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On the Rarity of Quasinormal Subgroups

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ABSTRACT - For each prime p and positive integer n, Berger and Gross have defined a finite p-group $G = HX$, where H is a core-free quasinormal subgroup of exponent p^{n-1} and X is a cyclic subgroup of order p^n. These groups are universal in the sense that any other finite p-group, with a similar factorisation into subgroups with the same properties, embeds in G. In our search for quasinormal subgroups of finite p-groups, we have discovered that these groups G have remarkably few of them. Indeed when p is odd, those lying in H can have exponent only p, p^{n-2} or p^{n-1}. Those of exponent p are nested and they all lie in each of those of exponent p^{n-2} and p^{n-1}.

1. Introduction.

A subgroup Q of a group G, such that $QH = HQ$ for all subgroups H of G, is said to be *quasinormal* (sometimes *permutable*) in G and we write Q *qn* G. The concept was introduced by Ore in 1937 (see[6]) and in [7] he proved that, in finite groups, quasinormal subgroups are always subnormal. He also proved that quasinormal subgroups are modular. Recall that a subgroup M of a group G is *modular* if, for all subgroups X and Y of G,

$$\langle X, M \rangle \cap Y = \langle X, M \cap Y \rangle \quad \text{if } X \leqslant Y$$

and

$$\langle X, M \rangle \cap Y = \langle X \cap Y, M \rangle \quad \text{if } M \leqslant Y.$$

Indeed *a subgroup Q of a finite group G is quasinormal in G if and only if Q is modular and subnormal in G*. Thus the concepts of modularity and

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quasinormality coincide in finite p-groups. In 1973 Maier and Schmid proved in [5] that if G is a finite group and Q *qn* G with Q core-free, i.e. $Q_G = 1$, then Q *lies in the hypercentre of* G. It follows fairly easily from this that the complexities of the embedding of Q in G reduce to the case where G is a p-group, i.e. where Q is a modular subgroup. Bearing in mind that modularity is a property invariant under subgroup lattice isomorphisms, the quasinormal subgroups of a finite p-group G are surely relevant when the structure of G is approached via its lattice of subgroups.

It was shown in [10] that given Q *qn* G and Q *abelian*, then Q^n *qn* G, provided n is odd or divisible by 4 (even when G is infinite). Apart from this, very little appears to be known about which subgroups of Q are also quasinormal in G.

Clearly Q *qn* G if and only if QX is a subgroup for each cyclic subgroup X of G. Thus the situation $G = QX$ is an obvious starting point for investigations. The case when G here is a finite p-group (for p an odd prime), and Q is an abelian quasinormal subgroup of G (with X cyclic), is studied in [2]. It is shown that there are two composition series of G passing through Q, all the members of which are quasinormal in G. The attempt to remove the hypothesis that Q is abelian has produced the present work. We have discovered a situation very far removed from the abelian case. In fact there is a family of finite p-groups G (for each odd prime p) with Q *qn* G such that

(i) there are subgroups $L \lhd M \leqslant Q$, with L, M *qn* G;
(ii) G has no quasinormal subgroup strictly between L and M;
(iii) L has exponent p and M has exponent p^m; and
(iv) m and the nilpotency class (even derived length) of M/L can be greater than any given positive integer.

These groups G were constructed by Berger and Gross in [1] and they are defined as follows. Let n be a positive integer and p be an odd prime. We define Γ_n to be the additive group $\mathbb{Z}$ of integers modulo $p^n\mathbb{Z}$. Let x_n be the permutation of Γ_n given by

$$x_n : \; p^n\mathbb{Z} + \ell \mapsto p^n\mathbb{Z} + \ell + 1.$$

Then x_n belongs to the symmetric group of degree p^n. For $0 \leqslant m \leqslant n$, define

$$x_{n,m} = x_n^{p^{n-m}}, \;\text{ of order } p^m.$$

Let $X_n = \langle x_n \rangle$, a cyclic group of order p^n; and let $X_{n,m} = \langle x_{n,m} \rangle$, the subgroup of order p^m. Define $\Delta_{n,m}$ to be the set of elements in the additive group Γ_n of order p^m. So

$$\Delta_{n,m} = \{\lambda p^{n-m} | 0 \leqslant \lambda \leqslant p^m - 1,\ p \nmid \lambda\} \text{ for } 1 \leqslant m \leqslant n$$

and $\Delta_{n,0} = \{0\}$. Then $|\Delta_{n,m}| = p^{m-1}(p-1)$ for $1 \leqslant m \leqslant n$. The permutation $x_{n,m}$ shifts by p^{n-m} and fixes the set $\Delta_{n,m+1}$, i.e. the set of λp^{n-m-1} ($p \nmid \lambda$) between 0 and $p^n - 1$, for $0 \leqslant m \leqslant n-1$. Thus $x_{n,m}$ acting on the set $\Delta_{n,m+1}$ (of cardinality $p^m(p-1)$) is the product of $p-1$ disjoint cycles $\pi_{n,m,i}$ of length p^m, $1 \leqslant i \leqslant p-1$, each cycle adding p^{n-m}. So

$$\pi_{n,m,i} = (ip^{n-m-1},\ ip^{n-m-1} + p^{n-m}, \ldots,\ ip^{n-m-1} + (p^m - 1)p^{n-m}).$$

Now for $0 \leqslant m \leqslant n-1$, define

$$A_{n,m} = \left\{ \prod_{i=1}^{p-1} \pi_{n,m,i}^{c_i} \,\middle|\, \sum_{i=1}^{p-1} c_i = 0 \right\}.$$

Each permutation in this set is a product of disjoint cycles, each containing an integer of the form λp^{n-m-1} ($p \nmid \lambda$) and shifting by a multiple of p^{n-m}. In fact *$A_{n,m}$ is an abelian group isomorphic to the direct product of $p-2$ cyclic groups of order p^m*. (See [1], Lemma 3.1 (1).) Its elements fix every integer not in $\Delta_{n,m+1}$.

Berger and Gross define the group

$$G_n = \langle x_n, A_{n,m} | 0 \leqslant m \leqslant n-1 \rangle,$$

and H_n to be the stabiliser of 0 in G_n. So

$$G_n = H_n X_n,$$

a finite p-group. Also H_n has exponent p^{n-1}. The main result of [1] is:-

THEOREM. *The subgroup H_n is core-free and quasinormal in G_n. Moreover, if H^* is a core-free quasinormal subgroup of a finite p-group $G^* = H^*\langle x^* \rangle$, where $\langle x^* \rangle$ is a cyclic group of order p^n, then there is a unique embedding ψ of G^* in G_n such that $\psi(x^*) = x_n$ and $\psi(H^*) \leqslant H_n$.*

Our main results concern the quasinormal subgroups of G_n that lie in H_n. In the Berger-Gross Theorem above, the prime p is arbitrary. However, the case $p = 2$ requires much additional analysis. Accordingly in all our work here we assume that

the prime p is odd.

We shall prove

THEOREM 1. *Let $n \geqslant 2$ and let Q be a non-trivial quasinormal subgroup of G_n lying in H_n of exponent p^k. Then one of the following holds:-*

(i) $k = 1$;
(ii) $n \geqslant 4$ *and* $k = n - 2$;
(iii) $n \geqslant 3$ *and* $k = n - 1$.

Moreover for $n \geqslant 4$, the quasinormal subgroups of exponent p are nested and are contained in all those of exponent p^{n-2} and exponent p^{n-1}. Thus there is a maximal quasinormal subgroup L of exponent p and a quasinormal subgroup M of exponent p^{n-2} such that $L \leqslant M$ and

there are no quasinormal subgroups of G_n in the interval $[M/L]$.

Clearly $L \lhd M$. Also M/L has exponent p^{n-3} and there is no bound to its derived length as n increases. (See [9].)

In Theorem 2, we find all the quasinormal subgroups of G_n lying in H_n and of exponent p. Theorem 3 gives first some necessary conditions for subgroups of G_n, lying in H_n and of exponent p^{n-1}, to be quasinormal in G_n; then necessary and sufficient conditions for certain subgroups of exponent p^{n-1} to be quasinormal in G_n. Theorem 4 is similar, dealing with subgroups of exponent p^{n-2}. Finally in Theorem 5 we show that there are no quasinormal subgroups of G_n ($n \geqslant 5$), lying in H_n and of exponent p^k, for $2 \leqslant k \leqslant n-3$. Theorem 1 follows from these results.

