

Regularity for elliptic pairs over $\mathbb{C}[[\hbar]]$

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ABSTRACT - We extend the results of Schapira and Schneiders [12] on relative regularity, finiteness and duality (in the smooth case) of elliptic pairs to the framework of $\mathcal{D}[[\hbar]]$ -modules and constructible sheaves of $\mathbb{C}[[\hbar]]$ -modules.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 35A27, 32C38, 46L65.

KEYWORDS. Modules over the ring of formal differential operators, Elliptic pairs, Regularity, Finiteness and duality properties.

Introduction

The notion of elliptic pair goes back to [12], where the authors consider a morphism of complex analytic manifolds $f : X \rightarrow Y$ and say that a coherent \mathcal{D}_X -module \mathcal{M} and an \mathbb{R} -constructible sheaf of \mathbb{C} -modules F form an f -elliptic pair if the f -characteristic variety of \mathcal{M} and the micro-support of F do not intersect outside the zero-section of the cotangent bundle T^*X .

Consider the constant map $a_X : X \rightarrow \{\text{pt}\}$. If X is the complexification of a real analytic manifold M and \mathcal{M} is an elliptic system on M , then $(\mathcal{M}, \mathbb{C}_M)$ is an a_X -elliptic pair. Elliptic pairs can thus be regarded as a generalization of elliptic systems. The functorial properties of elliptic pairs are studied in [12], where theorems of regularity, finiteness and duality are proved for these objects. As the authors point out, such theorems gen-

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The research of the author was supported by Fundação para a Ciência e Tecnologia and by Fundação Calouste Gulbenkian (Programa Estímulo à Investigação).

eralize several classical results of \mathcal{D}_X -modules theory, complex analytic geometry and elliptic systems theory.

The purpose of this paper is to extend the main results of [12] to the framework of modules over the ring $\mathcal{D}_X[[\hbar]]$, the ring of differential operators with a formal parameter \hbar . This ring appeared first in [7] as an example of an algebra of formal deformation. Therefore, the machinery developed in [7] to perform the study of deformation-quantization modules also apply to the study of $\mathcal{D}[[\hbar]]$ -modules. Such study has been performed in [1] and [9].

Denote by $\mathbb{C}[[\hbar]]$ the ring of formal power series on \mathbb{C} . Consider the right exact functor that maps each sheaf of $\mathbb{C}[[\hbar]]$ -modules F to $F/\hbar F$ regarded as a sheaf of \mathbb{C} -modules. Consider also its left derived functor denoted by gr_{\hbar} .

In this work we introduce the notion of f -elliptic pair over $\mathbb{C}[[\hbar]]$ in a natural way: if \mathcal{M} is a coherent $\mathcal{D}_X[[\hbar]]$ -module and F is an \mathbb{R} -constructible sheaf of $\mathbb{C}[[\hbar]]$ -modules, then (\mathcal{M}, F) is an f -elliptic pair over $\mathbb{C}[[\hbar]]$ if and only if $(\mathrm{gr}_{\hbar}(\mathcal{M}), \mathrm{gr}_{\hbar}(F))$ is an elliptic pair in the sense of [12]. This allow us to prove theorems for f -elliptic pairs over $\mathbb{C}[[\hbar]]$ using properties of gr_{\hbar} given by [7] to reduce the proofs to the theorems of [12].

Let us mention our main results. In Theorems 3.1 and 3.7 we prove regularity properties for f -elliptic pairs over $\mathbb{C}[[\hbar]]$. These regularity theorems generalize a classical regularity property of elliptic systems in the real analytic setting: the complex of real analytic solutions of an elliptic system is isomorphic to the complex of hyperfunctions solutions.

In Theorem 3.9 we use the regularity of f -elliptic pairs over $\mathbb{C}[[\hbar]]$ to give a finiteness criteria. Denote by $f_{\rightarrow, \hbar}$ the proper direct image in the framework of $\mathcal{D}_X[[\hbar]]$ -modules introduced in [9]. The statement of the theorem is the following: given an f -elliptic pair (\mathcal{M}, F) , such that f is proper when restricted to $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ and such that \mathcal{M} is good, then the cohomologies of the direct image $f_{\rightarrow, \hbar}(\mathcal{M} \otimes_{\mathcal{D}_X[[\hbar]]}^{\mathbb{L}} F)$ are coherent over $\mathcal{D}_Y[[\hbar]]$.

In Theorem 3.16 we prove a duality result in the case of a smooth morphism. It states that the direct image and the duality functor for $\mathcal{D}_X[[\hbar]]$ -modules commute when applied to f -elliptic pairs that satisfy the finiteness criteria. The reason why we must restrict to the smooth case is that two fundamental properties hold in this case: the transfer module is coherent over $\mathcal{D}_X[[\hbar]]$ and the extension rings $f^{-1}(\mathcal{D}_X[[\hbar]])$ and $(f^{-1}\mathcal{D}_X)[[\hbar]]$ are isomorphic. These properties are necessary to our construction of the duality morphism. We note that the smooth case includes the interesting case $f = a_X$.

In the last part of the paper we illustrate our results in some particular cases. For example, in the case where X is the complexification of a real analytic manifold M , we obtain regularity, finiteness and duality properties on the sheaves of formal analytic functions and formal hyperfunctions on M .

1. Complements on formal extensions.

In view of our purpose it suffices to work on the complex analytic setting, although some results hold in a more general situation. In the sequel X denotes a complex analytic manifold of dimension d_X .

We follow the notations of [5]. Namely, if \mathcal{R} is a sheaf of rings on X , we denote by $\text{Mod}(\mathcal{R})$ the category of left \mathcal{R} -modules and by $\text{D}^b(\mathcal{R})$ the bounded derived category of $\text{Mod}(\mathcal{R})$. If \mathcal{R} is coherent, we denote by $\text{Mod}_{\text{coh}}(\mathcal{R})$ the full abelian subcategory of $\text{Mod}(\mathcal{R})$ of coherent objects and by $\text{D}_{\text{coh}}^b(\mathcal{R})$ the full triangulated subcategory of $\text{D}^b(\mathcal{R})$ of objects with coherent cohomology groups. In the sequel, when the base ring is \mathbb{C} we may omit it.

Let \mathcal{R}_0 denote a sheaf of \mathbb{C}_X -algebras on X and set $\mathcal{R} := \mathcal{R}_0[[\hbar]] = \prod_{n \geq 0} \mathcal{R}_0 \hbar^n$. Then \mathcal{R} is a sheaf of algebras over the ring $\mathbb{C}[[\hbar]]$, the ring of formal power series with complex coefficients. One uses the abbreviation $\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]]$.

Consider the following left exact functor studied in [1]:

$$(1.1) \quad \begin{aligned} (\bullet)^{\hbar} : \text{Mod}(\mathcal{R}_0) &\rightarrow \text{Mod}(\mathcal{R}) \\ \mathcal{N} &\rightarrow \mathcal{N}^{\hbar} := \mathcal{N}[[\hbar]] = \varprojlim_{\hbar \geq 0} (\mathcal{N} \otimes_{\mathcal{R}_0} \mathcal{R} / \hbar^{n+1} \mathcal{R}). \end{aligned}$$

Recall that the sections of \mathcal{N}^{\hbar} on an open subset of X are formal power series of sections of \mathcal{N} on the same open subset.

One denotes by $(\bullet)^{\text{R}\hbar} : \text{D}^b(\mathcal{R}_0) \rightarrow \text{D}^b(\mathcal{R})$ the right derived functor of $(\bullet)^{\hbar}$. For each $F \in \text{D}^b(\mathcal{R}_0)$, $F^{\text{R}\hbar}$ is called the formal extension of F .

One says that $F \in \text{Mod}(\mathcal{R}_0)$ is \hbar -acyclic if $F^{\text{R}\hbar} \simeq F^{\hbar}$.

PROPOSITION 1.1 ([1], Proposition 2.5). *Let $\mathcal{N} \in \text{Mod}(\mathcal{R}_0)$ and suppose that \mathcal{B} is either a basis of open subsets of X , or a basis of compact subsets, such that $H^j(S; \mathcal{N}) = 0$ for all $j > 0$ and all $S \in \mathcal{B}$. Then, \mathcal{N} is \hbar -acyclic.*

From now on assume that \mathcal{R}_0 is an \hbar -acyclic \mathbb{C}_X -algebra.

