_A(f) \circ R_A(g)$ and $R_A(g) \circ L_A(f)$ is equal to the composition

$$(2.4)\qquad \begin{aligned} &L_A(f)R_A(g) = \\ &R_A(g)L_A(f) : A \xrightarrow{\cong} 1 \otimes (A \otimes 1) \xrightarrow{f\otimes(1\otimes g)} A \otimes (A \otimes A) \xrightarrow{m_A\circ(1\otimes m_A)} A \end{aligned}$$

i.e., we have $[L_A(f), R_A(g)] = 0$. We will denote by $m(f,g)$ the morphism $m(f,g) : 1 \xrightarrow{\cong} 1 \otimes 1 \xrightarrow{f\otimes g} A \otimes A \xrightarrow{m_A} A$. We now have the following result.

Proposition 2.2. *Let A be an associative monoid object in $(\mathbf{C}, \otimes, \mathbf{1})$. Then, the Lie transformation algebra $\mathscr{L}(A)$ of A is given by*

$$(2.5)\qquad \mathscr{L}(A) = \mathcal{L}(A) + \mathcal{R}(A)$$

Further, if A has no left (resp. right) absolute divisor of zero, then $D \in Hom(A,A)$ is an inner derivation if and only if $D = L_A(f) - R_A(f)$ for some $f : 1 \longrightarrow A$.

Proof. Let us choose elements $L_A(f) + R_A(g), L_A(f') + R_A(g') \in \mathcal{L}(A) + \mathcal{R}(A)$ for morphisms $f, f', g, g' : 1 \longrightarrow A$. Then, since $[L_A(f), R_A(g')] =$

$[L_A(f'), R_A(g)] = 0$, it follows that

(2.6) $$[L_A(f)+R_A(g), L_A(f')+R_A(g')] = [L_A(f), L_A(f')] + [R_A(g), R_A(g')]$$

We now notice that:

(2.7)
$$\begin{aligned}
&L_A(f)L_A(f')\\
&= \left(A \xrightarrow{\cong} 1\otimes A \xrightarrow{f'\otimes 1} A\otimes A \xrightarrow{m_A} A \xrightarrow{\cong} 1\otimes A \xrightarrow{f\otimes 1} A\otimes A \xrightarrow{m_A} A\right)\\
&= \left(A \xrightarrow{\cong} 1\otimes(1\otimes A) \xrightarrow{f\otimes(f'\otimes 1)} A\otimes(A\otimes A) \xrightarrow{m_A\circ(1\otimes m_a)} A\right)\\
&= \left(A \xrightarrow{\cong} (1\otimes 1)\otimes A \xrightarrow{(f\otimes f')\otimes 1} (A\otimes A)\otimes A \xrightarrow{m_A\circ(m_a\otimes 1)} A\right) = L_A(m(f,f'))
\end{aligned}$$

Hence, it follows that $[L_A(f), L_A(f')] = L_A(m(f,f') - m(f',f))$. Similarly, we may show that $R_A(g)R_A(g') = R_A(m(g',g))$ and hence $[R_A(g), R_A(g')] = R_A(m(g',g) - m(g,g'))$. From (2.6), it follows that

(2.8) $$[L_A(f)+R_A(g), L_A(f')+R_A(g')] = L_A(m(f,f') - m(f',f)) - R_A(m(g',g) - m(g,g'))$$

and hence $\mathcal{L}(A) + \mathcal{R}(A)$ is a Lie algebra. Since $\mathscr{L}(A)$ is the smallest Lie algebra containing $\mathcal{L}(A) + \mathcal{R}(A)$, we have $\mathscr{L}(A) = \mathcal{L}(A) + \mathcal{R}(A)$.

We now suppose that $L_A(f) + R_A(g) \in \mathscr{L}(A)$ is a derivation. Since $L_A(g) - R_A(g)$ is a derivation as mentioned before, so is $L_A(f+g) = (L_A(f) + R_A(g)) + (L_A(g) - R_A(g))$. Since A is associative, the following commutative diagram