We use standard notation for familiar concepts, together with that above introduced by Berger and Gross. Also we switch from multiplicative to additive notation for computation with modules over groups. In this connection there is a considerable amount of calculation involved and the following list should be helpful.

$(\alpha)_{\beta,\gamma}$: the cyclic permutation containing α, shifting by β, of length γ. Here α is a non-negative integer and β and γ are positive integers such that $\beta\gamma = p^n$.

$\{b_{i,k} | r_k \geqslant i \geqslant 1\}$: a basis (modulo $\Omega_{k-1}(H_n)$) of the elementary abelian group $\Omega_k(H_n)/\Omega_{k-1}(H_n)$.

$B_{\ell_k,k}$: the subgroup $\langle b_{i,k} | \ell_k \geqslant i \geqslant 1\rangle$.

B_k: the subgroup $B_{r_k,k}$.

C_k: the centraliser in G_n of the element $x_{n,k}$, i.e. $C_{G_n}(x_{n,k})$, chiefly when $k = 2$.

r_k: the rank of the elementary abelian group $\Omega_k(H_n)/\Omega_{k-1}(H_n)$, for $1 \leqslant k \leqslant n-1$.

X_n: the cyclic group $\langle x_n \rangle$ of order p^n.

$x_{n,m}$: the element $x_n^{p^{n-m}}$, $0 \leqslant m \leqslant n$.
$X_{n,m}$: the cyclic subgroup $\langle x_{n,m} \rangle$ of order p^m.
$\{w_i | r_1 \geqslant i \geqslant 1\}$: a basis of the elementary abelian group $\Omega_1(H_n)$, and $w_i = b_{1,i}$.
W_j: the subgroup $\langle w_i | j \geqslant i \geqslant 1 \rangle$ ($=B_{j,1}$).
Z_{p^m}: a cyclic group of order p^m.
$\Omega_k(G)$: the subgroup of the finite p-group G generated by the elements of order at most p^k.

2. Description of the groups G_n.

It is shown in [1], Lemma 3.14, that

(1) $$\Omega_k(G_n) = \langle x_n^{-i} A_{n,k} x_n^i | i \geqslant 0 \rangle \Omega_{k-1}(G_n),$$

for $1 \leqslant k \leqslant n-1$. Also

(2) $\Omega_k(G_n)/\Omega_{k-1}(G_n)$ *is elementary abelian of rank* $p^{n-k-1}(p-1)$.

We see this as follows. By [1], Lemma 3.3, if $n \geqslant 2$, then there is an epimorphism τ_n: $G_n \to G_{n-1}$, viz. $g \mapsto \tau_n(g)$ where

(3) $$\tau_n(g): \ a \bmod p^{n-1}\mathbb{Z} \mapsto b \bmod p^{n-1}\mathbb{Z}$$

if

$$g: \ a \bmod p^n\mathbb{Z} \mapsto b \bmod p^n\mathbb{Z},$$

for all integers a, b and $g \in G_n$. In particular τ_n maps H_n onto H_{n-1} and x_n to x_{n-1}. The kernel of τ_n is elementary abelian, by [1], Lemma 3.3 (9), and it is precisely $\Omega_1(G_n)$, by [1], Lemma 3.10. Also $|G_n| = p^{p^{n-1}}$, by [1], Corollary 3.15. Thus

$\Omega_1(G_n)$ *is elementary abelian of rank* $p^{n-1} - p^{n-2} = p^{n-2}(p-1)$.

Applying τ_n repeatedly for decreasing values of n, we see that

(4) $$G_n/\Omega_{k-1}(G_n) \cong G_{n-k+1},$$

for $2 \leqslant k \leqslant n$; and

(5) $$\Omega_1(G_n/\Omega_{k-1}(G_n)) = \Omega_k(G_n)/\Omega_{k-1}(G_n) \cong \Omega_1(G_{n-k+1}),$$

this last subgroup being elementary abelian of rank $p^{n-k-1}(p-1)$, proving (2). We shall use (4) and (5) repeatedly throughout our work.

By [1], Lemma 3.10,

$$\Omega_1(G_n) = \Omega_1(H_n)X_{n,1}.$$

Also, by (1),

$$\Omega_1(G_n) = \langle x_n^{-i} A_{n,1} x_n^i | i \geqslant 0 \rangle,$$

an elementary abelian group of rank $p^{n-2}(p-1)$, by (2). By [1], Lemma 3.2 (4),

$x_{n,1}$ *lies in the centre of* G_n.

Let

(6) $$r_1 = p^{n-2}(p-1) - 1 = \text{rank of } \Omega_1(H_n),$$

for $n \geqslant 2$. Recall that H_n is core-free in G_n and so $\Omega_1(G_n)$ is an indecomposable X_n-module. We find a canonical basis for $\Omega_1(G_n)$ as follows. Define

(7) $$w_{r_1} = x_n^{p^{n-2}-1} (\pi_{n,1,1}^{-1} \pi_{n,1,2}) x_n^{-(p^{n-2}-1)} \in \Omega_1(G_n).$$

We introduce the following shorthand notation for cyclic permutations of integers modulo $p^n\mathbb{Z}$ which just add a fixed integer:-

(8) $$(\alpha)_{\beta,\gamma} = (\alpha,\ \alpha+\beta,\ \alpha+2\beta, \dots,\ \alpha + (\gamma-1)\beta),$$

where $\beta\gamma = p^n$. Then

$$\pi_{n,1,1}^{-1}\pi_{n,1,2} = (p^{n-2})_{p^{n-1},p}^{-1} (2p^{n-2})_{p^{n-1},p} \in A_{n,1}.$$

Note that the two cycles here are disjoint, coming from different orbits of $x_{n,1}$. Also

$$w_{r_1} = (1)_{p^{n-1},p}^{-1} (1+p^{n-2})_{p^{n-1},p},$$

where again the two cycles are disjoint. Therefore w_{r_1} fixes 0 and so $w_{r_1} \in \Omega_1(H_n)$.

For $r_1 - 1 \geqslant i \geqslant 0$, define

(9) $$w_i = [w_{i+1}, x_n]$$

inductively. Then dropping the suffixes in (8) for the moment and writing $r = r_1$ and $x = x_n$, we have
$w_{r-1} = (1)(1+p^{n-2})^{-1}(2)^{-1}(2+p^{n-2}) = (1)(2)^{-1}(1+p^{n-2})^{-1}(2+p^{n-2})$,
$w_{r-2} = (1)^{-1}(2)^2(3)^{-1}(1+p^{n-2})(2+p^{n-2})^{-2}(3+p^{n-2})$,

. . . .

$w_{r-j} = (1)^{(-1)^{j+1}}(2)^{?} \dots (j+1+p^{n-2}),$

. . . .

$w_1 = (1)^{-1}(2)^{?} \dots (p^{n-1}-1),$
$w_0 = (1)(2)^{?} \dots (p^{n-1}).$

Any 2 cycles above are either the same or disjoint. Hence they commute. Also w_i, for $r_1\ (=r) \geqslant i \geqslant 1$, fixes 0 and so belongs to $\Omega_1(H_n)$. Clearly $w_{r_1}, w_{r_1-1}, \dots, w_0$ are linearly independent as elements of the module $\Omega_1(G_n)$, since each contains a cycle that does not occur in its predecessors. The rank of $\Omega_1(G_n)$ is r_1+1, by (2). Therefore

(10) $$\{w_i | r_1 \geqslant i \geqslant 1\} \text{ is a basis for } \Omega_1(H_n)$$

and

(11) $$\{w_i | r_1 \geqslant i \geqslant 0\} \text{ is a basis for } \Omega_1(G_n).$$

Clearly $\Omega_1(H_n)$ is an indecomposable X_n-module. For if not, then the centraliser of X_n in G_n is not cyclic and therefore intersects $\Omega_1(H_n)$ non-trivially. Then the normal closure of this intersection in G_n would lie in H_n, giving a contradiction. Also we must have $w_0 \in \langle x_{n,1} \rangle$ and $x_{n,1} = (1)(2)\dots,\ (p^{n-1})$, with cycles of length p, shifting by p^{n-1}. Thus

(12) $$w_0 = x_{n,1} = x_n^{p^{n-1}}.$$

Note. For any i, j, using additive module notation, we have

(13) $$[w_i, x_n^{p^j}] = w_i(x_n^{p^j} - 1) = w_i(x_n - 1)^{p^j} = w_{i-p^j}.$$

Here w_i, for $i<0$, is taken to be 1.

