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Commutativity of $*$-Prime Rings with Generalized Derivations

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ABSTRACT - Let R be a 2-torsion free $*$-prime ring and F be a generalized derivation of R with associated derivation d. If U is a $*-$Lie ideal of R then in the present paper, we shall show that $U \subseteq Z(R)$ if R admits a generalized derivation F(with associated derivation d) satisfying any one of the properties: $(i) F[u,v] = [F(u), v]$, $(ii) F(u \circ v) = F(u) \circ v$, $(iii) F[u,v] = [F(u), v] + [d(v), u]$, $(iv) F(u \circ v) = F(u) \circ v + d(v) \circ u$, $(v) F(uv) \pm uv = 0$ and $(vi) d(u)F(v) \pm uv = 0$ for all $u, v \in U$

1. Introduction.

Let R be an associative ring with centre $Z(R)$. R is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the Lie product and the Jordan product respectively. A ring R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. A ring with an involution '$*$' is said to $*$-prime if $aRb = aRb^* = 0$ or $a^*Rb = aRb = 0$ implies that either $a = 0$ or $b = 0$. Every prime ring with an involution is $*$-prime but the converse need not hold in general. An example due to Oukhtite [8] justifies the above statement that is, let R be a prime ring. Consider $S = R \times R^o$, where R^o is the opposite ring of R. Define involution $*$ on S as $(x, y)^* = (y, x)$. Since $(0, x)S(x, 0) = 0$, it follows that S is not prime. Further, it can be easily seen that if $(a, b)S(c, d) = (a, b)S(c, d)^* = 0$, then either $(a, b) = 0$ or $(c, d) = 0$. Hence S is $*$-prime but not prime. The set of symmetric and skew-sym-

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metric elements of a $*$-ring will be denoted by $S_*(R)$ i.e., $S_*(R) = \{x \in R \mid x^* = \pm x\}$. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. A Lie ideal is said to a $*$-Lie ideal if $U^* = U$. An additive mapping $d : R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \text{ for all } x, y \in R.$$

This observation leads to the following definition, given in [5]; an additive mapping $F : R \longrightarrow R$ is called a generalized derivation with associated derivation d if

$$F(xy) = F(x)y + xd(y) \text{ holds for all } x, y \in R.$$

Familiar examples of generalized derivations are derivations and generalized inner derivations and the later includes left multiplier i.e., an additive map $F : R \longrightarrow R$ satisfying $F(xy) = F(x)y$ for all $x, y \in R$. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is fixed element of R and d a derivation of R is a generalized derivation and if R has 1, all generalized derivations have this form.

Recently a number of authors have studied commutativity of rings satisfying certain differential identities (see [1], [2], [4] etc. where further refrences can be found). In the present paper our objective is to extend some earlier results for Lie ideals in $*$-prime rings involving generalized derivations. Infact, we shall show that a $*$-Lie ideal U is central if R admits a generalized derivation F with associated derivation d satisfying any one of the following properties $(i) F[u, v] = [F(u), v]$, $(ii) F(u \circ v) = F(u) \circ v$, $(iii) F[u, v] = [F(u), v] + [d(v), u]$, $(iv) F(u \circ v) = F(u) \circ v + d(v) \circ u$, $(v) F(uv) \pm uv = 0$, and $(vi) d(u)F(v) \pm uv = 0$ for all $u, v \in U$.

2. Preliminary Results.

We shall be frequently using the following identities without any specific mention,

$$[xy, z] = x[y, z] + [x, z]y$$
$$[x, yz] = [x, y]z + y[x, z]$$
$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$
$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z]$$

We begin with the following known results which shall be used throughout to prove our theorems:

LEMMA 2.1 ([11], Lemma 4). *If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of a 2-torsion free $*$-prime ring and $a, b \in R$ such that $aUb = 0 = a^*Ub$ then either $a = 0$ or $b = 0$.*

LEMMA 2.2 ([10], Lemma 2.3). *Let U be a non zero $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U, U] = 0$, then $U \subseteq Z(R)$.*

LEMMA 2.3 ([11], Lemma 3). *Let U be a non zero $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U, U] \neq 0$, then there exist a non zero $*$-ideal M of R such that $[M, R] \subseteq U$ and $[M, R] \not\subseteq Z(R)$.*