(2.9)
$$\begin{array}{ccccccc}
A\otimes A & \xrightarrow[\cong]{l_A\otimes 1} & (1\otimes A)\otimes A & \xrightarrow{((f+g)\otimes 1)\otimes 1} & (A\otimes A)\otimes A & \xrightarrow{=} & (A\otimes A)\otimes A\\
{\scriptstyle =}\downarrow & & {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle m_A\otimes 1}\\
A\otimes A & \xrightarrow[\cong]{l_{A\otimes A}} & 1\otimes(A\otimes A) & \xrightarrow{(f+g)\otimes(1\otimes 1)} & A\otimes(A\otimes A) & & A\otimes A\\
{\scriptstyle m_A}\downarrow & & {\scriptstyle 1\otimes m_A}\downarrow & & {\scriptstyle 1\otimes m_A}\downarrow & & {\scriptstyle m_A}\downarrow\\
A & \xrightarrow[\cong]{l_A} & 1\otimes A & \xrightarrow{(f+g)\otimes 1} & A\otimes A & \xrightarrow{m_A} & A
\end{array}$$

shows that $m_A \circ (L_A(f+g)\otimes 1) = L_A(f+g)\circ m_A : A\otimes A \longrightarrow A$. Since $L_A(f+g)$ is a derivation, it follows from (2.1) that $m_A \circ (1 \otimes L_A(f+g)) = 0$. On the other hand, we note that

(2.10) $$0 = m_A \circ (1\otimes L_A(f+g)) \circ (e_A \otimes 1) \circ l_A = L_A(m(e_A, f+g)) = L_A(e_A)L_A(f+g) = L_A(f+g)$$

and hence $L_A(f+g)=0$. Now, if A has no left absolute divisors of zero, it follows that $L_A(f+g)=0$ and the inner derivation $L_A(f)+R_A(g)\in\mathscr{L}(A)$ is actually of the form $L_A(f)+R_A(g)=R_A(g)-L_A(g)=L_A(-g)-R_A(-g)$. The result follows similarly for the case of no right absolute divisors of zero. □

PROPOSITION 2.3. *Let A be an object of $Mon(\mathbf{C})$. Then, the space $\mathscr{L}(A)\cap Der(A)$ of inner derivations on A is an ideal in the Lie algebra $Der(A)$.*

PROOF. We choose a morphism $f:1\longrightarrow A$ and some $D\in Der(A)$. Our first objective is to show that $[D,L_A(f)]=D\circ L_A(f)-L_A(f)\circ D=L_A(D\circ f)$. For this, we note that:

$$\begin{aligned}
D\circ L_A(f) &= D\circ m_A\circ(f\otimes 1)\circ l_A\\
&= m_A\circ(D\otimes 1+1\otimes D)\circ(f\otimes 1)\circ l_A\\
&= m_A\circ((D\circ f)\otimes 1)\circ l_A + m_A\circ(1\otimes D)\circ(f\otimes 1)\circ l_A\\
&= L_A(D\circ f)+m_A\circ(f\otimes 1)\circ(1\otimes D)\circ l_A\\
&= L_A(D\circ f)+m_A\circ(f\otimes 1)\circ l_A\circ D\\
&= L_A(D\circ f)+L_A(f)\circ D
\end{aligned}\tag{2.11}$$

It follows from (2.11) that $[D,\mathcal{L}(A)]\subseteq\mathcal{L}(A)$. Similarly, we may show that $[D,\mathcal{R}(A)]\subseteq\mathcal{R}(A)$. It follows, therefore, that for $\mathscr{M}_1=\mathscr{M}=\mathcal{L}(A)+\mathcal{R}(A)$, $[D,\mathscr{M}_1]\subseteq\mathscr{M}_1$.

We now suppose that $[D,\mathscr{M}_j]\subseteq\mathscr{M}_j$ for all $j\leq i$ for some given i. Then, given any element $D'\in\mathscr{M}_{i+1}$, by definition of $\mathscr{M}_{i+1}$ in (2.3), D' may be written as a sum $D'=\sum_{l=1}^{k}D'_l$ with each $D'_l\in[\mathscr{M}_1,\mathscr{M}_i]$. We now note that for each $1\leq l\leq k$, we have

$$[D,D'_l]\in[D,[\mathscr{M}_1,\mathscr{M}_i]]\subseteq[\mathscr{M}_1,[D,\mathscr{M}_i]]+[\mathscr{M}_i,[D,\mathscr{M}_1]]\subseteq [\mathscr{M}_1,\mathscr{M}_i]+[\mathscr{M}_i,\mathscr{M}_1]=\mathscr{M}_{i+1}\tag{2.12}$$

From (2.12), it follows that $[D,\mathscr{M}_{i+1}]\subseteq\mathscr{M}_{i+1}$ and hence $[D,\mathscr{M}_i]\subseteq\mathscr{M}_i$ for all $i\geq 1$ by induction. It follows that $[D,\mathscr{L}(A)]\subseteq\mathscr{L}(A)$. Since $[D,Der(A)]\subseteq Der(A)$, it follows that $[D,\mathscr{L}(A)\cap Der(A)]\subseteq\mathscr{L}(A)\cap Der(A)$. □