LEMMA 1.2. *Let $f : X \rightarrow Y$ be a morphism of complex manifolds and assume that $(f^{-1}\mathcal{R}_0)^\hbar \simeq f^{-1}\mathcal{R}$. Then, for each $\mathcal{N} \in \mathbf{D}^b(f^{-1}\mathcal{R}_0)$, we have a canonical morphism in $\mathbf{D}^b(\mathcal{R})$:*

$$\mathbf{R}f_!(\mathcal{N}^{\mathbf{R}\hbar}) \rightarrow \mathbf{R}f_!(\mathcal{N})^{\mathbf{R}\hbar}.$$

PROOF. Using Proposition 2.1 of [1] and formula (2.6.26) of [5] we have in $\mathbf{D}^b(\mathcal{R})$:

$$\begin{aligned} \mathbf{R}f_!(\mathcal{N}^{\mathbf{R}\hbar}) &\simeq \mathbf{R}f_! \mathbf{R}\mathcal{H}om_{f^{-1}\mathcal{R}_0}(f^{-1}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}), \mathcal{N}) \\ &\rightarrow \mathbf{R}\mathcal{H}om_{\mathcal{R}_0}(\mathbf{R}f_*f^{-1}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}), \mathbf{R}f_!\mathcal{N}) \\ &\rightarrow \mathbf{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathbf{R}f_!\mathcal{N}) \simeq (\mathbf{R}f_!\mathcal{N})^{\mathbf{R}\hbar}. \end{aligned}$$

□

Consider also the right exact functor that maps each $\mathcal{M} \in \text{Mod}(\mathcal{R})$ into $\mathcal{M}/\hbar\mathcal{M} \in \text{Mod}(\mathcal{R}_0)$. We shall use its left derived functor

$$\text{gr}_\hbar : \mathbf{D}^b(\mathcal{R}) \rightarrow \mathbf{D}^b(\mathcal{R}_0), \quad \mathcal{M} \rightarrow \text{gr}_\hbar(\mathcal{M}) := \mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \mathcal{R}_0$$

which was studied in detail in [7]. Recall that gr_\hbar commutes with tensor products, with $\mathbf{R}\mathcal{H}om$ and also with direct images, proper direct images and inverse images of sheaves. Recall also that $\mathcal{M} \in \text{Mod}(\mathcal{R})$ has no \hbar -torsion if $\text{gr}_\hbar(\mathcal{M})$ is concentrated in degree 0, that is, if $\hbar : \mathcal{M} \rightarrow \mathcal{M}$ is injective.

One says that $\mathcal{M} \in \text{Mod}(\mathcal{R})$ is \hbar -complete if the canonical morphism $\mathcal{M} \rightarrow \varprojlim_n \mathcal{M}/\hbar^{n+1}\mathcal{M}$ is an isomorphism. Set $\mathcal{R}^{\text{loc}} := \mathbb{C}((\hbar))_X \otimes_{\mathbb{C}^h_X} \mathcal{R}$, where $\mathbb{C}((\hbar))$ denotes the field of Laurent series with complex coefficients. One says that $\mathcal{M} \in \mathbf{D}(\mathcal{R})$ is cohomologically complete if $\mathbf{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\text{loc}}, \mathcal{M}) = 0$. The notions of \hbar -complete object and cohomologically complete object don't depend on the base ring.

We refer [7] for a comprehensive study of cohomologically complete objects. Let us just mention some facts that we shall use in the sequel: gr_\hbar is conservative when restricted to the triangulated subcategory of $\mathbf{D}^b(\mathcal{R})$ consisting of cohomologically complete objects (cf. Corollary 1.5.9 of [7]) and $\mathcal{N}^{\mathbf{R}\hbar}$ is cohomologically complete for every $\mathcal{N} \in \mathbf{D}^b(\mathcal{R}_0)$ (cf. Proposition 1.2 of [1]).

In view of Proposition 1.5 below let us fix a morphism of complex analytic manifolds $f : X \rightarrow Y$ and let \mathcal{R}_0 be an \hbar -acyclic \mathbb{C}^h -algebra on Y .

REMARK 1.3. There is a canonical morphism of sheaves of \mathbb{C}^\hbar -algebras $f^{-1}\mathcal{R} = f^{-1}\mathcal{R}_0^\hbar \rightarrow (f^{-1}\mathcal{R}_0)^\hbar$ induced by the morphisms

$$f^{-1}\mathcal{R}_0^\hbar \rightarrow f^{-1}(\mathcal{R}_0 \otimes \mathbb{C}_Y^\hbar / \hbar^{n+1} \mathbb{C}_Y^\hbar) \simeq f^{-1}\mathcal{R}_0 \otimes \mathbb{C}_X^\hbar / \hbar^{n+1} \mathbb{C}_X^\hbar,$$

and by the universal property of projective limits. Hence, there is a canonical functor $F \in \text{Mod}((f^{-1}\mathcal{R}_0)^\hbar) \mapsto F \in \text{Mod}(f^{-1}(\mathcal{R}_0)^\hbar)$.

ASSUMPTION 1.4. There exists a basis \mathcal{B} either of open subsets of Y or of compact subsets of Y such that $H^j(S; \mathcal{R}_0) = 0$, for all $j > 0$ and for all $S \in \mathcal{B}$.

PROPOSITION 1.5. Suppose that $f : X \rightarrow Y$ is smooth and that \mathcal{R}_0 verifies Assumption 1.4. Then $f^{-1}\mathcal{R} \xrightarrow{\sim} (f^{-1}\mathcal{R}_0)^\hbar$.

PROOF. Consider the canonical morphism from Remark 1.3. Note that the corresponding canonical morphism $\text{gr}_\hbar(f^{-1}\mathcal{R}) \rightarrow \text{gr}_\hbar((f^{-1}\mathcal{R}_0)^\hbar)$ is an isomorphism, since one has $\text{gr}_\hbar(f^{-1}\mathcal{R}) \simeq \text{gr}_\hbar((f^{-1}\mathcal{R}_0)^\hbar) \simeq f^{-1}\mathcal{R}_0$. Hence, in view of [7, Cor. 1.5.9], it suffices to show that both $f^{-1}\mathcal{R}$ and $(f^{-1}\mathcal{R}_0)^\hbar$ are cohomologically complete objects.

The ring $\mathcal{R} \simeq \mathcal{R}_0^{\text{R}\hbar}$ is cohomologically complete. Since f is smooth, \mathcal{R} is non-characteristic for f and $f^{-1}\mathcal{R}$ is cohomologically complete by Proposition 2.15 of [9].

We shall use Proposition 1.1 to prove that $(f^{-1}\mathcal{R}_0)^{\text{R}\hbar}$ is concentrated in degree 0, thus $(f^{-1}\mathcal{R}_0)^\hbar$ is cohomologically complete.

The result is now checked locally, so we can assume that $X = X' \times Y$ and that $f : X \rightarrow Y$ is the canonical projection.

Let us consider the case where \mathcal{B} is a basis of open subsets of Y . It is enough to show that there exists a basis \mathcal{B}' of open subsets of $X' \times Y$ such that $H^j(S', f^{-1}\mathcal{R}_0) = 0$ for all $j > 0$ and all $S' \in \mathcal{B}'$.

Consider the basis \mathcal{B}' formed by the open sets $V' \times V \subset X' \times Y$ such that V' is an open ball of X' (hence, contractible) and $V \in \mathcal{B}$. We are in the conditions to apply [5, Proposition 2.7.8]. Hence, for any $j > 0$ one has:

$$H^j(V' \times V, f^{-1}\mathcal{R}_0) \simeq H^j(f_{|V' \times V}^{-1}(V), f_{|V' \times V}^{-1}\mathcal{R}_0|_V) \simeq H^j(V, \mathcal{R}_0|_V) = 0.$$

The case where \mathcal{B} is a basis of compact subsets of X is similar, taking closed balls on X' instead of open balls. \square

\mathbb{R} -constructible sheaves of $\mathbb{C}[[\hbar]]$ -modules.

As usual denote by $\text{SS}(F)$ the micro-support of an object $F \in \text{D}^b(\mathbb{C}_X^\hbar)$, a closed involutive subset of the cotangent bundle T^*X . One has the estimate $\text{SS}(\text{gr}_\hbar(F)) \subset \text{SS}(F)$. The equality $\text{SS}(\text{gr}_\hbar(F)) = \text{SS}(F)$ holds if F is cohomologically complete (cf. Proposition 1.15 of [1]), then in such case one also has $\text{supp}(F) = \text{supp}(\text{gr}_\hbar(F))$.

Let \mathbb{K} be either \mathbb{C} or \mathbb{C}^\hbar . One denotes by $\text{D}_{\mathbb{R}-c}^b(\mathbb{K}_X)$ the bounded derived category of sheaves of \mathbb{K} -modules with \mathbb{R} -constructible cohomology.

Consider the constant map $a_X : X \rightarrow \{\text{pt}\}$ and denote by $\omega_X^\mathbb{K}$ the dualizing sheaf in the category $\text{D}^b(\mathbb{K}_X)$. Recall that one has $\omega_X^\mathbb{K} \simeq a_X^! \mathbb{K} \simeq \mathbb{K}_X[2d_X]$.