Next, for $2 \leqslant k \leqslant n-1$, we find a basis for $\Omega_k(G_n)/\Omega_{k-1}(G_n)$ as indecomposable X_n-module of rank

(14) $$p^{n-k-1}(p-1) = r_k + 1,$$

say (see (2)). Notation will necessarily becomes more complicated. By [1], Corollary 3.11,

(15) $$\Omega_k(G_n) = \Omega_k(H_n)X_{n,k}.$$

Thus by analogy with (7), define

$$b_{r_k,k} = x_n^{p^{n-k-1}}(\pi_{n,k,1}^{-1}\pi_{n,k,2})x_n^{-(p^{n-k-1}-1)}$$

for $1 \leqslant k \leqslant n-1$. Then using the notation (8), we have

$$\pi_{n,k,1}^{-1}\pi_{n,k,2} = (p^{n-k-1})^{-1}_{p^{n-k},p^k}(2p^{n-k-1})_{p^{n-k},p^k} \in A_{n,k}$$

and

(16) $$b_{r_k,k} = (1)^{-1}_{p^{n-k},p^k}(1+p^{n-k-1})_{p^{n-k},p^k}.$$

Since $b_{r_k,k}$ fixes $0 \in \Gamma_n$, it belongs to $\Omega_k(H_n)$. When $k=1$, we have

$$b_{r_1,1} = w_{r_1}.$$

Under the isomorphism (5), $b_{r_k,k}$ modulo $\Omega_{k-1}(G_n)$ corresponds to

(17) $$(1)^{-1}_{p^{n-k},p}(1+p^{n-k-1})_{p^{n-k},p}$$

in $\Omega_1(G_{n-k+1})$, since in the set $\mathbb{Z}/p^{n-k+1}\mathbb{Z}$ on which G_{n-k+1} acts, $p^{n-k+1} \equiv 0$. Also (17) is $b_{r_1,1}$ in G_{n-k+1}; and under the homomorphism (3), $x_n \mapsto x_{n-1}$. Therefore defining

(18) $$b_{i,k} = [b_{i+1,k}, x_n]$$

for $r_k - 1 \geqslant i \geqslant 0$, we see from (10) and (11) that

$$\{b_{i,k} | r_k \geqslant i \geqslant 1\} \text{ is a basis for } \Omega_k(H_n) \text{ modulo } \Omega_{k-1}(G_n),$$

even modulo $\Omega_{k-1}(H_n)$, since (as for $k=1$) $b_{i,k} \in H_n$ if $k \neq 0$. Also

(19) $$\{b_{i,k} | r_k \geqslant i \geqslant 0\} \text{ is a basis for } \Omega_k(G_n) \text{ modulo } \Omega_{k-1}(G_n).$$

Note that $b_{i,1} = w_i$, for all i. Also

(20) $$\tau_n \text{ maps } b_{i,2} \text{ in } G_n \text{ to } b_{i,1} \ (= w_i) \text{ in } G_{n-1},$$

for all i. We shall use the 'w' -notation when we wish to emphasise the fact that we are considering $\Omega_1(G_n)$ as X_n- (or G_n-)module.

By [1], Lemma 3.1 (4),

(21) $$b_{i,k} \text{ commutes with } x_{n,k},$$

for all $n-1 \geqslant k \geqslant 1$ and $r_k \geqslant i \geqslant 0$. Let

$$B_k = \langle b_{i,k} | r_k \geqslant i \geqslant 1 \rangle,$$

$1 \leqslant k \leqslant n-1$. Then B_k has exponent p^k and

(22) $$\left(\prod_{k=1}^{n-1} B_k\right) X_n = G_n.$$

LEMMA 1. *Let* $1 \leqslant k \leqslant n-1$. *Then*

(i) $\langle B_k, X_{n,k}\rangle \cong Z_{p^k} \times \cdots \times Z_{p^k}$ *(with* $r_k + 1$ *factors); and*
(ii) $[w_i, B_k X_{n,k}] = 1$, *for* $p^{n-k} - 1 \geqslant i \geqslant 1$, $2 \leqslant k \leqslant n-1$.

PROOF. (i) By (21), $[B_k, X_{n,k}] = 1$. Let $C_k = C_{G_n}(x_{n,k})$. So, writing $X = X_n$, $\langle B_k, X\rangle \leqslant C_k$ and $C_k = (H_n \cap C_k)X$. Then by (15),

$$\Omega_k(C_k) = \Omega_k(H_n \cap C_k)X_{n,k} = D_k,$$

say. Since D_k is a characteristic subgroup of C_k, it follows that the derived subgroup D_k' is normalised by X. But $D_k' \leqslant H_n$ and so $(D_k')^{G_n} \leqslant H_n$. Therefore since H_n is core-free in G_n, we must have $D_k' = 1$, i.e.

(23) $$D_k \text{ is abelian.}$$

However, $\langle B_k, X_{n,k}\rangle \leqslant D_k$ and so $\langle B_k, X_{n,k}\rangle$ is abelian of exponent p^k. Thus (i) follows from (19). Observe that

$$B_k = \langle b_{r_k,k}\rangle \times \langle b_{r_k-1,k}\rangle \times \cdots \times \langle b_{1,k}\rangle,$$

which is homogeneous of exponent p^k.

(ii) Let $k \geqslant 2$. By (13),

$$[w_i, x_{n,k}] = w_{i-p^{n-k}} = 1,$$

for $1 \leqslant i \leqslant p^{n-k} - 1$. Therefore $w_i \in D_k$ and (ii) follows from (23). □

We have bases (19) for the Ω-layers of G_n (as X_n-modules) and the homomorphism τ_n: $G_n \to G_{n-1}$ maps these bases to those of G_{n-1} (see (20)). In fact the bases that we have chosen have another very useful property within G_n itself. We claim first that

(24) $$b_{r_2,2}^p = b_{r_2,1}\ (= w_{r_2}).$$

For, from (16) with $k = 2$, where the two cycles are disjoint,

(25) $$b_{r_2,2}^p = \prod_{i=0}^{p-1} (1 + ip^{n-2})_{p^{n-1},p}^{-1}(1 + p^{n-3} + ip^{n-2})_{p^{n-1},p}.$$

The cycles in (25) are disjoint and hence commute. By (11) and (19) (with $k = 2$),

$$\Omega_2(G_n) = B_1 B_2 X_{n,2}.$$

Therefore by Lemma 1 (i), and using the notation of that Lemma,

$$D_2 = (D_2 \cap B_1)B_2 X_{n,2}$$

and D_2 is abelian, by (23), and is normalised by X_n. Thus again by Lemma 1 (i),

$$D_2^p = B_2^p X_{n,1} \text{ is elementary abelian}$$

of rank $r_2 + 1$ and is an X_n-submodule of $\Omega_1(G_n)$. Hence, setting $W_j = \langle w_i | j \geqslant i \geqslant 1 \rangle$ for $r_1 \geqslant j \geqslant 0$, we must have

(26) $$b_{r_2,2}^p \in D_2^p \cap H_n = W_{r_2}.$$

Now from the equations following (9) and a simple calculation, we have

(27) $$b_{r_2,1} = (1)_{p^{n-1},p}^{-1} (2)_{p^{n-1},p}^{?} \dots (1 + p^{n-3} + (p-1)p^{n-2})_{p^{n-1},p}.$$

Observe that the last cycle here is the same as the last cycle in (25). But from (26),

$$b_{r_2,2}^p = \prod_{i=0}^{r_2-1} b_{r_2-i,1}^{\alpha_i},$$

for integers α_i. Therefore, again by the equations following (9), we must have

$$\alpha_{r_2-1} = \alpha_{r_2-2} = \dots = \alpha_1 = 0$$

and so

$$b_{r_2,2}^p \in \langle b_{r_2,1} \rangle.$$

Thus (24) follows by comparing the exponents of $(1)_{p^{n-1},p}$ in (25) and (27).

