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The Schur Multiplier of a Generalized Baumslag-Solitar Group

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ABSTRACT - The structure of the Schur multiplier of an arbitrary generalized Baumslag-Solitar group is determined and applications to central extensions are described.

1. Introduction and Results.

A *generalized Baumslag-Solitar group*, or *GBS-group*, is the fundamental group of a finite connected graph of groups with infinite cyclic vertex and edge groups. In detail let Γ be a finite connected graph – multiple edges and loops are allowed – with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For each edge e we choose endpoints e^+ and e^-, and hence a direction for the edge. Infinite cyclic groups $\langle g_x \rangle$ and $\langle u_e \rangle$ are assigned to each vertex x and edge e. Injective homomorphisms $\langle u_e \rangle \to \langle g_{e^+} \rangle$ and $\langle u_e \rangle \to \langle g_{e^-} \rangle$ are defined by the assignments

$$u_e \mapsto g_{e^+}^{\omega^+(e)} \text{and} \quad u_e \mapsto g_{e^-}^{\omega^-(e)},$$

where $\omega^+(e),\ \omega^-(e) \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. Thus we have a weight function

$$\omega : E(\Gamma) \to \mathbb{Z}^* \times \mathbb{Z}^*$$

where $\omega(e) = (\omega^-(e), \omega^+(e))$. The weighted graph (Γ, ω) is called a *GBS-graph.*

The GBS-group determined by the weighted graph (Γ, ω) is the fundamental group

$$G = \pi_1(\Gamma, \omega).$$

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To obtain a presentation of G choose a maximal subtree T of Γ. Then G has generators

$$g_x, (x \in V(\Gamma)), \text{and } t_e,\ (e \in E(\Gamma)\backslash E(T)),$$

with defining relations

$$\begin{cases} g_{e^+}^{\omega^+(e)} & = g_{e^-}^{\omega^-(e)},\ \text{for } e \in E(T), \\ (g_{e^+}^{\omega^+(e)})^{t_e} & = g_{e^-}^{\omega^-(e)},\ \text{for } e \in E(\Gamma)\backslash E(T). \end{cases}$$

It is known that up to isomorphism G is independent of the choice of T: for this and other basic properties of graphs of groups see [1], [3], [10]. If Γ consists of a single loop with weight (n, m), then $\pi_1(\Gamma, \omega)$ is a Baumslag-Solitar group

$$BS(m, n) = \langle t, g \mid (g^m)^t = g^n \rangle.$$

It is easy to see that *GBS*-groups are torsion-free. They are obviously finitely presented, and in fact every finitely generated subgroup of a GBS-group is either free or a GBS-group ([6], 2.7: see also [4], 1.2), so such groups are coherent, i.e., all finitely generated subgroups are finitely presented. By an important result of Kropholler ([7]) the non-cyclic *GBS*-groups are exactly the finitely generated groups of cohomological dimension 2 which have an infinite cyclic subgroup commensurable with its conjugates. It is therefore natural to enquire about homology and cohomology of GBS-groups in dimensions 1 and 2.

Here we are concerned with integral homology: of course $H_1(G) \simeq G_{ab}$, the abelianization, while $H_2(G) = M(G)$ is the Schur multiplier of G. Our principal result describes the structure of the Schur multiplier of an arbitrary *GBS*-group.

THEOREM 1. *Let G be a generalized Baumslag-Solitar group. Then $M(G)$ is free abelian of rank $r_0(G) - 1$ where $r_0(G)$ is the torsion-free rank of G_{ab}.*

COROLLARY 1. *The Euler characteristic of a GBS-group is* 0.

This follows since the homology groups of a GBS-group G in dimensions 0, 1, 2 have torsion-free ranks 1, $r_0(G)$, $r_0(G) - 1$ respectively and the alternating sum of these is zero.

We remark that associated with any GBS-group there is a complex $K(\Gamma, \omega)$ defined in [4]. It can be shown that the Euler characteristic of this complex is zero and this observation is the basis for a topological – but not necessarily shorter – treatment of Theorem 1. Details will appear elsewhere ([5]).

T-dependence.

The structure of G_{ab}, and hence $r_0(G)$, can be found from the abelian presentation of G_{ab} arising from the standard presentation of the *GBS*-group G by the usual method of Smith normal form. However, this is a lengthy process and, as only $r_0(G)$ is required in order to compute $M(G)$, it is worthwhile to give a simpler method.