We shall use the duality functors below:

$$\begin{aligned} D'_{\mathbb{K}_X} : \text{D}^b(\mathbb{K}_X) &\rightarrow \text{D}^b(\mathbb{K}_X), & F &\mapsto \text{R.}\mathcal{H}om[\mathbb{K}_X](F, \mathbb{K}_X), \\ D_{\mathbb{K}_X} : \text{D}^b(\mathbb{K}_X) &\rightarrow \text{D}^b(\mathbb{K}_X), & F &\mapsto \text{R.}\mathcal{H}om[\mathbb{K}_X](F, \omega_X^\mathbb{K}). \end{aligned}$$

They induce functors $D'_{\mathbb{K}_X}, D_{\mathbb{K}_X} : \text{D}_{\mathbb{R}-c}^b(\mathbb{K}_X) \rightarrow \text{D}_{\mathbb{R}-c}^b(\mathbb{K}_X)$ which satisfy the microlocal relation: $\text{SS}(D'_{\mathbb{K}_X} F) = \text{SS}(D_{\mathbb{K}_X} F) = \text{SS}(F)^a$, where a denotes the opposite map on T^*X .

Any $F \in \text{D}_{\mathbb{R}-c}^b(\mathbb{C}_X^\hbar)$ is cohomologically complete by Proposition 1.6 of [1]. Hence, $\text{gr}_\hbar : \text{D}_{\mathbb{R}-c}^b(\mathbb{C}_X^\hbar) \rightarrow \text{D}^b(\mathbb{C}_X)$ is conservative and preserves the micro-support. It is also known that \mathbb{R} -constructible sheaves are \hbar -acyclic (Corollary 2.6 of [1]).

The next proposition shows that $F^\hbar \otimes_{\mathbb{C}_X^\hbar} \bullet$ is an exact functor in $\text{Mod}(\mathbb{C}_X^\hbar)$ for each $F \in \text{Mod}_{\mathbb{R}-c}(\mathbb{C}_X)$.

PROPOSITION 1.6. *Let $F \in \text{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$. Then we have $F^{\text{R}\hbar} \simeq F \otimes \mathbb{C}_X^\hbar$.*

PROOF. First we remark that by replacing F with an almost free resolution of F (in the sense of the Appendix of [6]) one easily reduces the proof to the case $F = \mathbb{C}_U$, U being an open subanalytic relatively compact subset of X . This reduction uses the fact that \mathbb{R} -constructible sheaves are \hbar -acyclic.

Therefore, it is enough to note that for each open subanalytic subset U we have a canonical isomorphism:

$$\mathbb{C}_U \otimes \mathbb{C}_X^\hbar \xrightarrow{\sim} \mathbb{C}_U^\hbar.$$

In fact, the stalks $(\mathbb{C}_U \otimes \mathbb{C}_X^\hbar)_x$ and $(\mathbb{C}_U^\hbar)_x$ are both isomorphic to \mathbb{C}^\hbar if $x \in U$, and both vanish if $x \notin U$. □

As a consequence of Proposition 1.6 above and Lemma 1.9 of [1], we have:

COROLLARY 1.7. *For $F \in \mathbf{D}_{\mathbb{R}-c}^b(C_X)$ and $G \in \mathbf{D}^b(C_X)$, there are isomorphisms in $\mathbf{D}^b(C_X^{\hbar})$:*

- (i) $\mathbf{R}\mathcal{H}om_{C_X^{\hbar}}(F^{\mathbf{R}\hbar}, G^{\mathbf{R}\hbar}) \simeq (\mathbf{R}\mathcal{H}om(F, G))^{\mathbf{R}\hbar}$
- (ii) $F^{\mathbf{R}\hbar} \otimes_{C_X^{\hbar}}^L G^{\mathbf{R}\hbar} \simeq F \otimes G^{\mathbf{R}\hbar}$.

2. Elliptic pairs over $\mathbb{C}[[\hbar]]$

$\mathcal{D}[[\hbar]]$ -modules.

Let \mathcal{O}_X , Ω_X and \mathcal{D}_X denote the ring of holomorphic functions on X , the sheaf of holomorphic forms of maximal degree on X , and the ring of linear differential holomorphic operators on X , respectively.

As usual, $\text{Mod}_{\text{gd}}(\mathcal{D}_X)$ denotes the full subcategory of $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ consisting of good \mathcal{D}_X -modules (in the sense of [4]) and $\mathbf{D}_{\text{gd}}^b(\mathcal{D}_X)$ denotes the full subcategory of $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_X)$ consisting of objects with good cohomology.

An object $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^{\hbar})$ is said to be good if $\text{gr}_{\hbar}(\mathcal{M}) \in \mathbf{D}_{\text{gd}}^b(\mathcal{D}_X)$. Denote by $\mathbf{D}_{\text{gd}}^b(\mathcal{D}_X^{\hbar})$ the full triangulated subcategory of $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^{\hbar})$ consisting of good objects.

We shall use the duality functors below:

$$\begin{aligned} \underline{D}_{\hbar, X} : \mathbf{D}^b((\mathcal{D}_X^{\hbar})^{\text{op}}) &\rightarrow \mathbf{D}^b((\mathcal{D}_X^{\hbar})^{\text{op}}), & \mathcal{M} &\mapsto \mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\hbar}) \\ \mathbf{D}'_{\hbar, X} : \mathbf{D}^b(\mathcal{D}_X^{\hbar}) &\rightarrow \mathbf{D}^b((\mathcal{D}_X^{\hbar})^{\text{op}}), & \mathcal{M} &\mapsto \mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{D}_X). \end{aligned}$$

Denote also by \underline{D}_X and \mathbf{D}'_X their counterparts for \mathcal{D} -modules.

The rings \mathcal{D}_X^{\hbar} and $(\mathcal{D}_X^{\hbar})^{\text{op}}$ are algebras of formal deformation in the sense of [7] and the machinery developed in loc. cit. apply to the study of \mathcal{D}_X^{\hbar} -modules. In particular, objects belonging to $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^{\hbar})$ are cohomologically complete and Theorem 1.6.4 of [7] provide a useful coherence criteria for \mathcal{D}_X^{\hbar} -modules. Recall also that coherent \mathcal{D}_X -modules are \hbar -acyclic (cf. Corollary 2.6 of [1]).

In the sequel most of our results are stated for right \mathcal{D}_X^{\hbar} -modules but one can get similar results for left \mathcal{D}_X^{\hbar} -modules. Indeed, the category of left \mathcal{D}_X^{\hbar} -modules and the category of right \mathcal{D}_X^{\hbar} -modules are equivalent. Hence, we don't need to distinguish left and right \mathcal{D}_X^{\hbar} -modules.

Direct images.

From now on, $f : X \rightarrow Y$ denotes a complex morphism between two complex analytic manifolds of dimensions d_X and d_Y , respectively.

Consider the usual transfer-module $\mathcal{D}_{X \rightarrow Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ with its structure of $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$ -bimodule. Denote by f_{*} and $f_{!}$ the functors of direct image and proper direct image in the \mathcal{D} -modules framework.

Set

$$\mathcal{D}_{X \rightarrow Y, \hbar} := \varprojlim_n (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y/\hbar^{n+1}\mathcal{D}_Y)).$$

Since each component of the projective limit has a natural structure of \hbar -torsion $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$ -bimodule, then $\mathcal{D}_{X \rightarrow Y, \hbar}$ is also a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$ -bimodule. Hereafter, when there is no risk of confusion, we use the following abbreviations: $\mathcal{K}_{\hbar} := \mathcal{D}_{X \rightarrow Y, \hbar}$ and $\mathcal{K} := \mathcal{D}_{X \rightarrow Y}$.

Note that \mathcal{K}_{\hbar} is also a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$ -bimodule (cf. Remark 1.3) and one has an isomorphism of bimodules $\mathcal{K}_{\hbar} \simeq \mathcal{K}^{\hbar}$. Moreover, $\mathcal{K}_{\hbar} \simeq \mathcal{K}^{\text{R}\hbar}$ is cohomologically complete, free of \hbar -torsion and $\text{gr}_{\hbar}(\mathcal{K}_{\hbar}) \simeq \mathcal{K}$ (Proposition 5.5 of [9]).

