GAGA for DQ-Algebroids

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ABSTRACT - Let $X$ be a smooth complex projective variety with associated compact complex manifold $X_{\text{an}}$. If $\mathcal{M}$ is a DQ-algebroid on $X$, then there is an induced DQ-algebroid $\mathcal{M}_{\text{an}}$ on $X_{\text{an}}$. We show that the natural functor from the derived category of bounded complexes of $\mathcal{M}$-modules with coherent cohomologies to the derived category of bounded complexes of $\mathcal{M}_{\text{an}}$-modules with coherent cohomologies is an equivalence.

Introduction.

Let $X$ be a projective scheme over the complex number field $\mathbb{C}$ with associated complex analytic space $X_{\text{an}}$. Serre’s GAGA paper [11] asserts that the category of coherent sheaves on $X$ is equivalent to the category of coherent analytic sheaves on $X_{\text{an}}$.

We consider the case where $X$ is a smooth algebraic variety over $\mathbb{C}$ or a complex manifold. In [8], the authors defined a DQ-algebra $\mathcal{M}$ on $X$ as a sheaf of $\mathbb{C}^h := \mathbb{C}[[h]]$-algebras locally isomorphic to $(\mathcal{O}_X[[h]], \ast)$ where $\ast$ is a star product. The authors also defined a DQ-algebroid as a $\mathbb{C}^h$-algebroid stack locally equivalent to the algebroid associated with a DQ-algebra. If $\mathcal{M}$ is a DQ-algebroid on $X$, then we have the notion of $\mathcal{M}$-modules. We denote by $\text{Mod}(\mathcal{M})$ the category of $\mathcal{M}$-modules and by $D^b(\mathcal{M})$ its bounded derived category. We will recall these notions and their properties from [8].

If $(X, \mathcal{M})$ is a smooth variety over $\mathbb{C}$ endowed with a DQ-algebroid, then there is an induced DQ-algebroid $\mathcal{M}_{\text{an}}$ on the complex manifold

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Mathematics Subject Classification: 18D05, 32C38, 46L65, 53D55.
$X_{\text{an}}$. Then we construct a functor $f^*: D^b_{\text{coh}}(\mathcal{X}) \to D^b_{\text{coh}}(\mathcal{X}_{\text{an}})$, where $D^b_{\text{coh}}(\mathcal{X})$ denotes the full triangulated subcategory of the bounded derived category $D^b(\mathcal{X})$ with coherent cohomologies and similarly $D^b_{\text{coh}}(\mathcal{X}_{\text{an}})$ denotes the full triangulated subcategory of the bounded derived category $D^b(\mathcal{X}_{\text{an}})$ with coherent cohomologies. By using Lemma 1.2, Corollary 1.4 and some results in [8], we prove the following theorem:

**Main Theorem** (See Theorem 4.12). Assume that $X$ is projective. Then the functor $f^*: D^b_{\text{coh}}(\mathcal{X}) \to D^b_{\text{coh}}(\mathcal{X}_{\text{an}})$ is an equivalence.

This paper is organized as follows: In section 1, we review Serre’s GAGA theorem and translate this theorem to the derived version. In section 2, we review some notions and results of DQ-modules from [8]. In particular, Remark 2.5 and Finiteness theorem (Theorem 2.13) are crucial for the paper. In section 3, we show how to induce an analytic DQ-algebroid from an algebraic DQ-algebroid on a smooth variety. In section 4, we prove the main theorem.

Throughout this paper, all varieties (or schemes) are over $\mathbb{C}$ if not otherwise specified.

1. Review on the GAGA Theorem.

Let $X$ be a scheme of finite type and let $X_{\text{an}}$ be the associated complex analytic space. We denote by Mod$(\mathcal{O}_X)$ (resp. Mod$(\mathcal{O}_{X_{\text{an}}})$) the category of sheaves on $X$ (resp. $X_{\text{an}}$). We also denote by Mod$_{\text{coh}}(\mathcal{O}_X)$ and Mod$_{\text{coh}}(\mathcal{O}_{X_{\text{an}}})$ the full subcategories of Mod$(\mathcal{O}_X)$ and Mod$(\mathcal{O}_{X_{\text{an}}})$ consisting of coherent sheaves, respectively. There is a continuous map $\varphi: X_{\text{an}} \to X$ of the underlying topological spaces and there is also a natural map of the structure sheaves $\varphi^{-1}\mathcal{O}_X \to \mathcal{O}_{X_{\text{an}}}$. To $\mathcal{F} \in$ Mod$(\mathcal{O}_X)$, one associates its complex analytic sheaf $\mathcal{F}^\text{an} := \mathcal{O}_{X_{\text{an}}} \otimes_{\varphi^{-1}\mathcal{O}_X} \varphi^{-1}\mathcal{F} \in$ Mod$(\mathcal{O}_{X_{\text{an}}})$. Hence we obtain a functor:

\[(*) \quad \gamma_X : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_{X_{\text{an}}}).\]

If $\mathcal{F}$ is a coherent sheaf, then $\mathcal{F}^\text{an}$ is also coherent.