REMARK 2.4. *It follows from Proposition 2.3 that if A is a monoid such that the derivation algebra $Der(A)$ is simple (as a Lie algebra) and there exist non zero inner derivations of A, then every derivation of A is inner.*

Given a monoid object A, we will say that a morphism $f : 1 \longrightarrow A$ is in the nucleus of A if each of the following three morphisms is identically 0:

(2.13)
$$A_0(f) := \left(A \otimes A \xrightarrow[\cong]{l_A \otimes 1} (1 \otimes A) \otimes A \xrightarrow{(f \otimes 1) \otimes 1} (A \otimes A) \otimes A \xrightarrow{m_A(m_A \otimes 1 - (1 \otimes m_A)a)} A\right)$$
$$A_1(f) := \left(A \otimes A \xrightarrow[\cong]{1 \otimes l_A} A \otimes (1 \otimes A) \xrightarrow{1 \otimes (f \otimes 1)} A \otimes (A \otimes A) \xrightarrow{m_A((m_A \otimes 1)a^{-1} - 1 \otimes m_A)} A\right)$$
$$A_2(f) := \left(A \otimes A \xrightarrow[\cong]{1 \otimes r_A} A \otimes (A \otimes 1) \xrightarrow{1 \otimes (1 \otimes f)} A \otimes (A \otimes A) \xrightarrow{m_A((m_A \otimes 1)a^{-1} - 1 \otimes m_A)} A\right)$$

where a is the natural isomorphism $a : (A \otimes A) \otimes A \xrightarrow{\cong} A \otimes (A \otimes A)$. The set of all morphisms in the nucleus of A will be denoted by $Nuc(A)$ (compare [3, § 1.13]). Clearly, if A is associative, $Nuc(A) = Hom(1, A)$.

Proposition 2.5. *Let A be an object of $Mon(\mathbf{C})$. Let D be a derivation on A and let $f : 1 \longrightarrow A$ be an element of the nucleus of A. Then, $D \circ f \in Nuc(A)$.*

Proof. Since $f \in Nuc(A)$, we know that $A_0(f) = A_1(f) = A_2(f) = 0$ as defined in (2.13). We proceed as follows:

$$\begin{aligned}
&A_0(D \circ f)\\
&= m_A(m_A \otimes 1 - (1 \otimes m_A)a)((D \circ f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= m_A(m_A \otimes 1 - (1 \otimes m_A)a)((D \otimes 1) \otimes 1)((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= m_A((Dm_A - m_A(1 \otimes D)) \otimes 1)((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&\quad - m_A(D \otimes 1)(1 \otimes m_A)a((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= (m_A(D \otimes 1)(m_A \otimes 1) - m_A(m_A \otimes 1)((1 \otimes D) \otimes 1)\\
&\quad - Dm_A(1 \otimes m_A)a + m_A(1 \otimes Dm_A)a) \circ ((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= (Dm_A(m_A \otimes 1) - m_A(1 \otimes D)(m_A \otimes 1) - m_A(m_A \otimes 1)((1 \otimes D) \otimes 1)\\
&\quad - Dm_A(1 \otimes m_A)a + m_A(1 \otimes Dm_A)a)((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= D \circ A_0(f) - m_A((m_A \otimes 1)((1 \otimes 1) \otimes D) + (m_A \otimes 1)((1 \otimes D) \otimes 1)\\
&\quad - (1 \otimes Dm_A)a)((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= -m_A(m_A \otimes 1)((1 \otimes D) \otimes 1 + (1 \otimes 1) \otimes D)((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&\quad + m_A(1 \otimes m_A)(1 \otimes (D \otimes 1) + 1 \otimes (1 \otimes D))a((f \otimes 1) \otimes 1)(l_A \otimes 1)\\
&= m_A((1 \otimes m_A)a - m_A \otimes 1)((f \otimes 1) \otimes 1)((1 \otimes D) \otimes 1)(l_A \otimes 1)\\
&\quad + m_A((1 \otimes m_A)a - m_A \otimes 1)((f \otimes 1) \otimes 1)((1 \otimes 1) \otimes D)(l_A \otimes 1)\\
&= -A_0(f) \circ (D \otimes 1) - A_0(f) \circ (1 \otimes D) = 0
\end{aligned}$$

Similarly, one may check that both $A_1(D \circ f)$ and $A_2(D \circ f)$ are 0. Hence, $D \circ f \in Nuc(A)$. □

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