The direct image functor and the proper direct image functor in the framework of \mathcal{D}^{\hbar} -modules are denoted respectively by f_{*}^{\hbar} and $f_{!}^{\hbar}$ and defined by:

$$\begin{aligned} f_{*}^{\hbar} &: \text{D}^b(\mathcal{D}_X^{\hbar}) \rightarrow \text{D}^b(\mathcal{D}_Y^{\hbar}), \quad \mathcal{M} \mapsto \text{R}f_* (\mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{K}_{\hbar}); \\ f_{!}^{\hbar} &: \text{D}^b(\mathcal{D}_X^{\hbar}) \rightarrow \text{D}^b(\mathcal{D}_Y^{\hbar}), \quad \mathcal{M} \mapsto \text{R}f_! (\mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{K}_{\hbar}). \end{aligned}$$

Direct images for \mathcal{D}^{\hbar} -modules are introduced in [9] (where the transfer module in the \hbar -setting is simply denoted by \mathcal{K}).

Clearly, for $\mathcal{M} \in \text{D}^b(\mathcal{D}_X^{\hbar})$ the isomorphisms $\text{gr}_{\hbar}(f_{*}^{\hbar}(\mathcal{M})) \simeq f_{*}(\text{gr}_{\hbar}(\mathcal{M}))$ and $\text{gr}_{\hbar}(f_{!}^{\hbar}(\mathcal{M})) \simeq f_{!}(\text{gr}_{\hbar}(\mathcal{M}))$ hold.

f-characteristic variety.

Following [12], the f -characteristic variety of $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X)$, denoted by $\text{char}_f(\mathcal{M})$, is a closed conic analytic subvariety of T^*X depending on f and satisfying the formula

$$(2.1) \quad \text{SS}(\mathcal{M} \otimes_{\mathcal{D}_X}^{\text{L}} \mathcal{K}) \subset \text{char}_f(\mathcal{M}).$$

Note that char_{α_X} is the usual characteristic variety of \mathcal{M} , denoted simply by $\text{char}(\mathcal{M})$. In this case, the inclusion (2.1) gives the well-known estimative: $\text{SS}(\text{Sol}(\mathcal{M})) \subset \text{char}(\mathcal{M})$. Here Sol denotes the solutions func-

tor in the \mathcal{D} -module framework, whose counterpart in the \mathcal{D}^\hbar -modules framework is the functor Sol_\hbar studied in [1]:

$$\text{Sol}_\hbar : \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)^{\text{op}} \rightarrow \mathbf{D}^b(\mathbb{C}_X^\hbar), \quad \mathcal{M} \mapsto \mathbf{R}\mathcal{H}om_{\mathcal{O}_X^\hbar}(\mathcal{M}, \mathcal{O}_X^\hbar).$$

DEFINITION 2.1. The f -characteristic variety of $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$ is denoted by $\text{char}_{f,\hbar}(\mathcal{M})$ and defined by $\text{char}_{f,\hbar}(\mathcal{M}) := \text{char}_f(\text{gr}_\hbar(\mathcal{M}))$.

For any $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$, $\text{char}_{a_X,\hbar}(\mathcal{M})$ coincides with the characteristic variety of \mathcal{M} denoted by $\text{char}_\hbar(\mathcal{M})$.

LEMMA 2.2. For any $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$, we have $\text{SS}(\mathcal{M} \otimes_{\mathcal{O}_X^\hbar}^{\mathbf{L}} \mathcal{H}_\hbar) \subset \text{char}_{f,\hbar}(\mathcal{M})$.

PROOF. Since \mathcal{M} is coherent and \mathcal{H}_\hbar is cohomologically complete, $\mathcal{M} \otimes_{\mathcal{O}_X^\hbar}^{\mathbf{L}} \mathcal{H}_\hbar$ is also cohomologically complete by Proposition 1.6.5 of [7]. Hence,

$$\text{SS}(\mathcal{M} \otimes_{\mathcal{O}_X^\hbar}^{\mathbf{L}} \mathcal{H}_\hbar) = \text{SS}(\text{gr}_\hbar(\mathcal{M}) \otimes_{\mathcal{O}_X^\hbar}^{\mathbf{L}} \mathcal{H}_\hbar).$$

Therefore, the result follows from estimative (2.1). □

We remark that $\text{SS}(\text{Sol}_\hbar(\mathcal{M})) = \text{char}_\hbar(\mathcal{M})$ is already proved in [1].

Elliptic pairs.

DEFINITION 2.3. A pair (\mathcal{M}, F) with $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^{\hbar,\text{op}})$ and $F \in \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X^\hbar)$ is an f -elliptic pair over \mathbb{C}^\hbar if $\text{char}_{f,\hbar}(\mathcal{M}) \cap \text{SS}(F) \subset T_X^*X$. If in addition $\mathcal{M} \in \mathbf{D}_{\text{gd}}^b(\mathcal{D}_X^{\hbar,\text{op}})$, then (\mathcal{M}, F) is said a good f -elliptic pair over \mathbb{C}^\hbar . The support of the pair (\mathcal{M}, F) is the intersection $\text{supp}(\mathcal{M}) \cap \text{supp}(F)$.

Since gr_\hbar preserves the micro-support of \mathbb{R} -constructible sheaves and the characteristic variety of coherent \mathcal{D}^\hbar -modules we have:

PROPOSITION 2.4. A pair (\mathcal{M}, F) is an f -elliptic pair over \mathbb{C}^\hbar if and only if $(\text{gr}_\hbar(\mathcal{M}), \text{gr}_\hbar(F))$ is an f -elliptic pair over \mathbb{C} (in the sense of [12]).

If (\mathcal{M}, F) is an f -elliptic pair over \mathbb{C}^\hbar , then $(\mathbf{D}_{\hbar,X}\mathcal{M}, D'_{\mathbb{C}^\hbar,X}F)$ is also an f -elliptic pair over \mathbb{C}^\hbar , the dual f -elliptic pair of (\mathcal{M}, F) .

Assume that X is the complexification of a real analytic manifold M .

DEFINITION 2.5. We say that a coherent \mathcal{D}_X^{\hbar} -module \mathcal{M} is an elliptic \mathcal{D}_X^{\hbar} -module if $(\mathcal{M}, \mathbb{C}_M^{\hbar})$ is an a_X -elliptic pair over \mathbb{C}^{\hbar} . We say that an operator $P \in \mathcal{D}_X^{\hbar}$ is an elliptic operator if $\mathcal{D}_X^{\hbar}/\mathcal{D}_X^{\hbar}P$ is an elliptic \mathcal{D}_X^{\hbar} -module.

In other words, $\mathcal{M} \in \text{Mod}_{\text{coh}}[\mathcal{D}_X^{\hbar}]$ is elliptic if $\text{char}_{\hbar}(\mathcal{M}) \cap T_M^*X \subset T_X^*X$. Moreover, one deduces from Lemma 3.5 of [1] that $P \in \mathcal{D}_X^{\hbar}$ is elliptic if and only if it is locally written as $P = P_0 + \hbar P'$ for some $P' \in \mathcal{D}_X^{\hbar}$ and P_0 an elliptic operator in the classical sense. Take a system of holomorphic coordinates $(x; \eta)$ on T^*X . P_0 is elliptic if its principal symbol satisfies $\sigma(P_0)((x; i\eta)) \neq 0$ for $\eta \neq 0$. Take, for example, $X = \mathbb{C}^n$, $M = \mathbb{R}^n$ and denote by Δ the Laplace operator. Then, $P = \Delta + \hbar P'$ is elliptic for any $P' \in \mathcal{D}_X^{\hbar}$.

The meaning of elliptic pairs on the real analytic setting illustrates why one can regard the theory of elliptic pairs (over \mathbb{C}^{\hbar}) as a natural generalization of the theory of elliptic systems on real analytic manifolds.

3. Theorems on elliptic pairs over $\mathbb{C}[[\hbar]]$

Regularity theorem.