The following theorem for a projective scheme is proved in Serre’s famous paper GAGA (see [11]) which is generalized by Grothendieck for a proper scheme (see [6 XII]).
**Theorem 1.1.** Let $X$ be a projective scheme. Then the functor $(\ast)$ induces an equivalence of categories

$$\text{Mod}_{\text{coh}}(\mathcal{O}_X) \xrightarrow{\sim} \text{Mod}_{\text{coh}}(\mathcal{O}_{X_{\text{an}}}).$$

Furthermore, for every coherent sheaf $\mathcal{F}$ on $X$, the natural maps

$$H^i(X; \mathcal{F}) \rightarrow H^i(X_{\text{an}}; \mathcal{F}_{\text{an}})$$

are isomorphisms, for all $i \geq 0$. \hfill \square

The following lemma is Theorem 2.2.8 in [1].

**Lemma 1.2.** Let $\mathcal{A}'$ and $\mathcal{B}'$ be thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor that takes $\mathcal{A}'$ to $\mathcal{B}'$. Assume furthermore that the following properties are satisfied:

1. $\mathcal{A}$ and $\mathcal{B}$ have enough injectives,
2. $\Phi$ is an equivalence of categories when restricted to $\mathcal{A}' \rightarrow \mathcal{B}'$,
3. $\Phi$ induces a natural isomorphism

$$\text{Ext}^i_{\mathcal{A}'}(F, G) \cong \text{Ext}^i_{\mathcal{B}'}(\Phi(F), \Phi(G))$$

for any $F, G \in \mathcal{A}'$ and any $i$.

Then the natural functor $\Phi : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$ induced by $\Phi$ is an equivalence of categories.

**Proof.** (i) We prove that the functor $\Phi$ is fully faithful, i.e., for any $F^\bullet, G^\bullet \in D^b(\mathcal{A})$, $\Phi$ induces an isomorphism

$$\text{Hom}_{D^b(\mathcal{A})}(F^\bullet, G^\bullet) \cong \text{Hom}_{D^b(\mathcal{B})}(\Phi(F^\bullet), \Phi(G^\bullet)).$$

We'll use a technique known as dévissage to prove it. The dévissage technique is the induction on the number $n(E^\bullet)$ defined as

$$n(E^\bullet) = \max \{ j - i | H^j(E^\bullet) \neq 0, H^i(E^\bullet) \neq 0 \}.$$

Hence we shall prove (1.1) by induction on $N = n(F^\bullet) + n(G^\bullet)$. If $N = -\infty$, then one of $F^\bullet$ or $G^\bullet$ is the zero complex, so there is nothing to prove. If $N = 0$, then there exist $F \in \mathcal{A}', G \in \mathcal{A}'$ such that $F^\bullet = F[a]$ and $G^\bullet = G[b]$ for some $a, b \in \mathbb{Z}$. Then

$$\text{Hom}_{D^b(\mathcal{A})}(F^\bullet, G^\bullet) = \text{Hom}_{D^b(\mathcal{A})}(F[a], G[b]) = \text{Ext}^b_{\mathcal{A}}(F, G)$$
and

\[ \text{Hom}_{D^b_G(B)}(\tilde{\Phi}(F^*), \tilde{\Phi}(G^*)) = \text{Hom}_{D^b_G(B)}(\tilde{\Phi}(F[a]), \tilde{\Phi}(G[b])) \]

\[ = \text{Ext}^{b-a}_G(\Phi(F), \Phi(G)). \]

Hence (1.1) follows from property 3 above.

Assume that \( \tilde{\Phi} \) induces an isomorphism

\[ \text{Hom}_{D^b_{\mathcal{A}}(A)}(F^*, G^*) \cong \text{Hom}_{D^b_G(B)}(\tilde{\Phi}(F^*), \tilde{\Phi}(G^*)) \]

for all \( F^*, G^* \in D^b_{\mathcal{A}}(A) \) with \( n(F^*) + n(G^*) < N \), and let \( F^*, G^* \) be objects of \( D^b_{\mathcal{A}}(A) \) with \( n(F^*) + n(G^*) = N > 0 \). We may assume that \( n(G^*) = N > 0 \) and that \( G^i = 0 \) for \( i < 0 \), and \( H^0(G^*) \neq 0 \).