Let $G = \pi_1(\Gamma, \omega)$ be a *GBS*-group and let T be the chosen maximal subtree of Γ. Suppose that $e = \langle x, y\rangle \in E(\Gamma)\backslash E(T)$ where $x \neq y$. Now there is a unique path in the tree T from x to y, say $x = x_0, x_1, \ldots, x_n = y$. By reading along this path, we obtain a relation $g_x^{p_1(e)} = g_y^{p_2(e)}$ where $p_1(e)$ and $p_2(e)$ are the respective products of the left and right weight values of the edges in the path from x to y. If the vector $(\omega^-(e), \omega^+(e))$ is a rational multiple of $(p_1(e), p_2(e))$, then e is said to be *T-dependent*, and otherwise e is *T-independent*. If e is a loop, then by convention $p_1(e) = 1 = p_2(e)$ and e is T-dependent precisely when $\omega^-(e) = \omega^+(e)$.

The definition of T-dependence may be restated as follows.

Lemma 1. *With the above notation, a non-tree edge $e = \langle x, y\rangle$ of a GBS-graph is T-dependent if and only if*

$$\frac{\omega^-(e)}{\omega^+(e)} = \frac{p_1(e)}{p_2(e)}.$$

If every non-tree edge of a GBS-graph is T-dependent, the GBS-graph is said to be *tree-dependent*. The torsion-free rank of the abelianization of a GBS-group can be computed from the following result.

Theorem 2. *Let $G = \pi_1(\Gamma, \omega)$ be a generalized Baumslag-Solitar group. Then*

$$r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \varepsilon(\Gamma, \omega)$$

where $\varepsilon(\Gamma, \omega) = 1$ if (Γ, ω) is tree-dependent and otherwise equals 0.

(A variant of this result with a different proof appears in [8], Theorem 1.1). We note that, as a consequence of Theorem 2, $r_0(G)$ can be found by simply inspecting the graph of the GBS-group G. Notice also that $\varepsilon(\Gamma, \omega)$ *depends only on the GBS-graph* (Γ, ω), not on the choice of maximal subtree. Thus the property of tree-dependence is independent of the maximal subtree selected.

We remark that the invariant ε is closely related to the centre of a GBS-group and is an important tool in the theory of GBS-groups: it is the subject of an ongoing investigation.

As is well known, knowledge of the structure of the Schur multiplier of a group allows one to draw conclusions about central extensions by the group. As a consequence of Theorem 1 one can determine when all central extensions by a GBS-group G split, i.e., they are direct products. It is shown in Corollary 4 below that *every central extension by a generalized Baumslag-Solitar group G splits if and only if G_{ab} is infinite cyclic.*

2. Proof of Theorem 2.

Let $G = \pi_1(\Gamma, \omega)$ be a *GBS*-group with T a maximal subtree of Γ. Then G has an abelian presentation with generators g_x, t_e, where $x \in V(\Gamma)$, $e \in E(\Gamma)\backslash E(T)$, subject to the defining relations $g_{e^-}^{\omega^-(e)} = g_{e^+}^{\omega^+(e)}$, $(e \in E(\Gamma))$. Put $G_0 = \langle g_x \mid x \in V(\Gamma)\rangle$; then $G_0 \simeq \pi_1(T, \omega)$ and $r_0(G_0) \leq 1$ since each pair of generators of G_0 is linearly dependent. Since G_0 has fewer relations than generators, it is infinite and $r_0(G_0) = 1$. Of course, the stable elements t_e, are linearly independent modulo the torsion subgroup of G_{ab}. Therefore

$$r_0(G) = |E(\Gamma)\backslash E(T)| + \varepsilon,$$

where $\varepsilon = 1$ if each vertex generator has infinite order modulo G' and otherwise $\varepsilon = 0$. If some non-tree edge e is T-independent, then, in the notation of Lemma 1, the relations $g_{e^-}^{\omega^-(e)} = g_{e^+}^{\omega^+(e)}$ and $g_{e^-}^{p_1(e)} = g_{e^+}^{p_2(e)}$ are independent, which forces each vertex generator to have finite order modulo G'; hence $\varepsilon = 0$. On the other hand, if all such edges are T-dependent, i.e., (Γ, ω) is tree-dependent, then all vertex generators have infinite order and $\varepsilon = 1$. Since