THEOREM 3.1. *Let (\mathcal{M}, F) be an elliptic pair over \mathbb{C}^{\hbar} . Then, the natural morphism below is an isomorphism in $\text{D}^b(\mathbb{C}_X^{\hbar})$:*

$$(3.1) \quad F \otimes_{\mathbb{C}_X^{\hbar}}^{\text{L}} (\mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{H}_{\hbar}) \rightarrow \text{R}\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(\text{D}'_{\mathbb{C}_X^{\hbar}} F, \mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{H}_{\hbar})$$

PROOF. The morphism (3.1) is induced by the isomorphism $F \simeq \text{D}'_{\mathbb{C}_X^{\hbar}} \text{D}'_{\mathbb{C}_X^{\hbar}} F$ and by the canonical morphism:

$$\text{D}'_{\mathbb{C}_X^{\hbar}} \text{D}'_{\mathbb{C}_X^{\hbar}} F \otimes_{\mathbb{C}_X^{\hbar}}^{\text{L}} (\mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{H}_{\hbar}) \rightarrow \text{R}\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(\text{D}'_{\mathbb{C}_X^{\hbar}} F, \mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{H}_{\hbar}).$$

Note that $\text{D}'_{\mathbb{C}_X^{\hbar}} F$ has \mathbb{R} -constructible cohomology and $\text{SS}(\text{D}'_{\mathbb{C}_X^{\hbar}} F) = \text{SS}(F)^a$. The transversality condition on the pair (\mathcal{M}, F) together with Lemma 2.2 entail:

$$\text{SS}(\mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\text{L}} \mathcal{H}_{\hbar}) \cap \text{SS}(\text{D}'_{\mathbb{C}_X^{\hbar}} F)^a \subset T_X^*X.$$

The conclusion follows by Proposition 5.4.14 of [5]. □

REMARK 3.2. The isomorphism (3.1) is written in the following form in terms of left \mathcal{D}_X^{\hbar} -modules:

$$\text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \text{D}'_{\mathbb{C}_X^{\hbar}} F \otimes_{\mathbb{C}_X^{\hbar}}^{\text{L}} \mathcal{H}_{\hbar}) \xrightarrow{\sim} \text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \text{R}\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(F, \mathcal{H}_{\hbar})).$$

We want to refine the regularity property in the case where F is the formal extension of an object in $\mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$. Let us start with some auxiliary results.

LEMMA 3.3. *Let $F, G \in \text{Mod}(\mathbb{C}_X)$. There is a natural morphism in $\text{Mod}(\mathbb{C}_X^\hbar)$:*

$$(3.2) \quad F \otimes G^\hbar \rightarrow (F \otimes G)^\hbar.$$

PROOF. It is enough to note that there is a projective system of morphisms $F \otimes G^\hbar \rightarrow F \otimes (G \otimes (\mathbb{C}^\hbar / \hbar^{n+1} \mathbb{C}^\hbar)) \xrightarrow{\sim} (F \otimes G) \otimes (\mathbb{C}^\hbar / \hbar^{n+1} \mathbb{C}^\hbar)$. \square

LEMMA 3.4. *Let $F, G \in \mathbf{D}^b(\mathbb{C}_X)$. Then there is a natural bifunctorial morphism in $\mathbf{D}^b(\mathbb{C}_X^\hbar)$:*

$$(3.3) \quad F \otimes G^{\text{R}\hbar} \rightarrow (F \otimes G)^{\text{R}\hbar}.$$

PROOF. (i) First, fix $F \in \text{Mod}(\mathbb{C}_X)$ and consider the two functors from $\text{Mod}(\mathbb{C}_X)$ to $\text{Mod}(\mathbb{C}_X^\hbar)$: $\theta_1: G \mapsto F \otimes G^\hbar$ and $\theta_2: G \mapsto (F \otimes G)^\hbar$. By Lemma 3.3, there is a morphism of functors $u: \theta_1 \rightarrow \theta_2$. Since both functors are left exact, this morphism u extends to the derived functors and we get the morphism (3.3) for a fixed $F \in \text{Mod}_{\mathbb{R}-c}(\mathbb{C}_X)$.

(ii) Now let us fix $G \in \mathbf{D}^b(\mathbb{C}_X)$ and consider the two functors from $\text{Mod}(\mathbb{C}_X)$ to $\mathbf{D}^b(\mathbb{C}_X^\hbar)$: $\lambda_1: F \mapsto F \otimes G^{\text{R}\hbar}$ and $\lambda_2: F \mapsto (F \otimes G)^{\text{R}\hbar}$. By (i) there exists a morphism of functors $v: \lambda_1 \rightarrow \lambda_2$. Both functors extend naturally to the category of bounded complexes $\mathbf{C}^b(\text{Mod}(\mathbb{C}_X))$ and send complexes quasi-isomorphic to zero to objects isomorphic to zero in $\mathbf{D}^b(\mathbb{C}_X^\hbar)$. Hence, both functors extend to $\mathbf{D}^b(\mathbb{C}_X)$ as well as the morphism of functors v . \square

LEMMA 3.5. *For $F, G \in \mathbf{D}^b(\mathbb{C}_X)$, we have a commutative diagram in $\mathbf{D}^b(\mathbb{C}_X^\hbar)$:*

$$(3.4) \quad \begin{array}{ccc} D'F \otimes G^{\text{R}\hbar} & \longrightarrow & (D'F \otimes G)^{\text{R}\hbar} \\ & \searrow & \downarrow \\ & & \mathbf{R}\mathcal{H}om(F, G)^{\text{R}\hbar}, \end{array}$$

such that the morphism in the horizontal arrow is the one given by Lemma 3.4.

PROOF. The oblique arrow is the composition of two canonical morphisms:

$$\mathbf{R}\mathcal{H}om(F, \mathbb{C}) \otimes G^{\text{R}\hbar} \rightarrow \mathbf{R}\mathcal{H}om(F, G^{\text{R}\hbar}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(F, G)^{\text{R}\hbar}.$$

The vertical arrow is defined by applying the functor $(\bullet)^{R\hbar}$ to the canonical morphism $R\mathcal{H}om(F, \mathbb{C}) \otimes G \rightarrow R\mathcal{H}om(F, G)$. The commutativity of the diagram is obvious. \square

REMARK 3.6. Assuming that $G \in \text{Mod}(\mathcal{D}_X^{\hbar})$ in Lemma 3.3, then the morphism (3.2) is \mathcal{D}_X^{\hbar} -linear. Similarly, if $G \in \text{D}^b(\mathcal{D}_X^{\hbar})$, then the morphism (3.3) is a morphism in $\text{D}^b(\mathcal{D}_X^{\hbar})$ and the diagram in Lemma 3.5 is a commutative diagram in $\text{D}^b(\mathcal{D}_X^{\hbar})$. In the sequel we shall use such diagram in the case $G = \mathcal{K}_{\hbar}$.

THEOREM 3.7. *Let $F \in \text{D}_{R-c}^b(C_X)$ and suppose that (\mathcal{M}, F^{\hbar}) is an f -elliptic pair over C^{\hbar} . Then there is a commutative diagram of isomorphisms in $\text{D}^b(C_X^{\hbar})$:*

$$(3.5) \quad \begin{array}{ccc} R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, D'_{C_X^{\hbar}} F^{\hbar} \overset{L}{\otimes}_{C_X^{\hbar}} \mathcal{K}_{\hbar}) & \xrightarrow{\sim} & R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, (D'F \otimes \mathcal{K})^{R\hbar}) \\ & \searrow \sim & \downarrow \sim \\ & & R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{K})^{R\hbar}). \end{array}$$

PROOF. First note that we obtain the diagram (3.5) by applying the functor $R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \bullet)$ to the commutative diagram provided by Lemma 3.5:

$$\begin{array}{ccc} D'F \otimes \mathcal{K}_{\hbar} & \longrightarrow & (D'F \otimes \mathcal{K})^{R\hbar} \\ & \searrow & \downarrow \\ & & R\mathcal{H}om(F, \mathcal{K})^{R\hbar}. \end{array}$$

We also use the fact that $D'_{C_X^{\hbar}} F^{\hbar} \overset{L}{\otimes}_{C_X^{\hbar}} \mathcal{K}_{\hbar}$ is isomorphic to $D'F \otimes \mathcal{K}_{\hbar}$.

Note that the oblique arrow is an isomorphism since it is the composition of two canonical isomorphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, D'_{C_X^{\hbar}} F^{\hbar} \overset{L}{\otimes}_{C_X^{\hbar}} \mathcal{K}_{\hbar}) &\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, R\mathcal{H}om_{C_X^{\hbar}}(F^{\hbar}, \mathcal{K}_{\hbar})) \\ &\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{K}_{\hbar})^{R\hbar}), \end{aligned}$$

The first one results from applying the regularity theorem 3.1 to the f -elliptic pair (\mathcal{M}, F^{\hbar}) (see also Remark 3.2) and the second isomorphism results from Lemma 1.7.

The vertical arrow is also an isomorphism in $\text{D}^b(C_X^{\hbar})$. In fact, we conclude from [7, Prop. 1.5.10] and [1, Prop. 2.2] that the objects on both sides of the vertical arrow are cohomologically complete. Hence, since gr_{\hbar} is

conservative on cohomologically complete objects, it is enough to prove that the canonical morphism below is an isomorphism:

$$(3.6) \quad \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{gr}_{\hbar}(\mathcal{M}), D'F \otimes \mathcal{H}) \rightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{gr}_{\hbar}(\mathcal{M}), \mathbf{R}\mathcal{H}om(F, \mathcal{H})).$$

In fact, $(\mathrm{gr}_{\hbar}(\mathcal{M}), F)$ is an f -elliptic pair over \mathbb{C} and the morphism (2.2) is precisely the regularity isomorphism for elliptic pairs over \mathbb{C} applied to $(\mathrm{gr}_{\hbar}(\mathcal{M}), F)$ (see Theorem 2.15 of [12]).