Let \( G^* \) be the complex with single non zero object \( H^0(G^*) \) in degree zero. From the morphism \( G^* \rightarrow G^* \), there exists a distinguished triangle \( G'' \rightarrow G^* \rightarrow G^* \rightarrow G'''[1] \). By the long exact cohomology sequence, one deduces \( n(G''') < n(G^*) \); also, from the assumption, \( n(G^*) = 0 < n(G^*) \).

From the long exact sequence of Hom’s; the five-lemma and the induction hypothesis, we conclude that

\[ \text{Hom}_{D^b_{\mathcal{A}}(A)}(F^*, G^*) \cong \text{Hom}_{D^b_G(B)}(\tilde{\Phi}(F^*), \tilde{\Phi}(G^*)) \]

which is what we needed to prove that \( \tilde{\Phi} \) is fully faithful. (The case when \( n(G^*) = 0 \) but \( n(F^*) > 0 \) follows in a similar way.)

(ii) Next, we shall prove that the functor \( \tilde{\Phi} \) is essentially surjective, i.e., any object \( G^* \) of \( D^b_G(B) \) is isomorphic to an object of the form \( \tilde{\Phi}(F^*) \) for some \( F^* \in D^b_{\mathcal{A}}(A) \). We prove this by induction on \( n = n(G^*) \). The case \( n = -\infty \) is trivial, and \( n = 0 \) follows from property 2.

So assume \( n > 0 \), and as before construct a distinguished triangle \( G''' \rightarrow G^* \rightarrow G^* \rightarrow G''''[1] \) where \( G^* = H^0(G^*) \neq 0 \) and we assume that \( G^* \) is zero in degrees \( < 0 \). Since \( \Phi \) is an equivalence of categories between \( \mathcal{A}' \) and \( B' \), we can find an \( F'' \in D^b_{\mathcal{A}}(A) \) such that \( \Phi(F'') \cong G^* \). Also, by induction hypothesis, we can find an \( F''' \in D^b_{\mathcal{A}}(A) \) such that \( \tilde{\Phi}(F''') \cong G''' \).

Since we proved that \( \tilde{\Phi} \) is fully faithful, we can find a map \( F'' \rightarrow F^* \) whose image by \( \Phi \) is the side of the distinguished triangle constructed before. Again, we have the distinguished triangle \( F''' \rightarrow F'' \rightarrow F^* \rightarrow F''''[1] \).

Since \( \Phi \) is a derived functor, it is a triangulated functor. Hence, we see that \( \tilde{\Phi}(F^*) \) is isomorphic to \( G^* \), as required.

The following proposition is well known (see, e.g., [9]).
**Proposition 1.3.** Let $\mathcal{R}$ be a sheaf of rings on a topological space $X$. Then the category $\text{Mod}(\mathcal{R})$ of $\mathcal{R}$-modules is a Grothendieck category. In particular, $\text{Mod}(\mathcal{R})$ has enough injectives.

Let $X$ be a scheme of finite type. Set $\mathcal{A} := \text{Mod}(\mathcal{O}_X)$, $\mathcal{B} := \text{Mod}(\mathcal{O}_{X_{an}})$, $\mathcal{A}' := \text{Mod}_{\text{coh}}(\mathcal{O}_X)$ and $\mathcal{B}' := \text{Mod}_{\text{coh}}(\mathcal{O}_{X_{an}})$. We also set $\text{D}_{\text{coh}}^b(\mathcal{A})$ and $\text{D}_{\text{coh}}^b(\mathcal{X}_{an}) := \text{D}_{\mathcal{B}}^b(\mathcal{X})$. Clearly, $\mathcal{A}'$ and $\mathcal{B}'$ are full thick subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively.

As an application of Lemma 1.2, we have the following.

**Corollary 1.4.** Let $X$ be a projective scheme. Then the functor $\gamma_X$ of (*) induces an equivalence (we keep the same notation)

$$\gamma_X : \text{D}_{\text{coh}}^b(X) \xrightarrow{\sim} \text{D}_{\text{coh}}^b(X_{an}).$$

**Proof.** We shall apply Lemma 1.2. First, note that $\mathcal{O}_{X_{an}}$ is a flat $\mathcal{O}_X$-module for each $x \in X$. Hence the functor $\gamma_X : \mathcal{A} \to \mathcal{B}$ is an exact functor. Both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives by Proposition 1.3. Hence condition 1 is satisfied and condition 2 is Theorem 1.1. To check condition 3, it is sufficient to prove that

$$\text{RHom}(\mathcal{F}, \mathcal{G}) \cong \text{RHom}(\mathcal{F}_{an}, \mathcal{G}_{an}) \text{ for } \mathcal{F}, \mathcal{G} \in \mathcal{A}'.
(1.2)$$