From (18) we deduce that

$$b_{r_2-1,2}^p = [b_{r_2,2}, x_n]^p = [b_{r_2,2}^p, x_n],$$

since $B_2^{X_n} \leqslant D_2$, which is abelian. Therefore, by (24) and (9),

$$b_{r_2-1,2}^p = [b_{r_2,1}, x_n] = b_{r_2-1,1}.$$

Continuing in this way, we see that

(28) $$b_{i,2}^p = b_{i,1} \ (= w_i),$$

$r_2 \geqslant i \geqslant 1$. Also, by (18),

(29) $$b_{0,2} = [b_{1,2}, x_n] \equiv x_{n,2} \mod \Omega_1(G_n),$$

using (12) and (20). Moreover $\langle b_{0,2}, x_{n,2} \rangle \leqslant C_2$ and

$$\Omega_1(G_n) \cap C_2 = W_{p^{n-2}-1} X_{n,1}$$

(by (13)). Therefore

(30) $$b_{0,2}^p = x_{n,1} \ (= b_{0,1}).$$

Combining the above results, we can now obtain useful information about adjacent Ω-layers in G_n. For $2\leqslant k\leqslant n-1$, we have from (4)

$$\Omega_2(G_{n-k+1})\cong\Omega_2(G_n/\Omega_{k-1}(G_n))=\Omega_{k+1}(G_n)/\Omega_{k-1}(G_n). \tag{31}$$

Thus we deduce from (28) and (30) that

$$b_{i,k+1}^p\equiv b_{i,k} \mod \Omega_{k-1}(H_n), \tag{32}$$

for $r_{k+1}\geqslant i\geqslant 1$; and

$$b_{0,k+1}^p\equiv x_{n,k}\equiv b_{0,k} \mod \Omega_{k-1}(G_n).$$

More information about adjacent Ω-layers of G_n will follow from our next key result.

Lemma 2. *Let $2\leqslant k\leqslant n-1$ and for $0\leqslant i\leqslant p^{k-2}(p-1)$, let*

$$V_i=W_{ip^{n-k}-1}.$$

Then B_k centralises the series

$$\Omega_1(H_n)=W_r=V_{p^{k-2}(p-1)}>\cdots>V_i>V_{i-1}>\cdots>V_0=1 \tag{33}$$

of length $p^{k-2}(p-1)$.

Proof. By Lemma 1 (ii), B_k centralises V_1. We argue by induction on i increasing and suppose that $[V_{i+1},B_k]\leqslant V_i$ for some $i\geqslant 0$. Then by (13)

$$[V_{i+2},X_{n,k}]=V_{i+1}X_{n,1},$$

and so

$$[V_{i+2},X_{n,k},B_k]\leqslant V_i.$$

Since $[X_{n,k},B_k]=1$, by Lemma 1 (i), the Three Subgroup Lemma (see for example [8], Lemma 2.13) implies

$$[B_k,V_{i+2},X_{n,k}]\leqslant V_iX_{n,1}\ (\lhd\, G_n).$$

Therefore $[B_k,V_{i+2}]\leqslant V_{i+1}X_{n,1}$, again by (13). Thus

$$[B_k,V_{i+2}]\leqslant V_{i+1}X_{n,1}\cap H_n=V_{i+1}$$

and the Lemma follows. □

Corollary 1. *For $0\leqslant i\leqslant n-2$, the factor group $\Omega_{i+2}(G_n)/\Omega_i(G_n)$ is nilpotent of class $p-1$ and hence regular.*

PROOF. By (31) we may assume that $i = 0$. Take $k = 2$ in Lemma 2. Then B_2 centralises the series (33) of length $p-1$. Also, modulo $X_{n,1}$, $X_{n,2}$ centralises this series (by (13)). Therefore, since$B_2X_{n,2}$ is abelian (by Lemma 1 (i)), the subgroup

$$\Omega_2(G_n) = B_1B_2X_{n,2}$$

is nilpotent of class at most $p-1$. In fact the class is exactly $p-1$, because of the action of $X_{n,2}$ on $\Omega_1(G_n) = W_rX_{n,1}$. The regularity follows, for example, from [3], Corollary 12.3.1. □

3. Subgroups of G_n.

We can now describe all the quasinormal subgroups of G_n lying in H_n and of exponent p.

THEOREM 2. *Let $n \geqslant 2$. Then a quasinormal subgroup of G_n of exponent p, lying in H_n, has the form W_i, for some i. Moreover W_i qn G_n if and only if* $1 \leqslant i \leqslant p^2 - 1$.

PROOF. Observe from (6) that when $n = 2$, $i \leqslant p-2$; and when $n = 3$, $i \leqslant p(p-1)-1$. Therefore in both these cases we are saying that W_i *qn* $G = G_n$ for all i for which W_i is defined.

Let Q *qn* G with Q of exponent p and lying in $H = H_n$. Then $Q \leqslant W_{r_1} = \Omega_1(H)$. Also QX_n is a subgroup and $\Omega_1(QX_n) = QX_{n,1}$. Thus $QX_{n,1}$ is an X_n-submodule of $\Omega_1(G)$ and so, since indecomposable modules are uniserial (see [4], Theorem VII, 5.3), $QX_{n,1} = W_iX_{n,1}$, for some i. Hence $Q = W_i$.

We establish first the sufficiency of the condition on i. Therefore *suppose that* $i \leqslant p^2 - 1$ and let $g = hy$, where $h \in H$ and $y \in X = X_n$, and put $K = \langle g \rangle$. We show that W_iK is a subgroup by considering 3 possibilities.

(i) *Suppose that* $y \in X^{p^2} = X_{n,n-2}$. Then y centralises W_i, by (13). Since $W_i \lhd H$, it follows that g normalises Q.

(ii) *Suppose that* $\langle y \rangle = X$. Then by [1], Lemma 3.12 (ii), $\Omega_1(K) = X_{n,1}$. Therefore $W_iK = W_iX_{n,1}K$ is a subgroup, since $W_iX_{n,1} \lhd G$.

(iii) Finally *suppose that* $\langle y \rangle = X^p$. Clearly we may assume that $n \geqslant 3$. By Corollary 1, $\Omega_{n-1}(G)/\Omega_{n-3}(G)$ is regular with elementary abelian derived subgroup. Therefore

$$g^p \equiv h^py^p \mod \Omega_{n-3}(G).$$

Also

$$h^p \in \langle b_{j,n-2} | 1 \leqslant j \leqslant p-2 \rangle \mod \Omega_{n-3}(H),$$

by (32) and (14). Continuing taking p-th powers in this way, we obtain

$$g^{p^{n-2}} \in W_{p-2} X_{n,1}. \tag{34}$$

Moreover $|HK : H| = |H\langle y \rangle : H| = |\langle y \rangle| = p^{n-1}$. Therefore $g^{p^{n-2}} \notin H$.

Now consider $W_i K$. If $i \leqslant p-1$, then y centralises W_i (by (13)) and so again g normalises W_i, i.e. $W_i K$ is a subgroup. On the other hand if $p \leqslant i$ $(\leqslant p^2-1)$, then by (34)

$$W_i K = W_i X_{n,1} K$$

and this is a subgroup as in (ii).

Conversely, suppose, for a contradiction, that $W_i\ qn\ G$ where $i \geqslant p^2$. Then by (6), $n \geqslant 4$. Let $g = b_{1,n-1} x_{n,n-2}$ and $K = \langle g \rangle$. We have $[b_{1,n-1}, x_{n,n-1}] = 1$, by (21). So

$$g^{p^{n-2}} = b_{1,n-1}^{p^{n-2}} = b_{1,1} = w_1, \tag{35}$$

using the argument of (iii) above. Consider the subgroup $W_i K$. Since $\Omega_1(G)$ is elementary abelian, an element of order p in $W_i K$ must have the form $w g_1$, where $w \in W_i$ and $g_1 \in \Omega_1(K) = W_1$ (by (35). Thus $\Omega_1(W_i K) = W_i \lhd W_i K$. Since $W_i \lhd H$, it follows that $x_{n,n-2}$ also normalises W_i. But $[w_{p^2}, x_{n,n-2}] = x_{n,1}$ (by (13)), a contradiction. This completes the proof of Theorem 2. □

For the next theorems we need to be able to identify certain subgroups of G_n. Let $n \geqslant 2$, $n-1 \geqslant k \geqslant 1$ and $r_k \geqslant \ell_k \geqslant 1$. Define

$$B_{\ell_k,k} = \langle b_{j,k} | 1 \leqslant j \leqslant \ell_k \rangle,$$

which is isomorphic to $Z_{p^k} \times \cdots \times Z_{p^k}$ (ℓ_k copies), by Lemma 1 (i).
Note that $B_{\ell_1,1} = W_{\ell_1}$.