We have proved that the oblique and vertical arrows in diagram (3.5) are isomorphisms. The commutativity of the diagram allow us to conclude that the horizontal arrow is also an isomorphism. \square

Using the functorial properties of $(\bullet)^{\mathrm{R}\hbar}$ we also get:

COROLLARY 3.8. *Let (\mathcal{M}, F) be an f -elliptic pair over \mathbb{C} . Then, $(\mathcal{M}^{\hbar}, F^{\hbar})$ is an f -elliptic pair over \mathbb{C}^{\hbar} and there are canonical isomorphisms in $\mathbf{D}^b(\mathbb{C}_X^{\hbar})$:*

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}^{\hbar}, D'_{\mathbb{C}_X^{\hbar}} F^{\hbar} \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{H}_{\hbar}) &\simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, D'F \otimes \mathcal{H})^{\mathrm{R}\hbar} \\ &\simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbf{R}\mathcal{H}om_{\mathbb{C}_X}(F, \mathcal{H}))^{\mathrm{R}\hbar}. \end{aligned}$$

Finiteness theorem.

THEOREM 3.9. *Let (\mathcal{M}, F) be a good f -elliptic pair over \mathbb{C}^{\hbar} and suppose that f is proper when restricted to the support of (\mathcal{M}, F) . Then, $f_{\downarrow, \hbar}(\mathcal{M} \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} F)$ is an object of $\mathbf{D}_{\mathrm{gd}}^b(\mathcal{D}_Y^{\hbar})$.*

PROOF. Note that by Theorem 1.6.4 of [7] it suffices to prove that:

- (i) $f_{\downarrow, \hbar}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M})$ is cohomologically complete;
- (ii) $\mathrm{gr}_{\hbar}(f_{\downarrow, \hbar}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M}))$ is an object of $\mathbf{D}_{\mathrm{gd}}^b(\mathcal{D}_Y^{\hbar})$.

Set $\mathcal{L} := (F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M}) \otimes_{\mathcal{D}_X^{\hbar}}^{\mathbf{L}} \mathcal{H}_{\hbar}$. The regularity theorem 3.1 yields the isomorphism:

$$\mathcal{L} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(D'_{\mathbb{C}_X^{\hbar}} F, \mathcal{M} \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{H}_{\hbar}).$$

Since $\mathcal{M} \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{H}_{\hbar}$ is cohomologically complete, \mathcal{L} is also cohomologically complete in view of Propositions 1.5.10 of [7]. Finally, $f_{\downarrow, \hbar}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M}) = \mathbf{R}f_!(\mathcal{L})$ is cohomologically complete by Proposition 1.5.12 of [7] and since $\mathbf{R}f_*$ and $\mathbf{R}f_!$ coincide by the hypothesis on the support.

On the other hand, note that $(\mathrm{gr}_{\hbar}(\mathcal{M}), \mathrm{gr}_{\hbar}(F))$ is an f -elliptic pair over \mathbb{C} that satisfies the conditions of the finiteness theorem for elliptic pairs over \mathbb{C} . Hence, $\mathrm{gr}_{\hbar}(f_{\underline{1}, \hbar}^{\mathrm{L}}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathrm{L}} \mathcal{M})) \simeq f_{\underline{1}}(\mathrm{gr}_{\hbar}(F) \otimes \mathrm{gr}_{\hbar}(\mathcal{M}))$ is an object of $\mathrm{D}_{\mathrm{gd}}^{\mathrm{b}}(\mathcal{D}_Y)$ by Theorem 4.2. of [12]. \square

Duality theorem.

The results in this paragraph are obtained assuming that $f : X \rightarrow Y$ is a smooth morphism of complex manifolds.

PROPOSITION 3.10. *If $f : X \rightarrow Y$ is smooth, then $f^{-1}\mathcal{D}_Y^{\hbar} \simeq (f^{-1}\mathcal{D}_Y)^{\hbar}$.*

PROOF. This is a particular case of Proposition 1.5 choosing for \mathcal{B} the family of compact Stein subsets of Y . \square

The rings $f^{-1}\mathcal{D}_Y^{\hbar}$ and $(f^{-1}\mathcal{D}_Y)^{\hbar}$ are not isomorphic in general: if $i : Y \hookrightarrow X$ is the embedding of a closed submanifold of dimension $d_Y < d_X$, then $i^{-1}\mathcal{D}_X^{\hbar} \rightarrow (i^{-1}\mathcal{D}_X)^{\hbar}$ is a monomorphism which is not an epimorphism.

Note that in the smooth case \mathcal{H}_{\hbar} is a coherent left \mathcal{D}_X^{\hbar} -module. Hence, for any coherent right \mathcal{D}_X^{\hbar} -module \mathcal{M} , one has:

$$(3.7) \quad \mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}} \mathcal{H}_{\hbar} \simeq \mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}} (\mathcal{D}_X^{\hbar} \otimes_{\mathcal{D}_X} \mathcal{H}) \simeq \mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{H}.$$

By the isomorphisms in (3.7), the object $\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{H}$ has a structure of $f^{-1}(\mathcal{D}_Y^{\hbar \mathrm{op}})$ -module. Passing to the derived category and applying $\mathrm{R}f_!$, one concludes that $f_{\underline{1}}(\mathcal{M})$ is an object of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^{\hbar \mathrm{op}})$ and that it is isomorphic to $f_{\underline{1}, \hbar}(\mathcal{M})$ for any $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar \mathrm{op}})$. In other words, the direct image of \mathcal{M} in the \hbar -setting coincides with its direct image as a \mathcal{D} -module. We use this fact in the sequel.

LEMMA 3.11. *Let $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. There is a morphism $f_{\underline{1}, \hbar}(\mathcal{M}^{\hbar}) \rightarrow f_{\underline{1}}(\mathcal{M})^{\mathrm{R}\hbar}$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^{\hbar \mathrm{op}})$.*

PROOF. First note that we have a chain of isomorphisms in $\mathrm{D}^{\mathrm{b}}(f^{-1}\mathcal{D}_Y^{\hbar \mathrm{op}})$:

$$\begin{aligned} \mathcal{M}^{\hbar} \otimes_{\mathcal{D}_X}^{\mathrm{L}} \mathcal{H} &\simeq \mathcal{M}^{\hbar} \otimes_{\mathcal{D}_X}^{\mathrm{L}} \mathrm{D}'_X \mathrm{D}'_X \mathcal{H} \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{D}'_X \mathcal{H}, \mathcal{M}_{\hbar}) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{D}'_X \mathcal{H}, \mathcal{M})^{\mathrm{R}\hbar}. \end{aligned}$$

The first and second isomorphisms follow from the coherence of \mathcal{H} over \mathcal{D}_X . The third follows from the properties of $(\bullet)^{\text{Rh}}$ (Lemma 2.3 of [1]).

Applying $\text{Rf}_!$ we get the following isomorphism in $\text{D}^b(\mathcal{D}_Y^{\text{op}})$:

$$f_{\underline{!},\hbar}(\mathcal{M}^{\hbar}) \simeq \text{Rf}_!((\mathcal{M} \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{H})^{\text{Rh}}).$$

Finally, Lemma 1.2 entails the following morphism in $\text{D}^b(\mathcal{D}_Y^{\hbar \text{op}})$:

$$f_{\underline{!},\hbar}(\mathcal{M}^{\hbar}) \rightarrow \text{Rf}_!((\mathcal{M} \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{H})^{\text{Rh}}) = f_{\underline{!}}(\mathcal{M})^{\text{Rh}}. \quad \square$$

If \mathcal{M} is a right $(\mathcal{D}_X, \mathcal{D}_X)$ -bimodule, its direct image as a bimodule is the object in the derived category $\text{D}^b(\mathcal{D}_Y^{\text{op}} \otimes \mathcal{D}_Y^{\text{op}})$ defined by:

$$f_{\underline{!}}(\mathcal{M}) := \text{Rf}_!((\mathcal{M} \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{H}) \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{H}).$$

In particular $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is a right $(\mathcal{D}_X, \mathcal{D}_X)$ -bimodule and one has the so-called differential trace morphism in $\text{D}^b(\mathcal{D}_Y^{\text{op}} \otimes \mathcal{D}_Y^{\text{op}})$:

$$(3.8) \quad \text{tr}_f : f_{\underline{!}}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X) \rightarrow \Omega_Y[d_Y] \otimes_{\mathcal{O}_Y} \mathcal{D}_Y.$$

Note that if \mathcal{M} is a right $(\mathcal{D}_X^{\hbar}, \mathcal{D}_X^{\hbar})$ -bimodule then $f_{\underline{!},\hbar}(\mathcal{M}) := f_{\underline{!}}(\mathcal{M})$ is an object of $\text{D}^b(\mathcal{D}_Y^{\hbar \text{op}} \otimes_{\mathbb{C}^{\hbar}} \mathcal{D}_Y^{\hbar \text{op}})$.