Since $\text{RHom}(\mathcal{F}, \mathcal{G}) \cong \text{R} \Gamma(X, \mathcal{F}^* \otimes^L \mathcal{G})$ and $\text{RHom}(\mathcal{F}_{an}, \mathcal{G}_{an}) \cong \text{R} \Gamma(X_{an}, (\mathcal{F}_{an}^*) \otimes \mathcal{G}_{an})$, where $\mathcal{F}^* = \text{R} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ and $(\mathcal{F}_{an}^*) = \text{R} \text{Hom}_{\mathcal{O}_{X_{an}}}((\mathcal{F}_{an}), (\mathcal{O}_{X_{an}})$, we reduce (1.2) to the following isomorphism

$$\text{R} \Gamma(X, \mathcal{F}) \cong \text{R} \Gamma(X_{an}, \mathcal{F}_{an}), \text{ where } \mathcal{F} \in \text{D}_{\text{coh}}^b(X).
(1.3)$$

Now, there is a morphism $\text{R} \Gamma(X, \mathcal{F}) \to \text{R} \Gamma(X_{an}, \mathcal{F}_{an})$ for $\mathcal{F} \in \text{D}_{\text{coh}}^b(X)$ and it is an isomorphism by Theorem 1.1. Hence the result follows. \qed


In this section, we recall some notions and results from [8].

Algebroid.

In this subsection, we denote by $X$ a topological space and by $K$ a commutative unital ring. If $A$ is a ring, an $A$-module means a left $A$-module. Recall
that the notion algebroid was first introduced by Kontsevich [10], see also [2] and [7]. A $K$-algebroid $\mathcal{A}$ on $X$ is a $K$-linear stack locally non empty and such that for any open subset $U$ of $X$, two objects of $\mathcal{A}(U)$ are locally isomorphic.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. In the sequel, we set $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$, etc.

Consider the data of

$$
\begin{cases}
\text{a $K$-algebroid } \mathcal{A} \text{ on } X \\
\sigma_i \in \mathcal{A}(U) \text{ and isomorphisms } \varphi_{ij} : \sigma_j|_{U_{ij}} \to \sigma_i|_{U_{ij}}.
\end{cases}
$$

(2.1)

To these data, we associate:

- $\mathcal{A}_i = \mathcal{E}нд_К(\sigma_i)$,
- $f_{ij} : \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ the $K$-algebra isomorphism $a \mapsto \varphi_{ij} \circ a \circ \varphi_{ij}^{-1}$,
- $a_{ijk}$, the invertible element of $\mathcal{A}_i(U_{ijk})$ given by $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ik}^{-1}$.

Then:

$$
\begin{cases}
f_{ij} \circ f_{jk} = \text{Ad}(a_{ijk}) \circ f_{ik} \\
a_{ijk}a_{jkl} = f_{ij}(a_{jkl})a_{i,jk}.
\end{cases}
$$

(2.2)

(Recall that $\text{Ad}(a)(b) = aba^{-1}$).

Conversely, let $\mathcal{A}_i$ be sheaves of $K$-algebras on $U_i$ ($i \in I$), let $f_{ij} : \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ ($i, j \in I$) be $K$-algebra isomorphisms, and let $a_{ijk}$ ($i, j, k \in I$) be invertible sections of $\mathcal{A}_i(U_{ijk})$ satisfying (2.2). One calls:

$$
(\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i, j \in I}, \{a_{ijk}\}_{i, j, k \in I})
$$

(2.3)
a gluing datum for $K$-algebroids on $\mathcal{U}$.

**Theorem 2.1 ([5]).** Assume that the topological space $X$ is paracompact. Considering a gluing datum (2.3) on $\mathcal{U}$. Then there exist an algebroid $\mathcal{A}$ on $X$ and $\{\sigma_i, \varphi_{ij}\}_{i, j \in I}$ as in (2.1) to which this gluing datum is associated. Moreover, the data $(\mathcal{A}, \sigma_i, \varphi_{ij})$ are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.

In general, if a topological space $X$ is not paracompact, for example for algebraic varieties, then Theorem 2.1 may be false. Hence we need another local description of such algebraic algebroids.