$$|E(\Gamma)\backslash E(T)| = |E(\Gamma)| - (|V(\Gamma)| - 1) = |E(\Gamma)| - |V(\Gamma)| + 1,$$

the result follows on setting $\varepsilon(\Gamma, \omega) = \varepsilon$. □

3. Proof of Theorem 1.

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group with T a maximal subtree of Γ. We recall the following inequality, which is valid for any finitely presented group H with n generators and r relators:

$$n - r \leq r_0(H) - d(M(H)),$$

where $d(X)$ is the minimal number of generators of a group X, (see, for example, [9], p.550). In the present situation we have $n = |V(\Gamma)| + |E(\Gamma)\backslash E(T)|$ and $r = |E(\Gamma)|$, so $n - r = 1$. Thus $d(M(G)) \leq r_0(G) - 1$ and it suffices to prove that $r_0(M(G)) \geq r_0(G) - 1$. The proof is by induction on $|E(\Gamma)|$, which may be assumed positive.

(i) *We can assume that $r_0(G) > 1$, so that Γ is not a tree.*

For if $r_0(G) = 1$, then $d(M(G)) = 0$. Note that if Γ is a tree, then $r_0(G) = 1$ since each pair of vertex generators is linearly independent.

(ii) *Case: Γ has a single non-tree edge.*

Let $e = \langle x, y \rangle$ be the edge which is not in T. Now $r_0(G) \leq 2$ by Theorem 2, so $r_0(G) = 2$ and $\varepsilon(\Gamma, \omega) = 1$; thus e must be T-dependent. Apply the five-term homology sequence for the exact sequence $G' \rightarrowtail G \twoheadrightarrow G_{ab}$ to get

$$M(G) \to M(G_{ab}) \to G'/[G', G] \to G_{ab} \to G_{ab} \to 1.$$

Note that $r_0(M(G_{ab})) = 1$ since $M(G_{ab}) \simeq G_{ab} \wedge G_{ab}$.

We claim that $G'/[G', G]$ is finite. To see this write $t = t_e$ and let $\omega(e) = (h, k)$, so that $(g_y^k)^t = g_x^h$. Also $\langle g_x \rangle \cap \langle g_y \rangle = \langle g_x^m = g_y^n \rangle$ where $m, n \in \mathbb{Z}^*$. By T-dependence (h, k) is a rational multiple of (m, n), say $ih = jm$ and $ik = jn$, with $i, j \in \mathbb{Z}^*$. Then

$$[g_y, t]^{ik} \equiv [g_y^{ik}, t] \equiv g_y^{-ik} g_x^{ih} \equiv g_y^{-jn} g_x^{jm} \equiv 1 \bmod [G', G].$$

Next for any vertex generator g_z we have $g_z^r = g_y^s$ for some $r, s \in \mathbb{Z}^*$, and hence

$$[g_z, t]^r \equiv [g_y^s, t] \equiv [g_y, t]^s \bmod [G', G].$$

Finally, $[g_u, g_v][G', G]$ has finite order for any vertex generators g_u, g_v. It follows that $G'/[G', G]$ is periodic, so it is finite.

Returning to the exact homology sequence above, we conclude that $M(G)$ must be infinite, so that $r_0(M(G)) \geq 1 = r_0(G) - 1$, as required.

(iii) *From now on we assume that Γ has at least two non-tree edges.*

Let $e = \langle x, y \rangle$ be one of the non-tree edges and let the unique path in T from x to y be

$$\langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \ldots, \langle x_{k-1}, x_k \rangle,$$

where $x = x_1$ and $y = x_k$. Define subgroups $G_1 = \langle t_e, g_{x_1}, \ldots, g_{x_k} \rangle$ and

$$G_2 = \langle t_f, g_x \mid x \in V(\Gamma),\ f \in E(\Gamma) \backslash E(T),\ f \neq e \rangle.$$

Then $G_i = \pi_1(\Gamma_i, \omega)$, $i = 1, 2$, where Γ_1, Γ_2 are subgraphs of Γ with $V(\Gamma_1) = \{x_1, \ldots, x_k\}$, $V(\Gamma_2) = V(\Gamma)$ and respective edge sets $\{e, \langle x_j, x_{j+1} \rangle \mid j = 1, 2, \ldots, k-1\}$ and $E(\Gamma) \backslash \{e\}$, with restricted weight functions. Furthermore

$$G = G_1 *_U G_2$$

where $U = \langle g_{x_1}, g_{x_2}, \ldots, g_{x_k} \rangle$. Since $U \simeq \pi_1(T_0, \omega)$, with T_0 the path $\langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \ldots, \langle x_{k-1}, x_k \rangle$, we have $r_0(U) = 1$ and $M(U) = 0$ by (i).