LEMMA 2.4 ([10], Theorem 1.1). *Let R be a 2-torsion free $*$-prime ring, U a non zero Lie ideal of R and d a non zero derivation of R which commutes with $*$. If $d^2(U) = 0$, then $U \subseteq Z(R)$.*

LEMMA 2.5 ([10], Lemma 2.4). *Let U be a $*$-Lie ideal of a 2-torsion free $*$-prime ring R and $d(\neq 0)$ be derivation of R which commutes with $*$. If $d(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.*

LEMMA 2.6 ([10], Lemma 2.5). *Let $d(\neq 0)$ be derivation of a 2-torsion free $*$-prime ring R which commutes with $*$. Let $U \not\subseteq Z(R)$ be a $*$-Lie ideal of R. If $t \in R$ satisfies $td(U) = 0$ or $d(U)t = 0$ then $t = 0$.*

We shall now prove the following :

LEMMA 2.7. *Let R be a 2-torsion free $*$-prime ring and U be a $*$-Lie ideal of R. If $a \in S_*(R) \cap R$ such that $[a, U] \subseteq Z(R)$ then either $U \subseteq Z(R)$ or $a \in Z(R)$.*

PROOF. Let $U \not\subseteq Z(R)$. The given hypothesis can be written as $I_a(U) \subseteq Z(R)$ where I_a is the inner derivation determined by a. Hence using Lemma 2.5, $I_a = 0$ and this gives that $a \in Z(R)$.

LEMMA 2.8. *Let R be a 2-torsion free $*$-prime ring and d be a non-zero derivation of R which commutes with $*$. If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of R such that $[a, d(U)] = 0$ for some $a \in S_*(R) \cap R$, then $a \in Z(R)$.*

PROOF. Replacing u by $[a,u]$ in $[a,d(U)]=0$ we have, $0= =[a,d[a,u]]=[a,[a,d(u)]]+[a,[d(a),u]]=[a,[d(a),u]]$ for all $u\in U$. Hence, $0=d[a,[d(a),u]]=[d(a),[d(a),u]]+[a,d[d(a),u]]$ for all $u\in U$, Now using the hypothesis $0=[d(a),[d(a),u]]$ for all $u\in U$, by Lemma 2.4, $d(a)\in Z(R)$. Therefore, $d[a,u]=[d(a),u]+[a,d(u)]=0$. Replacing u by $[a^2,u]$ in $[a,d(U)]=0$ we obtain,

$$\begin{aligned}
0 &= [a,d[a^2,u]]=[a,d(a[a,u]+[a,u]a)]\\
&= [a,d(a)[a,u]+ad[a,u]+d[a,u]a+[a,u]d(a)]\\
&= [a,d(a)[a,u]]+[a,[a,u]d(a)]\\
&= d(a)[a,[a,u]]+[a,d(a)][a,u]+[a,u][a,d(a)]+[a,[a,u]]d(a)\\
&= d(a)[a,[a,u]]+[a,[a,u]]d(a)\\
&= 2d(a)[a,[a,u]].
\end{aligned}$$

Since R is 2-torsion free, $d(a)[a,[a,u]]=0$ for all $u\in U$. Hence

$$0=d(a)[a,[a,u]]=d(a)U[a,[a,u]] \text{ for all } u\in U.$$

Since, $a\in S_*(R)\cap R$ so, $d(a)\in S_*(R)\cap R$. Thus, $0=d(a)U[a,[a,u]]= =(d(a))^*U[a,[a,u]]$ for all $u\in U$. Therefore, either $[a,[a,u]]=0$ for all $u\in U$ or $d(a)=0$. If $[a,[a,u]]=0$ for all $u\in U$ then $a\in Z(R)$. If $d(a)=0$, using Lemma 2.3 there exists an $*$-ideal M of R, let $[va,u]\in U$ where $v\in[M,R]$, hence $0=[a,d[va,u]]=[a,d(v)[a,u]+vd[a,u]+d[v,u]a+ +[v,u]d(a)]=[a,d(v)[a,u]+[v,u]d(a)]=d(v)[a,[a,u]]+[a,d(v)][a,u]+ +[v,u][a,d(a)]+[a,[v,u]]d(a)=d(v)[a,[a,u]]$ for all $v\in[M,R]$, $u\in U$. Therefore, $0=d[M,R][a,[a,u]]$ for all $u\in U$. Using Lemma 2.6, $0=[a,[a,u]]$ for all $u\in U$. Thus, $a\in Z(R)$.