**Definition 2.2.** Let $\mathcal{A}$ and $\mathcal{A}'$ be two sheaves of $K$-algebras. An $\mathcal{A} \otimes \mathcal{A}'$-module $\mathcal{L}$ is called bi-invertible if there exists locally a section $\omega$ of $\mathcal{L}$ such that $\mathcal{A} \ni a \mapsto (a \otimes 1)\omega \in \mathcal{L}$ and $\mathcal{A}' \ni a' \mapsto (a' \otimes 1)\omega \in \mathcal{L}$ give isomorphism of $\mathcal{A}$-modules and $\mathcal{A}'$-modules, respectively.
Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. Consider the data of
\begin{equation}
\begin{aligned}
\mathcal{A} & \overset{\mathcal{X}}{\rightarrow} \text{End}_K(\sigma_i), \\
\mathcal{L}_{ij} & \overset{\mathcal{X}}{\rightarrow} \text{Hom}_{\mathcal{A}_i|_{U_{ij}}, \mathcal{A}_j|_{U_{ij}}}.
\end{aligned}
\end{equation}

Hence we obtain:
\begin{equation}
\{(\mathcal{A}_i)_{i \in I}, \{\mathcal{L}_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I}\}
\end{equation}

an algebraic gluing datum for $K$-algebroids on $\mathcal{U}$.

**Theorem 2.3** ([8] Proposition 2.1.13). Consider an algebraic gluing datum (2.5) on $\mathcal{U}$. Then there exist an algebroid $\mathcal{A}$ on $X$ and $\{\sigma_i, \varphi_{ij}\}_{i,j \in I}$ as in (2.1) to which this gluing datum is associated. Moreover, the data $(\mathcal{A}, \sigma_i, \varphi_{ij})$ are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.

For an algebroid $\mathcal{A}$, one defines the Grothendieck $K$-linear abelian category $\text{Mod}(\mathcal{A})$, whose objects are called $\mathcal{A}$-modules, by setting:
\[ \text{Mod}(\mathcal{A}) := \text{Fct}_K(\mathcal{A}, \text{Mod}(K_X)). \]

Here $\text{Mod}(K_X)$ is the $K$-linear stack of sheaves of $K$-modules on $X$, and $\text{Fct}_K$ is the category of $K$-linear functors of stacks.

We have the well defined notion of tensor product for two $K$-algebroids $\mathcal{C}$ and $\mathcal{C}'$, say $\mathcal{C} \otimes_K \mathcal{C}'$. For a $K$-algebroid $\mathcal{A}$, $\text{Mod}(\mathcal{A} \otimes_K \mathcal{A}^{\text{op}})$ has a canonical object given by
\[ \mathcal{A} \otimes_K \mathcal{A}^{\text{op}} \ni (\sigma, \sigma') \mapsto \text{Hom}_{\mathcal{A}}(\sigma', \sigma) \in \text{Mod}(K_X). \]

We denote this object by the same letter $\mathcal{A}$.

For $K$-algebroids $\mathcal{A}_i$ ($i = 1, 2, 3$), we have functors:
\[ \otimes_{\mathcal{A}_2} : \text{Mod}(\mathcal{A}_1 \otimes_K \mathcal{A}_2^{\text{op}}) \times \text{Mod}(\mathcal{A}_2 \otimes_K \mathcal{A}_3^{\text{op}}) \rightarrow \text{Mod}(\mathcal{A}_1 \otimes_K \mathcal{A}_3^{\text{op}}) \]
and
\[ \text{Hom}_{\mathcal{A}_1}(\cdot, \cdot) : \text{Mod}(\mathcal{A}_1 \otimes_K \mathcal{A}_2^{\text{op}})^{\text{op}} \times \text{Mod}(\mathcal{A}_1 \otimes_K \mathcal{A}_3^{\text{op}}) \rightarrow \text{Mod}(\mathcal{A}_2 \otimes_K \mathcal{A}_3^{\text{op}}). \]

In particular, we have
\[ \otimes_{\mathcal{A}} : \text{Mod}(\mathcal{A}^{\text{op}}) \times \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(K_X) \]
and
\[ \mathcal{H}om_{\mathcal{A}}(\cdot, \cdot) : \text{Mod}(\mathcal{A})^{\text{op}} \times \text{Mod}(\mathcal{A}) \to \text{Mod}(\mathcal{K}_X). \]

Let \( Y \) be another topological space. Let \( f : X \to Y \) be a continuous map and let \( \mathcal{A} \) be a \( \mathbb{K} \)-algebroid on \( Y \). We denote by \( f^{-1} \mathcal{A} \) the \( \mathbb{K} \)-linear stack associated with the prestack \( \mathcal{E} \) given by
\[
\mathcal{E}(U) = \{ (\sigma, V) ; \text{ } V \text{ is an open subset of } Y \text{ such that } f(U) \subset V \text{ and } \\
\sigma \in \mathcal{A}(V) \} \text{ for any open subset } U \text{ of } X,
\]
\[
\text{Hom}_{\mathcal{E}(U)}((\sigma, V), (\sigma', V')) = \Gamma(U, f^{-1} \mathcal{H}om_{\mathcal{A}}(\sigma, \sigma')).
\]

Then \( f^{-1} \mathcal{A} \) is a \( \mathbb{K} \)-algebroid on \( X \).