PROPOSITION 3.12. *The morphism f induces a differential trace morphism in the derived category $\text{D}^b(\mathcal{D}_Y^{\hbar \text{op}} \otimes_{\mathbb{C}^{\hbar}} \mathcal{D}_Y^{\hbar \text{op}})$:*

$$\text{tr}_{f,\hbar} : f_{\underline{!},\hbar}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\hbar}) \rightarrow \Omega_Y[d_Y] \otimes_{\mathcal{O}_Y} \mathcal{D}_Y^{\hbar}.$$

PROOF. We have the following isomorphisms in $\text{D}^b(\mathcal{D}_Y^{\hbar \text{op}} \otimes_{\mathbb{C}^{\hbar}} \mathcal{D}_Y^{\hbar \text{op}})$:

$$\begin{aligned} f_{\underline{!},\hbar}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\hbar}) &\simeq f_{\underline{!},\hbar}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{H}^{\hbar}) \\ &\simeq f_{\underline{!},\hbar}((\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{H})^{\hbar}) \end{aligned}$$

The first one results from the associativity of tensor products. The second one results from $\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{H}^{\hbar} \simeq (\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{H})^{\hbar}$.

The morphism $\text{tr}_{f,\hbar}$ is then constructed composing the morphisms below:

$$f_{\underline{!},\hbar}((\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{H})^{\hbar}) \rightarrow f_{\underline{!}}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{H})^{\text{Rh}} \rightarrow \Omega_Y[d_Y] \otimes_{\mathcal{O}_Y} \mathcal{D}_Y^{\hbar}.$$

The first morphism is an application of Lemma 3.11 which, in this case, preserves the bimodule structures involved. The second morphism is the formal extension of the classical trace morphism (3.8). \square

REMARK 3.13. Consider the absolute case $f = a_X$. In this case one gets $\mathcal{K}_{\hbar} \simeq \mathcal{O}_X^{\hbar}$ and the following isomorphisms hold in $\mathbf{D}^b(\mathbb{C}^{\hbar})$:

$$\underline{a}_{X, \hbar}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\hbar}) \simeq \mathbf{R}\Gamma_c(X; \mathbb{C}_X^{\hbar}[2d_X]) \simeq \mathbf{R}\Gamma_c(X; \omega_X^{\hbar}).$$

Thus, the trace morphism $\mathrm{tr}_{a_X, \hbar}$ coincides with the morphism $\mathbf{R}\Gamma_c(X, \omega_X^{\hbar}) \rightarrow \mathbb{C}^{\hbar}$ induced by the natural transformation $\mathrm{Ra}_X! a_X^! \rightarrow \mathrm{id}$.

LEMMA 3.14. Let $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_X^{\hbar \mathrm{op}})$. We have a canonical morphism in $\mathbf{D}^b(\mathcal{D}_Y^{\hbar \mathrm{op}})$:

$$(3.9) \quad f_{\underline{1}, \hbar}(\underline{\mathbf{D}}_{\hbar, X} \mathcal{M}) \rightarrow \underline{\mathbf{D}}_{\hbar, Y}(f_{\underline{1}, \hbar} \mathcal{M}).$$

PROOF. First note that there is a basis change morphism in $\mathbf{D}^b(\mathcal{D}_Y^{\hbar \mathrm{op}})$:

$$\begin{aligned} f_{\underline{1}, \hbar}(\underline{\mathbf{D}}_{\hbar, X}(\mathcal{M})) &\simeq \mathbf{R}f_!(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{K}_{\hbar})) \\ &\rightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y^{\hbar}}(\mathbf{R}f_!(\mathcal{M} \otimes_{\mathcal{D}_X^{\hbar}}^{\mathbf{L}} \mathcal{H}), \mathbf{R}f_!(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{K}_{\hbar}) \otimes_{\mathcal{D}_X^{\hbar}}^{\mathbf{L}} \mathcal{H}) \\ &\simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y^{\hbar}}(f_{\underline{1}, \hbar}(\mathcal{M}), f_{\underline{1}, \hbar}(\Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\hbar})). \end{aligned}$$

We obtain (3.9) composing the above morphism with the morphism induced by Proposition 3.12 on the second term of the $\mathbf{R}\mathcal{H}om$. \square

COROLLARY 3.15. For $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_X^{\hbar \mathrm{op}})$ and $F \in \mathbf{D}^b(\mathbb{C}_X^{\hbar})$, there is a canonical morphism in $\mathbf{D}^b(\mathcal{D}_Y^{\hbar \mathrm{op}})$:

$$(3.10) \quad f_{\underline{1}, \hbar}(\mathbf{D}'_{\mathbb{C}_X^{\hbar}} F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \underline{\mathbf{D}}_{\hbar, X}(\mathcal{M})) \rightarrow \underline{\mathbf{D}}_{\hbar, Y}(f_{\underline{1}, \hbar}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M})).$$

PROOF. Note that there is a canonical morphism in $\mathbf{D}^b(\mathcal{D}_X^{\hbar \mathrm{op}})$:

$$(3.11) \quad \mathbf{D}'_{\mathbb{C}_X^{\hbar}} F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \underline{\mathbf{D}}_{\hbar, X}(\mathcal{M}) \rightarrow \underline{\mathbf{D}}_{\hbar, X}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M}).$$

The morphism (3.10) is then obtained as a composition of morphisms:

$$\begin{aligned} f_{\underline{1}, \hbar}(\mathbf{D}'_{\mathbb{C}_X^{\hbar}} F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \underline{\mathbf{D}}_{\hbar, X}(\mathcal{M})) &\rightarrow f_{\underline{1}, \hbar}(\underline{\mathbf{D}}_{\hbar, X}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M})) \\ &\rightarrow \underline{\mathbf{D}}_{\hbar, Y}(f_{\underline{1}, \hbar}(F \otimes_{\mathbb{C}_X^{\hbar}}^{\mathbf{L}} \mathcal{M})), \end{aligned}$$

where the first arrow results from (3.11) and the second arrow results from (3.9). \square

THEOREM 3.16. *Let (\mathcal{M}, F) be a good f -elliptic pair over \mathbb{C}^\hbar and suppose that f is proper when restricted to the support of (\mathcal{M}, F) . Then, the canonical morphism*

$$(3.12) \quad f_{\underline{1}, \hbar}(\mathbf{D}'_{\mathbb{C}_X^\hbar} F \otimes_{\mathbb{C}_X^\hbar}^{\mathbf{L}} \mathbf{D}_{\hbar, X}(\mathcal{M})) \rightarrow \mathbf{D}_{\hbar, Y}(f_{\underline{1}, \hbar}(F \otimes_{\mathbb{C}_X^\hbar}^{\mathbf{L}} \mathcal{M}))$$

is an isomorphism in $\mathbf{D}_{\text{gd}}^{\text{b}}(\mathcal{S}_Y^{\hbar, \text{op}})$.

PROOF. Set $\mathcal{L}_1 = f_{\underline{1}, \hbar}(\mathbf{D}'_{\mathbb{C}_X^\hbar} F \otimes_{\mathbb{C}_X^\hbar}^{\mathbf{L}} \mathbf{D}_{\hbar, X}(\mathcal{M}))$ and $\mathcal{L}_2 = \mathbf{D}_{\hbar, Y}(f_{\underline{1}, \hbar}(F \otimes_{\mathbb{C}_X^\hbar}^{\mathbf{L}} \mathcal{M}))$.