3. Main Results.

We facilitate our discussion by proving the following theorem

THEOREM 3.1. *Let R be a 2-torsion free $*$-prime ring and $F:R\to R$ be a generalized derivation with associated non zero derivation d which commutes with $*$. If U is a $*$-Lie ideal of R such that $F[u,v]=[F(u),v]$ for all $u,v\in U$ then $U\subseteq Z(R)$.*

PROOF. Replacing u by $[u,ru]$ in $F[u,v]=[F(u),v]$ for all $u,v\in U$ we have

$$F[[u,r]u,v]=[F([u,r]u),v] \quad \text{for all } u,v\in U, r\in R.$$

This implies that $F([u,r][u,v]+[[u,r],v]u)=[F[u,r]u+[u,r]d(u),v]$ for all $u,v\in U, r\in R$. Using the hypothesis we obtain $[u,r]d[u,v]= = [u,r][d(u),v]$ for all $u,v\in U, r\in R$.

This gives us $[u,r][u,d(v)]=0$ for all $u,v\in U, r\in R$. Replacing r by rs for some s in R we get

$$[u,R]R[u,d(v)]=0 \text{ for all } u,v\in U. \tag{3.1}$$

If $u\in S_*(R)\cap U$, then $[u,R]R[u,d(U)]=[u,R]^*R[u,d(U)]$. Thus, for some $u\in S_*(R)\cap U$ either $[u,R]=0$ or $[u,d(U)]=0$. But for any $u\in U$, $u-u^*, u+u^*\in S_*(R)\cap U$. Therefore, for some $u\in U$ either $[u-u^*,R]=0$ or $[u-u^*,d(U)]=0$. If $[u-u^*,R]=0$ then from equation (3.1) we obtain that $[u,R]R[u,d(U)]=[u,R]^*R[u,d(U)]=0$ for all $u\in U$ hence either $[u,R]=0$ or $[u,d(U)]=0$. Let $L=\{u\in U \mid [u,R]=0\}$ and $K=\{u\in U\mid [u,d(U)]=0\}$. Then it can be seen that L and K are two additive subgroups of U whose union is U. Using Brauer's trick we have either $L=U$ or $K=U$. If $L=U$, then $[u,R]=0$ for all $u\in U$ that is $U\subseteq Z(R)$ and if $K=U$, then $[u,d(U)]=0$ for all $u\in U$, which implies that $U\subseteq Z(R)$ by Lemma 2.8. If $[u-u^*,d(U)]=0$, then again by (3.1) we obtain that $[u,R]R[u,d(U)]=[u,R]R[u,d(U)]^*=0$ for all $u\in U$. This gives us either $[u,R]=0$ or $[u,d(U)]=0$. If $[u,d(U)]=0$ then using Lemma 2.7 we obtain $U\subseteq Z(R)$. Hence in any case we obtain that $U\subseteq Z(R)$.

THEOREM 3.2. *Let R be a 2-torsion free $*$-prime ring and $F: R\to R$ be a generalized derivation with associated non zero derivation d which commutes with $*$. If U is a $*$-Lie ideal of R such that $F(u\circ v)=F(u)\circ v$ for all $u,v\in U$ then $U\subseteq Z(R)$.*

PROOF. Replacing u by $[u,ru]$ in $F(u\circ v)=F(u)\circ v$ for all $u,v\in U$, $r\in R$ we have

$$F([u,r]u\circ v)=F([u,r]u)\circ v \quad \text{for all } u,v\in U, r\in R.$$

This yeilds that,

$$F(([u,r]\circ v)u+[u,r][u,v])=F[u,r]u\circ v+[u,r]d(u)\circ v \text{ for all } u,v\in U, r\in R.$$