**Notations:** For the rest of this section, we denote by \( X \) a complex manifold or a smooth variety and by \( \mathcal{C}^h := \mathbb{C}[[h]] \) the power series algebra.

**Invertible \( \mathcal{O}_X \)-algebroids.**

**Definition 2.4.** A \( \mathcal{C} \)-algebroid \( \mathcal{P} \) on \( X \) is called an invertible \( \mathcal{O}_X \)-algebroid if for any open subset \( U \) of \( X \) and any \( \sigma \in \mathcal{P}(U) \), there is a \( \mathcal{C} \)-algebra isomorphism \( \text{End}_\mathcal{P}(\sigma) \simeq \mathcal{O}_U \).

We shall state some properties for invertible \( \mathcal{O}_X \)-algebroids.

Let \( \mathcal{P} \) be an invertible \( \mathcal{O}_X \)-algebroid. Then for any \( \sigma, \sigma' \in \mathcal{P}(U) \), \( \mathcal{H}om(\sigma, \sigma') \) is an invertible \( \mathcal{O}_U \)-module.

For two invertible \( \mathcal{O}_X \)-algebroids \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \). We denote by \( \mathcal{P}_1 \otimes_{\mathcal{O}_X} \mathcal{P}_2 \) the \( \mathcal{C} \)-linear stack associated with the prestack whose objects over an open set \( U \) is \( \mathcal{P}_1(U) \times \mathcal{P}_2(U) \), and \( \mathcal{H}om(\mathcal{P}_1(\sigma_1, \sigma_2), (\sigma'_1, \sigma'_2)) = \mathcal{H}om(\sigma_1, \sigma'_1) \otimes_{\mathcal{O}_X} \mathcal{H}om(\sigma_2, \sigma'_2) \). Then \( \mathcal{P}_1 \otimes_{\mathcal{O}_X} \mathcal{P}_2 \) is an invertible \( \mathcal{O}_X \)-algebroid. Note that the set of equivalence classes of invertible \( \mathcal{O}_X \)-algebroids has structure of an additive group by the operation \( \cdot \otimes_{\mathcal{O}_X} \cdot \), and this group is isomorphic to \( H^2(X, \mathcal{O}_X^\times) \) ([4], [9]).

The following remark is due to Prof. Joseph Oesterle and is crucial for the paper.

**Remark 2.5.** For a smooth algebraic variety \( X \) as Zariski topology over \( \mathbb{C} \), the group \( H^2(X, \mathcal{O}_X^\times) \) is trivial. Hence any invertible \( \mathcal{O}_X \)-algebroid \( \mathcal{P} \) is equivalent to \( \mathcal{O}_X \).

We sketch the proof of it. Let \( K \) be the field of rational functions on \( X \)
and let $K^\times_X$ be the constant sheaf with stalk the abelian group $K^\times$. Denote by $X_1 = \{ x \in X | \dim \mathcal{O}_{X,x} = 1 \}$ (or the set of closed irreducible hypersurfaces of $X$). Let $x \in X_1$, since $X$ is a variety, the ring $\mathcal{O}_{X,x}$ is a DVR with valuation $v_x$ and quotient field $K$. Let $Z_x = (i_x)_*(\mathbb{Z})$ where $i_x : x \to X$ and let $U \subset X$ be an open set, then $Z_x(U) = 0$ if $x \notin U$ and $Z_x(U) = Z$ if $x \in U$. Consider the sheaf $\bigoplus_{x \in X_1} Z_x$, then $\left( \bigoplus_{x \in X_1} Z_x \right)(U) = \bigoplus_{x \in X_1} Z_x(U) = Z_{U \cap X_1}$. Hence we can define a morphism of sheaves

$$v : K^\times_X \to \bigoplus_{x \in X_1} Z_x$$

by: $v(f) = (v_x(f))_{x \in X_1 \cap U}$ where $U$ is a nonempty open subset of $X$ and $f \in K^\times_X(U) = K^\times$. Then one has an exact sequence

$$0 \to \mathcal{O}_X^\times \xrightarrow{u} K^\times_X \xrightarrow{v} \bigoplus_{x \in X_1} Z_x \to 0$$

where $u$ is the natural morphism. Since $K^\times_X$ is constant, it is a flabby sheaf for the Zariski topology. On the other hand, the sheaf $\bigoplus_{x \in X_1} Z_x$ is also flabby. It follows that $H^j(X; \mathcal{O}_X^\times)$ is zero for $j > 1$.