Lemma 3. *Let* $n \geqslant 2$, $r_k \geqslant \ell_k \geqslant p^{n-k-1} - 1$ *for* $1 \leqslant k \leqslant n-3$, $r_{n-2} \geqslant \ell_{n-2} \geqslant \ell_{n-1}$ *and* $r_{n-1} \geqslant \ell_{n-1} \geqslant 1$. *Then*

$$L = B_{\ell_1,1} B_{\ell_2,2} \ldots B_{\ell_{n-1},n-1} X_{n,n-1} \tag{36}$$

is a subgroup of G_n; *and*

$$B_{\ell_1,1} B_{\ell_2,2} \ldots B_{\ell_{n-1},n-1} = L \cap H_n \tag{37}$$

is a subgroup of H_n.

PROOF. If $n=2$, then $L=W_{\ell_1}X_{n,1} \lhd G_2$. If $n=3$, then $L=W_{\ell_1}B_{\ell_2,2}X_{n,2}$ with $r_1=p(p-1)-1\geqslant\ell_1\geqslant\ell_2$ and $r_2=p-2\geqslant\ell_2\geqslant 1$. By Lemma 1 (i), $B_{\ell_2,2}X_{n,2}$ is a subgroup. Also $W_{\ell_1}X_{n,1} \lhd G_3$. Since L is the product of these two subgroups, again L is a subgroup.

Now suppose that the Lemma is true for $n-1$, some $n\geqslant 4$, and argue by induction on n. Then from (4) we obtain

$$L\Omega_1(G_n) = W_{r_1}L$$

is a subgroup. Recall that $C_2 = C_{G_n}(x_{n,2})$. It follows from (13), (22) and Lemma 1 (i), that

$$C_2 = W_{p^{n-2}-1}B_{r_2,2}\dots B_{r_{n-1},n-1}X_n.$$

Then

$$W_{r_1}L\cap C_2 = W_{r_1}B_{\ell_2,2}\dots B_{\ell_{n-1},n-1}X_{n,n-1}\cap C_2$$
$$= W_{p^{n-2}-1}B_{\ell_2,2}\dots B_{\ell_{n-1},n-1}X_{n,n-1}$$

is a subgroup. Since $\ell_1\geqslant p^{n-2}-1$, forming the product of this subgroup with the normal subgroup $W_{\ell_1}X_{n,1}$, we obtain L. So L is a subgroup.

Equation (37) follows from Dedekind's Intersection Lemma. □

The quasinormal subgroups of exponent p^{n-1} that we shall exhibit in Theorem 3 all have the form $L\cap H_n$ given by (37); and those of exponent p^{n-2} in Theorem 4 have the form $\Omega_{n-2}(L\cap H_n)$. As a consequence of the next result,we see that these subgroups are actually normal in H_n.

LEMMA 4. *Let* $n\geqslant 3$, $r_k\geqslant\ell_k\geqslant r_k-p^2$ *for* $1\leqslant k\leqslant n-3$, $r_{n-2}\geqslant\ell_{n-2}\geqslant\ell_{n-1}$ *and* $r_{n-1}\geqslant\ell_{n-1}\geqslant 1$. *Define* L *as in* (36). *Then*

(i) *L is a subgroup of G_n; and*

(ii) *for* $1\leqslant k\leqslant n-1$,

(38) $$\Omega_k(L) = B_{\ell_1,1}B_{\ell_2,2}\dots B_{\ell_k,k}X_{n,k} = L_k,$$

say; and $L_k \lhd G_n$ *for* $1\leqslant k\leqslant n-2$.

Moreover, if $r_k\geqslant\ell_k\geqslant r_k-p$ *for* $1\leqslant k\leqslant n-2$ *and* $r_{n-1}\geqslant\ell_{n-1}\geqslant 1$, *then*

(iii) *again L is a subgroup of G_n and $L\lhd G_n$.*

PROOF. (i) The hypotheses here imply those of Lemma 3. For, with $n\geqslant 3$ and $1\leqslant k\leqslant n-3$,

$$r_k-p^2-(p^{n-k-1}-1) = p^{n-k-1}(p-1)-p^2-p^{n-k-1} = p^{n-k-1}(p-2)-p^2$$
$$\geqslant p^2(p-2)-p^2\geqslant 0.$$

Therefore L is a subgroup of G_n.

(ii) It is easy to check that, for $1 \leqslant k \leqslant n-2$, $\ell_k \geqslant \ell_{k+1}$. We claim that, for $2 \leqslant k \leqslant n-1$,

$$(B_{\ell_k,k}X_{n,k})^p \subset B_{\ell_1,1}B_{\ell_2,2}\dots B_{\ell_{k-1},k-1}X_{n,k-1}. \tag{39}$$

We do not know yet that the right side of (39) is a subgroup. In order to prove (39), suppose that $k=2$. Then $B_{\ell_k,k}X_{n,k} \cong Z_{p^2} \times \cdot \cdot \cdot \times Z_{p^2}$ ($\ell_k + 1$ copies), by Lemma 1 (i). Thus (39) follows from (28), since $\ell_1 \geqslant \ell_2$. Therefore (39) is true for $n=3$. We suppose that (39) is true for G_{n-1} ($n \geqslant 4$) and $2 \leqslant k \leqslant n-2$ and argue by induction on n. Then for $3 \leqslant k \leqslant n-1$, we have from (20)

$$(B_{\ell_k,k}X_{n,k})^p \subset W_{r_1}B_{\ell_2,2}\dots B_{\ell_{k-1},k-1}X_{n,k-1}. \tag{40}$$

However, the left side of (40) and all but the first factor on the right belong to C_2 $(= C_{G_n}(x_{n,2}))$. Thus intersecting (40) with C_2 gives

$$\begin{aligned}(B_{\ell_k,k}X_{n,k})^p &\subset W_{p^{n-2}-1,1}B_{\ell_2,2}\dots B_{\ell_{k-1},k-1}X_{n,k-1}\\ &\subset B_{\ell_1,1}B_{\ell_2,2}\dots B_{\ell_{k-1},k-1}X_{n,k-1},\end{aligned}$$

since one checks easily that

$$\ell_1 \geqslant p^{n-2}-1. \tag{41}$$

Therefore (39) is true.

Now we can prove (38). Clearly $\Omega_{n-1}(L) = L = L_{n-1}$. We proceed by induction on k decreasing and suppose that (38) is true for some $k \leqslant n-1$. Certainly

$$\Omega_{k-1}(L) \supset B_{\ell_1,1}\dots B_{\ell_{k-1},k-1}X_{n,k-1}. \tag{42}$$

If the inclusion (42) is strict, then there is an element $g \in B_{\ell_k,k}X_{n,k}$ with $|g| = p^{k-1}$ and $g \notin L_{k-1}$. But by Lemma 1 (i), $g \in (B_{\ell_k,k}X_{n,k})^p$, contradicting (39). Therefore (42) is an equality. Thus our induction argument goes through and we have proved (38).

The second part of (ii) will follow from

$$L_{n-2} \lhd G_n. \tag{43}$$

When $n=3$, $L = W_{\ell_1}B_{\ell_2,2}X_{3,2}$ and $L_1 = W_{\ell_1}X_{3,1} \lhd G_3$. Therefore we suppose that $n \geqslant 4$ and that (43) is true for G_{n-1} and proceed by induction. Again from (20)

$$W_{r_1}B_{\ell_2,2}\dots B_{\ell_{n-2},n-2}X_{n,n-2} \lhd G_n;$$

and intersecting with C_2 gives

$$W_{p^{n-2}-1}B_{\ell_2,2}\dots B_{\ell_{n-2},n-2}X_{n,n-2} \lhd C_2.$$

Thus from (41) and $W_{\ell_1}X_{n,1} \lhd G_n$, we see that L_{n-2} is normalised by C_2. However, $[W_{r_1}, L_{n-2}] \leqslant W_{r_1-p^2}X_{n,1}$, by Lemma 2, and so W_{r_1} normalises L_{n-2}. Since $W_{r_1}C_2 = G_n$, (43) follows.