Thus we obtain,

$$\begin{aligned} F([u,r]\circ v)u+([u,r]\circ v)d(u)+F[u,r][u,v]+[u,r]d[u,v] \\ =(F[u,r]\circ v)u+F[u,r][u,v]+([u,r]\circ v)d(u) \\ +[u,r][d(u),v] \text{ for all } u,v\in U, r\in R. \end{aligned}$$

Using our hypothesis we find that $[u, r]d[u, v] = [u, r][d(u), v]$ for all $u, v \in U, r \in R$.Hence, we obtain, $[u, r][u, d(v)] = 0$ for all $u, v \in U, r \in R$. Replacing r by rs for some s in R we get $[u, R]R[u, d(v)] = 0$ for all $u, v \in U$. This leads to equation (3.1). Hence, proceeding on the same way as above, we obtain that $U \subseteq Z(R)$.

THEOREM 3.3. *Let R be a 2-torsion free $*$-prime ring and $F : R \to R$ be a generalized derivation with associated non zero derivation d which commutes with $*$. If U is a $*$-Lie ideal of R such that $F[u, v] = = [F(u), v] + [d(v), u]$ for all $u, v \in U$ then $U \subseteq Z(R)$.*

PROOF. We have $F[u, v] = [F(u), v] + [d(v), u]$ for all $u, v \in U$. Now replacing u by $[u, ru]$ we get

$$F[[u, r]u, v] = [F([u, r]u), v] + [d(v), [u, r]u] \text{ for all } u, v \in U, r \in R.$$

This gives us

$$F([u, r][u, v] + [[u, r], v]u) = [F[u, r]u + [u, r]d(u), v] + [d(v), [u, r]u] \quad \text{for all } u, v \in U, r \in R.$$

Hence

$$\begin{aligned} &F[u, r][u, v] + [u, r]d[u, v] + F[[u, r], v]u + [[u, r], v]d(u) \\ &= F[u, r][u, v] + [F[u, r], v]u + [u, r][d(u), v] + [[u, r], v]d(u) \\ &\quad + [u, r][d(v), u] + [d(v), [u, r]]u \text{ for all } u, v \in U, r \in R. \end{aligned}$$

Using the hypothesis we obtain, $[u, r]d[u, v] = [u, r][d(v), u] + [u, r][d(u), v]$ for all $u, v \in U, r \in R$. This gives us $[u, r][u, d(v)] = 0$ for all $u, v \in U, r \in R$. This is same as equation (3.1) hence, continuing in the same manner as above we obtain that $U \subseteq Z(R)$.

THEOREM 3.4. *Let R be a 2-torsion free $*$-prime ring and $F : R \to R$ be a generalized derivation with associated non zero derivation d which commutes with $*$. If U is a $*$-Lie ideal of R such that $F(u \circ v) = = F(u) \circ v + d(v) \circ u$ for all $u, v \in U$ then $U \subseteq Z(R)$.*

PROOF. Replacing u by $[u, ru]$ in $F(u \circ v) = F(u) \circ v + d(v) \circ u$ for all $u, v \in U$, we obtain, $F([u, r]u \circ v) = F([u, r]u) \circ v + d(v) \circ [u, r]u$ for all $u, v \in U, r \in R$. This gives us, $F(([u, r] \circ v)u + [u, r][u, v]) =$

$= F[u,r]u \circ v + [u,r]d(u) \circ v + d(v) \circ [u,r]u$ for all $u, v \in U, r \in R$. Thus,

$$\begin{aligned} &F([u,r] \circ v)u + ([u,r] \circ v)d(u) + F[u,r][u,v] + [u,r]d[u,v] \\ &= (F[u,r] \circ v)u + F[u,r][u,v] + ([u,r] \circ v)d(u) + [u,r][d(u),v] \\ &\quad + (d(v) \circ [u,r])u + [u,r][d(v),u] \text{ for all } u, v \in U, r \in R. \end{aligned}$$

Using our hypothesis $[u,r]d[u,v] = [u,r][d(u),v] + [u,r][d(v),u]$ for all $u, v \in U, r \in R$. This gives us $[u,r][u,d(v)] = 0$ for all $u, v \in U, r \in R$. Replacing r by rs for some s in R we get $[u,R]R[u,d(v)] = 0$ for all $u, v \in U$ which is equation (3.1). Therefore, proceeding in the same way as above we obtain that $U \subseteq Z(R)$.