First note that both (\mathcal{M}, F) and $(\mathbf{D}_{\hbar, X} \mathcal{M}, \mathbf{D}'_{\mathbb{C}_X^\hbar} F)$ are f -elliptic pairs that verify the conditions of the finiteness theorem 3.9. Therefore \mathcal{L}_1 and \mathcal{L}_2 are objects of $\mathbf{D}_{\text{gd}}^{\text{b}}(\mathcal{S}_Y^{\hbar, \text{op}})$. Hence, by the conservativity of gr_{\hbar} on coherent objects, it suffices to check that the induced morphism $\text{gr}_{\hbar}(\mathcal{L}_1) \rightarrow \text{gr}_{\hbar}(\mathcal{L}_2)$ is an isomorphism in $\mathbf{D}_{\text{gd}}^{\text{b}}(\mathcal{S}_Y^{\text{op}})$. Consider the isomorphisms:

$$\begin{aligned} \text{gr}_{\hbar}(\mathcal{L}_1) &\simeq f_{\underline{1}}(\mathbf{D}'_{\mathbb{C}_X} \text{gr}_{\hbar}(F) \otimes_{\mathbb{C}_X} \mathbf{D}_X(\text{gr}_{\hbar}(\mathcal{M}))) \\ \text{gr}_{\hbar}(\mathcal{L}_2) &\simeq \mathbf{D}_Y(f_{\underline{1}}(\text{gr}_{\hbar}(F) \otimes_{\mathbb{C}_X} \text{gr}_{\hbar}(\mathcal{M}))). \end{aligned}$$

By the construction of the duality morphism, $\text{gr}_{\hbar}(\mathcal{L}_1) \rightarrow \text{gr}_{\hbar}(\mathcal{L}_2)$ is precisely the duality morphism for f -elliptic pairs over \mathbb{C} applied to $(\text{gr}_{\hbar} \mathcal{M}, \text{gr}_{\hbar} F)$. We conclude that $\text{gr}_{\hbar}(\mathcal{L}_1) \rightarrow \text{gr}_{\hbar}(\mathcal{L}_2)$ is an isomorphism in $\mathbf{D}_{\text{gd}}^{\text{b}}(\mathcal{S}_Y^{\text{op}})$ since the pair $(\text{gr}_{\hbar} \mathcal{M}, \text{gr}_{\hbar} F)$ is in the conditions of the duality theorem of [12] (see [12, Theorem 5.15]).

Remarks on particular cases.

We can apply our theorems in some particular cases. This is similar to what is done in [12] and we leave out the details.

(i) For each $\mathcal{M} \in \mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{S}_X^{\hbar, \text{op}})$ the pair $(\mathcal{M}, \mathbb{C}_X^\hbar)$ forms an f -elliptic pair over \mathbb{C}^\hbar . Theorems 3.9 and 3.16 in this particular case give finiteness and duality theorems (in the smooth case) for \mathcal{S}_X^{\hbar} -modules.

(ii) The classical finiteness theorem for coherent \mathcal{O}_X -modules (Grauert's theorem) can be generalized to the \hbar -framework using similar arguments to those employed in the proof of Theorem 3.9. A relative duality theorem for coherent \mathcal{O}_X^\hbar -modules can also be proved under the condition $f^{-1} \mathcal{O}_Y^\hbar \xrightarrow{\sim} (f^{-1} \mathcal{O}_Y)^\hbar$. In fact, with this condition one is able to construct a duality morphism using the one from the \mathcal{O}_X -modules theory. Again the general idea is similar (in fact it is easier) to that of Theorem 3.16, thus we don't give further details. Let us just remark that the iso-

morphism $f^{-1}\mathcal{O}_Y^{\hbar} \xrightarrow{\sim} (f^{-1}\mathcal{O}_Y)^{\hbar}$ holds if f is smooth (in Proposition 1.5, choose for \mathcal{B} the family of open Stein subsets of Y).

On the other hand, we can apply our theorems to elliptic pairs of the form $(\mathcal{F} \otimes_{\mathcal{O}_X^{\hbar}} \mathcal{D}_X^{\hbar}, \mathbb{C}_X^{\hbar})$, \mathcal{F} being an object of $\mathbf{D}_{\text{coh}}^b(\mathcal{O}_X^{\hbar})$. However, unlike the \mathcal{D} -modules case in [12], it is not clear if we obtain the finiteness and duality results for \mathcal{O}_X^{\hbar} -modules mentioned above. In other words, we don't know what is the relation between $f_{\ast, \hbar}(\mathcal{F} \otimes_{\mathcal{O}_X^{\hbar}} \mathcal{D}_X^{\hbar})$ and $\mathbf{R}f_!(\mathcal{F}) \otimes_{\mathcal{O}_Y^{\hbar}} \mathcal{D}_Y^{\hbar}$. Only in the case $f = a_X$ it becomes obvious these objects are isomorphic.

Note that in the case $f = a_X$ one gets absolute finiteness and duality results for \mathcal{O}_X^{\hbar} -modules which are contained in the finiteness and duality results for deformation-quantization modules of [7].

(iii) Consider the case $f = a_X$. Let (\mathcal{M}, F) be a good a_X -elliptic pair with compact support. Then, Theorem 3.9 says that the cohomology modules of the complex of global solutions of \mathcal{M} on the generalized sheaf of holomorphic functions associated to F are finitely generated over \mathbb{C}^{\hbar} . In particular, we get the following statement: if \mathcal{M} is a good \mathcal{D}_X^{\hbar} -module with compact support, then the cohomologies of $\mathbf{R}\Gamma(X; \text{Sol}_{\hbar}(\mathcal{M}))$ are finitely generated over \mathbb{C}^{\hbar} .

(iv) Let X be the complexification of a real analytic manifold M and let \mathcal{A}_M denote the sheaf of real analytic functions on M . Consider the following sheaves of real analytic functions with \hbar -parameter:

$$\mathcal{A}_{M, \hbar} := \mathbb{C}_M^{\hbar} \otimes_{\mathbb{C}_X^{\hbar}} \mathcal{O}_X^{\hbar} \simeq \mathbb{C}_M \otimes \mathcal{O}_X^{\hbar}, \quad \mathcal{A}_M^{\hbar} := (\mathbb{C}_M \otimes \mathcal{O}_X)^{\hbar}.$$

One has the isomorphism $\mathcal{A}_M^{\hbar} \simeq \mathcal{A}_M^{\text{R}\hbar}$, since \mathcal{A}_M verifies the hypothesis of Proposition 1.1 taking for \mathcal{B} the family of all open subsets of M . Hence, both $\mathcal{A}_{M, \hbar}$ and \mathcal{A}_M^{\hbar} are concentrated in degree 0 and we can identify them to usual sheaves. There is a natural monomorphism $\mathcal{A}_{M, \hbar} \hookrightarrow \mathcal{A}_M^{\hbar}$ in $\text{Mod}(\mathcal{D}_X^{\hbar})$ which is not an epimorphism (this morphism is a particular case of Lemma 3.4).

Let us consider the c -soft sheaf of hyperfunctions on M , defined by $\mathcal{B}_M := \mathbf{R}\mathcal{H}om(\mathbb{C}_M, \mathcal{O}_X)$. Since c -soft objects are $\Gamma(K; _)$ -injective for any compact $K \subset X$, \mathcal{B}_M is \hbar -acyclic by Proposition 1.1. Hence:

$$\mathcal{B}_M^{\text{R}\hbar} \simeq \mathcal{B}_M^{\hbar} \simeq \mathbf{R}\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(\mathbf{D}'_{\mathbb{C}_X^{\hbar}} \mathbb{C}_M^{\hbar}, \mathcal{O}_X^{\hbar}).$$

Applying our theorems to a_X -elliptic pairs of the form $(\mathcal{M}, \mathbb{C}_M^{\hbar})$ we obtain:

COROLLARY 3.17. *Let \mathcal{M} be an elliptic \mathcal{D}_X^{\hbar} -module on M .*

(a) *There is a commutative diagram of isomorphisms in $\mathbf{D}^b(\mathbb{C}_X^{\hbar})$:*

$$\begin{array}{ccc}
 \mathrm{R}\mathcal{H}om_{\mathcal{O}_X^{\hbar}}(\mathcal{M}, \mathcal{A}_{M, \hbar}) & \xrightarrow{\sim} & \mathrm{R}\mathcal{H}om_{\mathcal{O}_X^{\hbar}}(\mathcal{M}, \mathcal{A}_M^{\hbar}) \\
 & \searrow \sim & \downarrow \sim \\
 & & \mathrm{R}\mathcal{H}om_{\mathcal{O}_X^{\hbar}}(\mathcal{M}, \mathcal{B}_M^{\hbar}).
 \end{array}$$

(b) If M is compact and $\mathcal{M} \in \mathrm{D}_{\mathrm{gd}}^{\mathrm{b}}(\mathcal{O}_X^{\hbar})$, then $\mathrm{R}\Gamma(M; \Omega_M^{\hbar} \otimes_{\mathcal{O}_M^{\hbar}}^{\mathrm{L}} \mathcal{M})$ is the dual of $\mathrm{R}\Gamma(M; \mathrm{R}\mathcal{H}om_{\mathcal{O}_M^{\hbar}}(\mathcal{M}, \mathcal{B}_{M, \hbar}))$ and both objects belong to $\mathrm{D}_f^{\mathrm{b}}(\mathbb{C}^{\hbar})$.

Acknowledgments. We thank Pierre Schapira for proposing the subject of this paper and for making useful remarks and suggestions. We thank Teresa Monteiro Fernandes for her supervision, helpful discussions and constant incentive. We also thank Ana Rita Martins for her comments.

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