Next we form the Mayer-Vietoris sequence for the generalized free product $G = G_1 *_U G_2$, ([2], p. 51),

$$0 = M(U) \to M(G_1) \oplus M(G_2) \to M(G) \to U_{ab} \to$$

$$(*) \qquad (G_1)_{ab} \oplus (G_2)_{ab} \to G_{ab} \to 1.$$

At this point we must distinguish two cases.

(iv) *Case: the graph Γ has a non-tree edge e which is T-dependent.*

Apply the Mayer-Vietoris sequence above for the edge e. Since Γ_1 has just one non-tree edge e and it is T-dependent in Γ_1, we conclude that $r_0(G_1) = 2$ and $M(G_1) \simeq \mathbb{Z}$ by (ii). Also UG_1'/G_1' is infinite, so the image of $(G_1)_{ab}$ in the exact sequence $(*)$ has infinite projection into $(G)_{ab}$. Therefore

$$r_0(G) \leq r_0(G_1) + r_0(G_2) - 1 = r_0(G_2) + 1$$

and $r_0(G_2) \geq r_0(G) - 1$. By induction on $|E(\Gamma)|$ the result is true for G_2, so we have

$$r_0(M(G)) \geq r_0(M(G_1) \oplus M(G_2)) \geq 1 + (r_0(G) - 2) = r_0(G) - 1,$$

as required.

We are now left with the situation:

(v) *Case: all non-tree edges in Γ are T-independent.*

Choose any non-tree edge e and apply the sequence $(*)$ in (iii) for this edge. Since e is T-independent, $r_0(G_1) = 1$ and $M(G_1) = 0$. Also UG_1'/G_1' is finite because e is T-independent. By (iii) there is non-tree edge $f \neq e$ and

$\bar{t} = t_f \in G_2$. Since f is T-independent, $r_0(\langle \bar{t}, U \rangle) = 1$ and also UG_2'/G_2' is finite. Consequently the image of U_{ab} in $(G_1)_{ab} \oplus (G_1)_{ab}$ is finite. Since U_{ab} is infinite, it follows from the sequence $(*)$ that the cokernel of the map $M(G_1) \oplus M(G_2) \to M(G)$ is infinite.

By induction hypothesis the result holds for G_2, so we conclude that

$$r_0(M(G)) \geq r_0(M(G_1)) + r_0(M(G_2)) + 1 = (r_0(G_2) - 1) + 1 = r_0(G_2),$$

since $M(G_1) = 0$. Finally, the image of U_{ab} in the sequence $(*)$ being finite, we obtain

$$r_0(G) = r_0((G_1)_{ab} \oplus (G_2)_{ab}) = r_0(G_1) + r_0(G_2) = 1 + r_0(G_2).$$

Hence $r_0(G_2) = r_0(G) - 1$ and $r_0(M(G)) \geq r_0(G) - 1$, as required. □

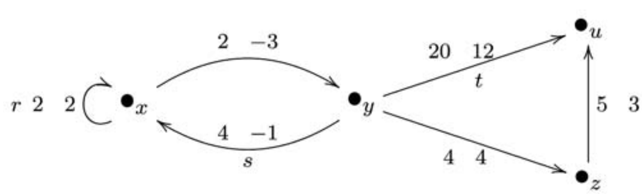
COROLLARY 2. *The GBS-group $\pi_1(\Gamma, \omega)$ has trivial Schur multiplier if and only if Γ is either a tree or else a tree with one further edge and Γ is not tree-dependent.*

PROOF. By the theorem $M(G) = 0$ if and only if $r_0(G) = 1$. This condition requires there to be at most one non-tree edge and by Theorem 2 it must be T-independent. □

COROLLARY 3. *Every GBS-group has deficiency* 1.