THEOREM 3.5. *Let R be a 2-torsion free $*$-prime ring and U a $*$-Lie ideal of R. If d and g are any two derivations such that both of them are non-zero which commute with $*$. If $[g(U), d(U)] = 0$ then $U \subseteq Z(R)$.*

PROOF. In view of Lemma 2.8 the proof is clear.

THEOREM 3.6. *Let R be a 2-torsion free $*$-prime ring and U a $*$-Lie ideal of R. If F is a generalized derivation with associated non zero derivation d which commutes with $*$ such that $F(uv) \pm uv = 0$ for all $u, v \in U$ then $U \subseteq Z(R)$.*

PROOF. Let $U \not\subseteq R$. We have $F(uv) \pm uv = 0$ for all $u, v \in U$. This can be rewritten as

$$F(u)v + ud(v) \pm uv = 0 \text{ for all } u, v \in U. \tag{3.2}$$

Replacing u by $[u, ru]$ in (3.2) and using (3.2) we get

$$[u,r]ud(v) = 0 \text{ for all } u, v \in U, r \in R. \tag{3.3}$$

Substituting rs in place of r for some $s \in R$ in (3.3) we get $[u,R]Rud(U) = 0$ for all $u \in U$. If $u \in S_*(R) \cap U$, then either $[u,R] = 0$ or $ud(U) = 0$ for each fixed $u \in S_*(R) \cap U$. For any $v \in U$ we have $v - v^* \in S_*(R) \cap U$ and $v + v^* \in S_*(R) \cap U$, thus $2v \in S_*(R) \cap U$. Thus for some fixed $v \in U$, either $[2v, R] = 0$ or $2vd(U) = 0$. As R is 2-torsion free we have for some fixed $v \in U$, either $[v,R] = 0$ or $vd(U) = 0$. Let $A = \{v \in U \mid [v,R] = 0\}$ and $B = \{v \in U \mid vd(U) = 0\}$. It can be easily seen that A and B are two additive subgroups of U whose union is U thus using Brauer's trick we get

$A = U$ or $B = U$. If $A = U$, then $U \subseteq Z(R)$. If $B = U$, then using Lemma 2.6 we obtain that $U \subseteq Z(R)$ or $U = 0$. Thus in every case we obtain that $U \subseteq Z(R)$.

THEOREM 3.7. *Let R be a 2-torsion free $*$-prime ring and U a $*$-Lie ideal of R. If F is a generalized derivation with associated non zero derivation d which commutes with $*$ such that $d(u)F(v) \pm uv = 0$ for all $u, v \in U$ then $U \subseteq Z(R)$.*

PROOF. We have $d(u)F(v) \pm uv = 0$ for all $u, v \in U$. Replacing v by $[v, rv]$, $r \in R$ we obtain

$$d(u)[v, r]d(v) = 0 \text{ for all } u, v \in U, r \in R. \tag{3.4}$$

Using Lemma 2.6 we find that, $[v, r]d(v) = 0$ for all $v \in U, r \in R$. Substituting rs for r where $s \in R$ we have

$$[v, R]Rd(v) = 0 \text{ for all } v \in U. \tag{3.5}$$

If $v \in S_*(R) \cap U$, then either $[v, R] = 0$ or $d(v) = 0$. For any $u \in U$, $u - u^* \in S_*(R) \cap U$. Thus from above $[u - u^*, R] = 0$ or $d(u - u^*) = 0$ that is either $[u, R] = [u, R]^*$ or $d(u) = (d(u))^*$. If $[u, R] = [u, R]^*$ then from (3.5) we have $[u, R]Rd(u) = [u, R]^*Rd(u) = 0$ for all $u \in U$. Thus either $[u, R] = 0$ or $d(u) = 0$. Now if $d(u) = (d(u))^*$, then again equation (3.5) yields that $[u, R] = 0$ or $d(u) = 0$. In both cases, by using Brauer's trick and since $d \neq 0$, we conclude that $U \subseteq Z(R)$.

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