(iii) It is easy to check that the hypotheses here imply those of parts (i) and (ii). So L is a subgroup. Now however

$$[W_{r_1}, L] \leqslant W_{r_1-p}X_{n,1} \leqslant L,$$

by Lemma 2, and so W_{r_1} normalises L. Also when $n = 3$, by considering L modulo $\Omega_1(G_3)$ and intersecting with C_2, we see that $L \lhd G_n$. Then the induction argument used to prove (43) can be applied here to give $L \lhd G_n$ for all n. □

4. Quasinormal subgroups of large exponent.

We shall see that the quasinormal subgroups Q of G_n lying in H_n, other than those of Theorem 2, all have exponent p^m for $m = n-2$ or $n-1$. Also $\Omega_k(Q)$ modulo $\Omega_{k-1}(H_n)$ has 'large' rank, for all $1 \leqslant k \leqslant m-1$. We begin with those of exponent p^{n-1}.

THEOREM 3. (i) *Let $n \geqslant 3$ and Q be a quasinormal subgroup of G_n of exponent p^{n-1} and lying in H_n. Then for $1 \leqslant k \leqslant n-1$,*

$$\Omega_k(Q) \equiv B_{\ell_k,k} \ mod\ \Omega_{k-1}(H_n); \tag{44}$$

and $\ell_k \geqslant r_k - p$, for $1 \leqslant k \leqslant n-2$, and $\ell_{n-1} \geqslant 1$.

(ii) *Again let $n \geqslant 3$ and*

$$Q = \prod_{k=1}^{n-1} B_{\ell_k,k}$$

with $\ell_k \geqslant r_k - p$ for $1 \leqslant k \leqslant n-2$ and $\ell_{n-1} \geqslant 1$. Then Q is a subgroup of H_n of exponent p^{n-1}. Moreover Q qn G_n if and only if

$$\ell_k \geqslant r_k - p + 1, \tag{45}$$

for $1 \leqslant k \leqslant n-2$.

PROOF. (i) Since QX_n is a subgroup, we see that $\Omega_1(QX_n)$ is an X_n-module contained in $\Omega_1(G_n)$ and therefore has the form $W_iX_{n,1}$. Also there is an element $hb_{1,n-1} \in Q$ with $h \in \Omega_{n-2}(H_n)$, by Theorem 2. Thus since W_{r_1} must normalise Q,

$$[W_{r_1}, hb_{1,n-1}] \leqslant Q \cap W_{r_1-p},$$

by Lemma 2. However

this commutator subgroup does not belong to W_{r_1-p-1}.

For, otherwise we would have $[W_{r_1}, b_{1,n-1}] \leqslant W_{r_1-p-1}$. Then conjugating by x_n, we would obtain (using (18))

$$[W_{r_1}, b_{1,n-1}b_{0,n-1}] \leqslant W_{r_1-p-1}X_{n,1}. \tag{46}$$

But $b_{0,n-1} \equiv x_{n,n-1} \mod \Omega_{n-2}(G_n)$, using (29) and the homomorphism τ_n. Thus $x_{n,n-1}$ would centralise $W_{r_1}X_{n,1}/W_{r_1-p-1}X_{n,1}$, a contradiction.

Thus (44) is true for $k = 1$. The rest of (i) follows by induction on n from $G_n/\Omega_1(G_n) \cong G_{n-1}$.

(ii) The hypotheses here imply those of Lemma 4. Thus Q is a subgroup of H_n of exponent p^{n-1} (of the form (37)).

Now suppose that Q qn G_n. Assume for the moment that

$$\ell_1 \geqslant r_1 - p + 1. \tag{47}$$

Then we obtain (45), again by induction on n. So it suffices to prove (47).

Let $g = w_{r_1}x_n$ and $K = \langle g \rangle$. Then

$$\Omega_1(K) = \langle x_{n,1} \rangle, \tag{48}$$

by [1], Lemma 3.12 (2). In fact this follows easily from a simple module calculation. Thus using additive notation,

$$\begin{aligned} g^p = (w_{r_1}x_n)^p &= w_{r_1}(1 + x_n^{-1} + x_n^{-2} + \cdots + x_n^{-(p-1)})x_{n,n-1} \\ &= w_{r_1}(x_n^{-1} - 1)^{p-1}x_{n,n-1} \equiv w_{r_1}(x_n - 1)^{p-1}x_{n,n-1} \mod W_{r_1-p}X_{n,1} \\ &\equiv w_{r_1-p+1}x_{n,n-1} \mod W_{r_1-p}X_{n,1}. \end{aligned} \tag{49}$$

(The appearance of w_{r_1-p+1} here is the clue to establishing (47).) Similarly

$$g^{p^2} \equiv w_{r_1-p^2+1}x_{n,n-2} \mod W_{r_1-p^2}X_{n,1}.$$

(We can obtain (48) by computing $g^{p^{n-1}}$ in this way.) Since $\ell_1 > r_1 - p^2 + 1$, it follows from (48) that

$$x_{n,n-2} \in QK. \tag{50}$$

Also $b_{1,n-1} \in Q$, by hypothesis, and so QK contains

$$[b_{1,n-1}, g] = [b_{1,n-1}, x_n][b_{1,n-1}, w_{r_1}]^{x_n}.$$

However, by Lemma 2,

$$[b_{1,n-1}, w_{r_1}]^{x_n} \in W_{\ell_1}^{x_n} = W_{\ell_1}^g \leqslant QK.$$

Hence

(51) $$QK \text{ contains } [b_{1,n-1}, x_n] = g_1,$$

say. Now $g_1 \in Q^{G_n} \leqslant QX_{n,n-1}$, since $QX_{n,n-1} \lhd G_n$, by Lemma 4 (iii). From (4) and (12) we have

$$g_1 = x_{n,n-1}g_2,$$

where $g_2 \in \Omega_{n-2}(G_n) = \Omega_{n-2}(H_n)X_{n,n-2}$. Therefore $g_2 = hy$, with $h \in \Omega_{n-2}(H_n)$ and $y \in X_{n,n-2}$. Thus $y \in QK$, by (50). Then

$$g_1 \in QX_{n,n-1} \Rightarrow g_2 \in QX_{n,n-1} \Rightarrow h \in QX_{n,n-1} \cap H_n = Q \Rightarrow g_2 \in QK \Rightarrow$$
$$\Rightarrow x_{n,n-1} \in QK,$$

by (51).

Finally from (49) we now obtain

$$w_{r_1-p+1} \in QK \cap H_n = Q.$$

Thus $\ell_1 \geqslant r_1 - p + 1$, as required.

Conversely, suppose that $\ell_k \geqslant r_k - p + 1$ for $1 \leqslant k \leqslant n-2$ and $\ell_{n-1} \geqslant 1$. We show that Q *qn* G_n. Let $g = hy$, where $h \in H_n$, $y \in X_n$, and let $K = \langle g \rangle$. It suffices to show that QK is a subgroup. Let $y \in X_{n,k} \setminus X_{n,k-1}$, where $0 \leqslant k \leqslant n$. (Assume that $X_{n,-1} = 1$.) We claim that

(52) $$g^p \in QX_{n,k-1}.$$

Note that $QX_{n,n-1} \lhd G_n$, by Lemma 4 (iii). Thus $QX_{n,i}$ is a subgroup for all i; and

(53) $$Q \lhd H_n.$$

Let $h = wb_2b_3 \dots b_{n-1}$, where $w \in W_{r_1}$ and $b_i \in B_i$, $2 \leqslant i \leqslant n-1$. Considering $\Omega_1(G_n)$ as G_n-module, we have

$$g^p = (wb_2 \dots b_{n-1}y)^p = w((b_2 \dots b_{n-1}y)^{-1} - 1)^{p-1}(b_2 \dots b_{n-1}y)^p.$$

Thus

(54) $$g^p \in W_{r_1-p+1}C_2,$$

by Lemma 1. We prove (52) by induction on n.