Let $f : X \to Y$ be a morphism of complex manifolds or smooth varieties. For an invertible $\mathcal{O}_Y$-algebroid $\mathcal{P}_Y$, we denote by $f^* \mathcal{P}_Y$ the $\mathbb{C}$-linear stack on $X$ associated with the prestack whose objects on $U$ are the objects of $(f^{-1} \mathcal{P}_Y)(U)$ and $\mathcal{H}om(\sigma, \sigma') = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{H}om_{f^{-1}(\mathcal{P}_Y)}(\sigma, \sigma')$. Then $f^* \mathcal{P}_Y$ is an invertible $\mathcal{O}_X$-algebroid.

**Star-products.**

Let $X$ be a complex manifold (or a smooth variety). We denote by $\delta_X : X \hookrightarrow X \times X$ the diagonal embedding and we set $\Delta_X = \delta_X(X)$. We denote by $\mathcal{O}_X$ the structure sheaf on $X$, by $\Omega_X$ the sheaf of differential forms of maximal degree and by $\Theta_X$ the sheaf of vector fields. As usual, we denote by $\mathcal{D}_X$ the sheaf of rings of differential operators on $X$. Recall that a bi-differential operator $P$ on $X$ is a $\mathbb{C}$-bilinear morphism $\mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ which is obtained as the composition $\delta_X^{-1} \circ \bar{P}$ where $\bar{P}$ is a differential operator on $X \times X$ defined on a neighborhood of the diagonal and $\delta^{-1}$ is the restriction to the diagonal:

$$P(f, g)(x) = (\bar{P}(x_1, x_2; \partial_{x_1}, \partial_{x_2})(f(x_1)g(x_2)))_{|x_1=x_2=x}.$$  

Hence the sheaf of bi-differential operators is isomorphic to $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. 

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where the both $\mathcal{D}_X$ are regarded as $\mathcal{O}_X$-modules by the left multiplications.

**Definition 2.6.** A **star algebra** on $\mathcal{O}_X[[\hbar]]$ is a $\mathcal{C}_h$-bilinear sheaf morphism

$$\star : \mathcal{O}_X[[\hbar]] \times \mathcal{O}_X[[\hbar]] \to \mathcal{O}_X[[\hbar]]$$

satisfying the following conditions:

(i) the star product makes $\mathcal{O}_X[[\hbar]]$ into a sheaf of associated unital $\mathcal{C}_h$-algebra with unit $1 \in \mathcal{O}_X$.

(ii) there is a sequence $P_i : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ of bi-differential operators, such that for any two local sections $f, g \in \mathcal{O}_X$ one has

$$f \star g = fg + \sum_{i=1}^{\infty} P_i(f, g)\hbar^i.$$ 

Note that $f \star g \equiv fg \mod \hbar$, and $P_i(f, 1) = P_i(1, f) = 0$ for all $f$ and $i > 0$. We call $(\mathcal{O}_X[[\hbar]], \star)$ a star algebra.

**DQ-algebras.**

**Definition 2.7.** A DQ-algebra $\mathcal{A}$ on $X$ is a $\mathcal{C}_h$-algebra locally isomorphic to a star-algebra $(\mathcal{O}_X[[\hbar]], \star)$ as a $\mathcal{C}_h$-algebra.

Clearly, a DQ-algebra is a sheaf of $\hbar$-adically complete flat $\mathcal{C}_h$-algebra on $X$ satisfying $\mathcal{A}/\hbar, \mathcal{A} \simeq \mathcal{O}_X$. Note also that for an algebraic variety $X$, a DQ-algebra $\mathcal{A}$ is called deformation quantization of $\mathcal{O}_X$ in [3] and [12].

**Remark 2.8.** For a smooth projective variety $X$, there exists a DQ-algebra $\mathcal{A}_X$ on $X$. For details, one refers to [3].

**DQ-algebroids.**

**Definition 2.9.** A DQ-algebroid $\mathcal{A}$ on $X$ is a $\mathcal{C}_h$-algebroid such that for each open set $U \subset X$ and each $\sigma \in \mathcal{A}(U)$, the $\mathcal{C}_h$-algebra $\mathcal{Hom}_\mathcal{A}(\sigma, \sigma)$ is a DQ-algebra on $U$.

Let $\mathcal{A}_X$ be a DQ-algebroid on $X$. For an $\mathcal{A}_X$-module $\mathcal{M}$, the local notions of being coherent or locally free, etc. make sense.
The category $\text{Mod}(\mathcal{A}_X)$ is a Grothendieck category and we denote by $\text{D}(\mathcal{A}_X)$ its derived category and by $\text{D}^b(\mathcal{A}_X)$ its bounded derived category. We also denote by $\text{D}^b_{\text{coh}}(\mathcal{A}_X)$ the full triangulated subcategory of $\text{D}^b(\mathcal{A}_X)$ consisting of objects with coherent cohomologies.