PROOF. Recall that the deficiency $\mathrm{def}(G)$ of group G is equal to $\sup\{n - r\}$ where n and r are the respective numbers of generators and relations in an arbitrary finite presentation. If G is a GBS-group, then $1 \leq \mathrm{def}(G) \leq r_0(G) - d(M(G)) = 1$. □

EXAMPLE. Consider the *GBS*-group G arising from the following *GBS*-graph,



where the maximal subtree chosen is the path x, y, z, u and the stable elements are r, s, t as indicated. Then G has a presentation with generators

$$r, s, t, g_x, g_y, g_z, g_u$$

and relations

$$(g_x^2)^r = g_x^2,\ g_x^2 = g_y^{-3},\ g_y^4 = g_z^4,\ g_z^5 = g_u^3,\ (g_u^{12})^t = g_y^{20},\ (g_x^4)^s = g_y^{-1}.$$

All non-tree edges with the exception of $\langle y, x\rangle$ are T-dependent. Therefore (Γ, ω) is not tree-dependent, $\varepsilon(\Gamma, \omega) = 0$ and $r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 = 3$. Thus $M(G) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

4. Applications to Central Extensions.

We will now apply our results to yield information about central extensions by GBS-groups. Let G be a *GBS*-group and C an abelian group regarded as a trivial G-module. Denote by F the periodic subgroup of G_{ab}; thus $G_{ab} \simeq F \oplus \mathbb{Z}^{r_0(G)}$ where F is finite. By the Universal Coefficients Theorem

$$H^2(G, C) \simeq \mathrm{Ext}\,(G_{ab}, C) \oplus \mathrm{Hom}\,(M(G), C) \simeq \mathrm{Ext}\,(F, C) \oplus \mathrm{Dr}\ C^{r_0(G)-1}.$$

First we determine when all central extensions of C by G are direct products, i.e., when $H^2(G, C) = 0$.

THEOREM 3. *Let G be a generalized Baumslag-Solitar group and let $C \neq 1$ be an abelian group regarded as a trivial G-module. Then $H^2(G, C) = 0$ if and only if $r_0(G) = 1$ and C is divisible by all primes $p \in \pi(G_{ab})$.*

PROOF. With the notation used above, $H^2(G, C) = 0$ if and only if $\mathrm{Ext}\,(F, C) = 0$ and $r_0(G) = 1$. Since F is finite and $\mathrm{Ext}\,(\mathbb{Z}_n, C) \simeq C/C^n$, it follows that $\mathrm{Ext}\,(F, C) = 0$ if and only if $C = C^p$ for all $p \in \pi(G_{ab})$. (For the elementary properties of Ext used here see [9], 7.2). □

COROLLARY 4. *The following conditions on a generalized Baumslag-Solitar group G are equivalent*

(i) $H^2(G, \mathbb{Z}) = 0$;
(ii) $G_{ab} \simeq \mathbb{Z}$;
(iii) $H^2(G, C) = 0$ *for all abelian groups C.*

PROOF. Clearly condition (i) implies that $\pi(G_{ab})$ is empty and so (ii) holds. Also (ii) implies (iii), while trivially (iii) implies (i). □

For example, if $G = BS(m, n)$, then $G_{ab} \simeq \mathbb{Z} \oplus \mathbb{Z}_{|m-n|}$, so that G has the property of Corollary 4 if and only if $|m - n| = 1$.

There are corresponding results for homology, which can be proved in an analogous way by using the Universal Coefficients Theorem for homology,

$$H_2(G, C) \simeq \mathrm{Tor}(G_{ab}, C) \oplus (M(G) \otimes C),$$

and elementary properties of Tor, (see [9], 7.1).

THEOREM 4. *Let G be a generalized Baumslag-Solitar group and let $C \neq 1$ an abelian group regarded as a trivial G-module. Then $H_2(G, C) = 0$ if and only if $r_0(G) = 1$ and $C_p = 1$ for all primes $p \in \pi(G_{ab})$.*

COROLLARY 5. *The following conditions on a generalized Baumslag-Solitar group G are equivalent:*

(i) $H_2(G, \mathbb{Q}/\mathbb{Z}) = 0$;
(ii) $G_{ab} \simeq \mathbb{Z}$;
(iii) $H_2(G, C) = 0$ *for all abelian groups C.*

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