Suppose $n = 3$. If $k \leqslant 2$, then $g^p \in W_{p-2}X_{3,k-1}$, by Corollary 1, and so $g^p \in QX_{3,k-1}$. If $k = 3$, then by (54)

$$g^p \in W_{r_1}X_{3,2} \cap W_{r_1-p+1}C_2 = (W_{r_1} \cap W_{r_1-p+1}C_2)X_{3,2} \leqslant QX_{3,2}.$$

So (52) holds for $n = 3$. We proceed by induction on n, assuming that $n \geqslant 4$ and that (52) holds in G_{n-1}. Thus by induction (using $G_n/\Omega_1(G_n) \cong G_{n-1}$) and (54), we have

$$\begin{aligned} g^p &\in W_{r_1}X_{n,1}QX_{n,k-1} \cap W_{r_1-p+1}C_2 \leqslant (W_{r_1}Q \cap W_{r_1-p+1}C_2)X_{n,1}X_{n,k-1} \\ &= W_{r_1-p+1}(W_{r_1}Q \cap C_2)X_{n,1}X_{n,k-1} \leqslant W_{r_1-p+1}(W_{p^{n-2}-1}Q)X_{n,1}X_{n,k-1}, \end{aligned}$$

by the argument used in Lemma 3. Therefore $g^p \in QX_{n,1}X_{n,k-1}$. So (52) follows if $k \geqslant 2$. But if $k \leqslant 1$, then $g^p \in H_n$ and so $g^p \in Q$ $(= QX_{n,k-1})$. Thus we have established (52).

Now from (52) we obtain $Q\langle g^p\rangle \leqslant QX_{n,k-1}$. But

$$|Q\langle g^p\rangle| = |Q||\langle g^p\rangle : Q \cap \langle g^p\rangle| = |Q|p^{k-1} = |QX_{n,k-1}|.$$

Therefore

$$Q\langle g^p\rangle = QX_{n,k-1}, \tag{55}$$

a subgroup. By Lemma 4

$$\Omega_{k-1}(QX_{n,n-1}) = \Omega_{k-1}(Q)X_{n,k-1} = N,$$

say, and $N \lhd G_n$. If $k \leqslant n-1$, then by Lemma 1 (i), $X_{n,k}$ centralises Q modulo N and therefore g normalises Q (by (53)). Otherwise $k = n$ and then $N = QX_{n,n-1} = Q\langle g^p\rangle$, by (55); i.e. QK is a subgroup.

This completes the proof of Theorem 3. □

Next we deal with the quasinormal subgroups of exponent p^{n-2}. The argument follows that of Theorem 3, though there added complications. Establishing the necessity of our conditions for quasinormality involves considering the product QK for $K = \langle w_{r_1}b_{1,n-1}x_{n,2}\rangle$ rather than $\langle w_{r_1}x_n\rangle$; and the prime 3 causes problems.

THEOREM 4. (i) *Let* $n \geqslant 4$ *and* Q *qn* G_n, $Q \leqslant H_n$ *and* Q *have exponent* p^{n-2}. *Then for* $1 \leqslant k \leqslant n-2$,

$$\Omega_k(Q) \equiv B_{\ell_k,k} \mod \Omega_{k-1}(H_n); \tag{56}$$

and $\ell_k \geqslant r_k - p^2$, *for* $1 \leqslant k \leqslant n-3$, *and* $\ell_{n-2} \geqslant 1$.

(ii) *Again let* $n \geqslant 4$ *and*

$$Q = \prod_{k=1}^{n-2} B_{\ell_k,k},$$

with $\ell_k \geqslant r_k - p^2$ *for* $1 \leqslant k \leqslant n-3$ *and* $\ell_{n-2} \geqslant 1$. *Then* Q *is a subgroup of* H_n

of exponent p^{n-2}. *Moreover* Q *qn* G_n *if and only if*

$$\ell_k \geqslant r_k - p(p-1)$$

for $1 \leqslant k \leqslant n-3$ *and* $\ell_{n-2} \geqslant p-2$.

PROOF. (i) We argue exactly as in Theorem 3 (i) to obtain the form (56). Then for the restrictions on ℓ_k we use the fact that there is an element $hb_{1,n-2} \in Q$, with $h \in \Omega_{n-3}(H_n)$. Thus $[W_{r_1}, hb_{1,n-2}]$ lies in $Q \cap W_{r_1-p^2}$ (by Lemma 2), but not in $W_{r_1-p^2-1}$.

(ii) With L defined as in Lemma 4, we have

$$Q = \Omega_{n-2}(L) \cap H_n.$$

Hence Q is a subgroup.

Now suppose that Q *qn* G_n. We shall show that

(57) $$\ell_1 \geqslant r_1 - p(p-1)$$

and

(58) $$\ell_2 \geqslant p-2 \text{ when } n=4.$$

Then the necessity of our conditions will follow by induction on n, as in Theorem 3. We shall prove that

(59) $$w_{p^{n-2}} \in Q,$$

i.e. $\ell_1 \geqslant p^{n-2}$. Assume this for the moment and let

$$g = w_{r_1} b_{1,n-1} x_{n,2}, \quad K = \langle g \rangle.$$

Then since QK is a subgroup, (59) implies

$$[w_{p^{n-2}}, g] \in QK.$$

Therefore substituting for g and expanding (using (13)), we get

$$[w_{p^{n-2}}, b_{1,n-1}] x_{n,1} \in QK.$$

Thus $x_{n,1} \in QK$. Considering actions on $\Omega_1(G_n)$ and using Lemma 2, we obtain

$$g^p = w b_{1,n-1}^p x_{n,1}^{\varepsilon},$$

where

(60) $$w \in W_{r_1-p(p-1)} \setminus W_{r_1-p(p-1)-1}.$$

The appearance of the suffix $r_1 - p(p-1)$ in (60) is the clue to establishing

(57). Recall that $b_{1,n-1}$ and $x_{n,2}$ commute, by Lemma 1 (i). Also ε is not divisible by p, since $g^p \notin H_n$. By (39), $b_{1,n-1}^p \in Q$. Therefore $wx_{n,1}^{\varepsilon} \in QK$ and so $w \in QK$. But if (57) is false, then $w \notin Q$. Thus $\langle w \rangle \times \langle x_{n,1} \rangle$ is disjoint from Q and embeds in the chain $[QK/K]$, a contradiction. Then (57) will follow from (59).

We have

$$\ell_1 \geqslant r_1 - p^2, \tag{61}$$

from (i), i.e.

$$\ell_1 \geqslant p^{n-2}(p-2) - 1. \tag{62}$$

Thus if $p \geqslant 5$, then $\ell_1 \geqslant p^{n-2}$ and we have (59). Therefore we may suppose that $p = 3$.

Let $g = w_{r_1} x_n$ and $K = \langle g \rangle$. Then QK is a subgroup and we have

$$\begin{aligned} g^{p^{n-2}} &= w_{r_1}(x_n^{-1} - 1)^{p^{n-2}-1} x_{n,2} \\ &\in QK \cap (W_{r_1 - p^{n-2}+1} X_{n,2} \setminus W_{r_1 - p^{n-2}} X_{n,2}). \end{aligned} \tag{63}$$

Also $b_{1,2} \in Q$ (by definition) and thus QK contains

$$[b_{1,2}, w_{r_1} x_n] = [b_{1,2}, x_n][b_{1,2}, w_{r_1}]^{x_n}.$$

Here the first factor on the right has the form $wx_{n,2}$, where

$$w \in W_{p^{n-2}-1} \leqslant Q,$$

by (29), the fact that C_2 contains $b_{1,2}$ and x_n, and (62); and by Lemma 2 the second factor lies in $W_{r_1 - p^{n-2}} X_{n,1} \leqslant QK$, by (61) and the fact that

$$\Omega_1(K) = X_{n,1} \tag{64}$$

(see [1], Lemma 3.12 (2)). Therefore $x_{n,2} \in QK$ and hence, by (63),

$$W_{r_1 - p^{n-2}+1} \leqslant QK \cap H_n = Q$$

(again using (64)). Thus $\ell_1 \geqslant r_1 - p^{n-2} + 1 = p^{n-2}(p-2)$ and (59) follows.

It remains to prove (58). Here $n = 4$. Let $g = b_{p-2,3} x_{4,2}$. Then

$$g^p = w b_{p-2,2} x_{4,1}$$

where $w \in W_{p^2-1}$ (using (28) and the facts that $b_{p-2,3}$ and $x_{4,2}$ commute and $g \in C_2$). Thus

$$g^{p^2} = w_{p-2}. \tag{65}$$

Let $K = \langle g \rangle$ and consider the subgroup QK. If uv has order p with $u \in Q$ and $v \in K$, then assuming (58) is false, we must have

$$v \in \langle g^{p^2} \rangle.$$

But by (57), $\ell_1 \geqslant r_1 - p(p-1) = p(p-1)^2 - 1 > p - 2$ and hence by (65), $v \in Q$. Therefore $uv \in \Omega_1(Q) = W_{\ell_1} = \Omega_1(QK) \lhd QK$ and so g normalises W_{ℓ_1}. Thus $x_{4,2}$ normalises W_{ℓ_1}. However, $W_{p^2} \leqslant W_{\ell_1}$ (by (59)), contradicting $[W_{p^2}, x_{4,2}] = X_{4,1}$. This proves (58) and we have established the necessary conditions in (ii).

Conversely suppose that

$$\ell_k \geqslant r_k - p(p-1), \text{ for } 1 \leqslant k \leqslant n-3, \text{ and } \ell_{n-2} \geqslant p-2.$$

Let $g = hy$, $h \in H_n$, $y \in X_n$, and $K = \langle g \rangle$. We show that

(66) $$QK \text{ is a subgroup.}$$

By Lemma 4 (ii),

(67) $$QX_{n,n-2} \lhd G_n$$

and so $Q \lhd H_n$. Thus we may suppose that $y \notin X_{n,1}$. Suppose that $y \in X_{n,k} \setminus X_{n,k-1}$, $2 \leqslant k \leqslant n-1$. It follows from (67) that $QX_{n,k-1}$ is a subgroup. Then by analogy with (52) in the proof of Theorem 3 (ii), we have

(68) $$g^p \in QX_{n,k-1}.$$

For, the argument used to prove (54) gives $g^p \in W_{r_1 - p(p-1)}C_2$. Then (68) follows by induction on n, as in Theorem 3 (ii). Analogous to (55) we obtain

$$Q\langle g^p \rangle = QX_{n,k-1}$$

and in the same way we deduce that QK is a subgroup.

Finally suppose that $K = X_n$. Then $g^p \in H_n X_{n,n-1}$ and, by the above, $Q\langle g^{p^2} \rangle = QX_{n,n-2} \lhd G_n$. Thus (66) follows and Q *qn* G_n. □

Our final result is surely the most striking, showing that a finite p-group can be remarkably devoid of quasinormal subgroups throughout most of its structure.

THEOREM 5. *Let $n \geqslant 5$. Then there are no quasinormal subgroups of G_n, lying in H_n and of exponent p^k, for $2 \leqslant k \leqslant n-3$.*

PROOF. Using $G_n/\Omega_1(G_n) \cong G_{n-1}$ and induction, it suffices to prove that there are no quasinormal subgroups of exponent p^2. Thus suppose that

Q qn G_n, $Q \leqslant H_n$ and Q has exponent p^2. As we saw in Theorems 3 (i) and 4 (i), we must have

$$\Omega_1(Q) = W_i \ \text{ and } \ Q \equiv B_{\ell_2,2} \ \text{mod} \ W_{r_1}$$

for some i, $\ell_2 \geqslant 1$. Since w_{r_1} must normalise Q and there is an element $b_{1,2}w \in Q$, with $w \in W_{r_1}$, we also have $[w_{r_1}, b_{1,2}w] \in Q$. Therefore $w_{r_1 - p^{n-2}} \in \Omega_1(Q)$, by Lemma 2 and the argument used to establish (46). Thus

(69) $$i \geqslant r_1 - p^{n-2} = p^{n-2}(p-2) - 1.$$

When $p \geqslant 5$, then it follows that

(70) $$i > p^{n-2}.$$

Assume for the moment that (70) is true even for $p = 3$. Let $g = b_{1,n-1}x_{n,2}$ $(= x_{n,2}b_{1,n-1}$, by Lemma 1 (i)) and put $K = \langle g \rangle$. Consider the subgroup QK. By (70)

$$\Omega_1(QK) \geqslant \Omega_1(Q) \geqslant W_{p^{n-2}}.$$

Also g must normalise $\Omega_1(QK)$ and hence

$$[w_{p^{n-2}}, g] = [w_{p^{n-2}}, b_{1,n-1}]x_{n,1} \in \Omega_1(QK).$$

Then since $[w_{p^{n-2}}, b_{1,n-1}] \in \Omega_1(Q)$, we have

$$x_{n,1} \in \Omega_1(QK).$$

Thus $x_{n,1} = uv$, with $u \in Q$, $v \in K$, and clearly we must have

$$v = g^{p(1+\lambda p)} = b_{1,n-1}^{p+\lambda p^2} x_{n,1},$$

for some integer λ. Therefore

$$x_{n,1} = u b_{1,n-1}^{p+\lambda p^2} x_{n,1},$$

and so $b_{1,n-1}^{p+\lambda p^2} = u^{-1} \in Q$ of exponent p^2. Since $b_{1,n-1}^{p+\lambda p^2}$ has order p^{n-2} $(\geqslant p^3)$, we have a contradiction. Thus Q cannot exist.

It remains to prove that (70) holds for $p = 3$. Indeed we shall prove that

(71) $$i \geqslant r_1 - p^{n-4}(p-1) - p^{n-3}(p-1),$$

for all odd p. Then

$$i \geqslant p^{n-1} - 2p^{n-2} + p^{n-4} - 1 = p^{n-2}(p - 2 + 1/p^2 - 1/p^{n-2}) > p^{n-2}$$

for all odd p and so (70) is true.

In order to prove (71), we take $g = w_{r_1}^s b_{1,4}^{-1} x_{n,3}$, where p does not divide s. Observe that $b_{1,4}$ and $x_{n,3}$ commute, by Lemma 1 (i). Also w_{r_1} and $x_{n,3}$ commute modulo $W_{r_1-p^{n-3}} X_{n,1}$. Thus working in H_n modulo $W_{r_1-p^{n-3}}$, we see (from Lemma 2) that

$$g^p = w' b_{1,4}^{-p} x_{n,2},$$

where $w' \in W_j$, for $j = r_1 - p^{n-4}(p-1)$, but $w' \notin W_{j-1}$, again by the argument used to prove (46). Similarly

$$g^{p^2} = w'' b_{1,4}^{-p^2} x_{n,1},$$

where $w'' \in W_k$, for $k = j - p^{n-3}(p-1)$, but $w'' \notin W_{k-1}$. Here we are using

$$b_{1,4}^{-p} \equiv b_{1,3}^{-1} \mod \Omega_2(C_2),$$

by (32) etc. Since, in the same way, we have

$$b_{1,4}^{-p^2} \equiv b_{1,2}^{-1} \mod \Omega_1(C_2),$$

we may write

$$g^{p^2} = w''' b_{1,2}^{-1} x_{n,1}$$

for $w''' \in W_k \setminus W_{k-1}$. Recalling that there is an element $b_{1,2} w \in Q$ $(w \in W_{r_1})$ and putting $K = \langle g \rangle$, it follows that

$$w''' w x_{n,1} = g^{p^2} b_{1,2} w \in \Omega_1(QK). \tag{72}$$

Also we can choose s such that

$$w''' w \notin W_{k-1}. \tag{73}$$

However, $p^{n-2}(p-1) \geqslant 2p^{n-2} \Rightarrow i \geqslant r_1 - p^{n-2}$ (by (69))

$$\geqslant p^{n-2} - 1 > p^{n-3} \Rightarrow w_{p^{n-3}} \in Q.$$

Thus $[w_{p^{n-3}}, g] = [w_{p^{n-3}}, b_{1,4}^{-1}] x_{n,1}$ implies $x_{n,1} \in \Omega_1(QK)$. Now by (28) and Corollary 1,

$$g^{p^3} = b_{1,2}^{-p} = w_1^{-1} \in \Omega_1(Q),$$

and so $\Omega_1(QK) = \Omega_1(Q) \times X_{n,1}$. Then from (72) we have

$$w''' w \in \Omega_1(QK) \cap H_n = \Omega_1(Q).$$

Hence $i \geqslant k$ by (73), i.e. we have proved (71). This completes the proof of Theorem 5. □

Concluding remarks. When we embarked on this work, we were interested to know if every quasinormal subgroup of a finite p-group was a term of a composition series consisting of quasinormal subgroups. Now it is difficult to imagine how that supposition could have been further from the truth. So we finish with the following question. Is there a non-trivial subgroup-theoretic property $\mathfrak{X}$ (see [8], page 9) of finite p-groups such that

(i) $\mathfrak{X}$ is invariant under subgroup lattice isomorphisms, and

(ii) every chain of $\mathfrak{X}$-subgroups of a finite p-group can be refined to a composition series of $\mathfrak{X}$-subgroups?

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