**Graded modules.**

Let $\mathcal{A}_X$ be a DQ-algebroid on $X$. Let us denote by $\text{gr}(\mathcal{A}_X)$ the C-algebroid associated with the prestack $\mathcal{\Xi}$ given by

$$\text{Ob}(\mathcal{\Xi}(U)) = \text{Ob}(\mathcal{A}_X(U)) \text{ for an open subset } U \text{ of } X,$$

$$\text{Hom}_{\mathcal{\Xi}}(\sigma, \sigma') = \text{Hom}_{\mathcal{A}_X}(\sigma, \sigma')/\text{hHom}_{\mathcal{A}_X}(\sigma, \sigma') \text{ for } \sigma, \sigma' \in \mathcal{A}_X(U).$$

Then it is easy to see that $\text{gr}(\mathcal{A}_X)$ is an invertible $\mathcal{O}_X$-algebroid and the left derived functor of the right exact functor $\text{Mod}(\mathcal{A}_X) \to \text{Mod}(\text{gr}(\mathcal{A}_X))$ given by $\mathcal{M} \to \mathcal{M}/\text{h} \mathcal{M}$ is denoted by $\text{gr} : \text{D}^b(\mathcal{A}_X) \to \text{D}^b(\text{gr}(\mathcal{A}_X))$.

The functor $\text{gr}$ induces a functor (we keep the same notation):

(2.6) \[ \text{gr} : \text{D}^b_{\text{coh}}(\mathcal{A}_X) \to \text{D}^b_{\text{coh}}(\text{gr}(\mathcal{A}_X)). \]

The following lemma is in [8].

**Lemma 2.10.** The functor $\text{gr}$ of (2.6) is conservative (i.e., a morphism in $\text{D}^b_{\text{coh}}(\mathcal{A}_X)$ is an isomorphism as soon as its image by $\text{gr}$ is an isomorphism in $\text{D}^b_{\text{coh}}(\text{gr}(\mathcal{A}_X))$).

Denote by $\text{D}^b_f(C^h) := \text{D}^b_{\text{coh}}(C^h)$ and $\text{D}^b_f(C) := \text{D}^b_{\text{coh}}(C)$ the full triangulated subcategories of $\text{D}^b(C^h)$ and $\text{D}^b(C)$ consisting of objects with finitely generated cohomologies, respectively.

Hence we have a well defined functor $C \otimes_{C^h}^L : \text{D}^b_f(C^h) \to \text{D}^b_f(C)$. As an application of Lemma 2.10, we get the following.

**Corollary 2.11.** The functor $C \otimes_{C^h}^L : \text{D}^b_f(C^h) \to \text{D}^b_f(C)$ is conservative.

**Proof.** Applying the functor $\text{gr}$ in Lemma 2.10 to $X = \{\text{pt}\}$. \[ \square \]

The following proposition is in [8] which will be used in Theorem 4.2.

**Proposition 2.12.** Let $(X_i, \mathcal{A}_{X_i})$ be complex manifolds or smooth varieties endowed with DQ-algebroids $\mathcal{A}_{X_i}$ ($i = 1, 2, 3$).
(i) Let $\mathcal{K}_i \in D^b(\mathcal{A}_{X_i \times X_{i+1}})$ ($i = 1, 2$). Then
\[
\text{gr} (\mathcal{K}_1 \otimes_{\mathcal{A}_2} \mathcal{K}_2) \simeq \text{gr} (\mathcal{K}_1) \otimes_{\text{gr}(\mathcal{A}_2)} \text{gr} (\mathcal{K}_2).
\]

(ii) Let $\mathcal{K}_i \in D^b(\mathcal{A}_{X_i \times X_{i+1}})$ ($i = 1, 2$). Then
\[
\text{gr} R\text{Hom}_{\mathcal{A}_2}(\mathcal{K}_1, \mathcal{K}_2) \simeq R\text{Hom}_{\text{gr}(\mathcal{A}_2)}(\text{gr}(\mathcal{K}_1), \text{gr}(\mathcal{K}_2)).
\]

**Finiteness for DQ-kernels.**

Recall that we have the following Finiteness theorem.

**Finiteness Theorem 2.13 ([8]).** Let $(X, \mathcal{A}_X)$ be a compact complex manifold or a smooth projective variety endowed with a DQ-algebroid $\mathcal{A}_X$. Let $\mathcal{M}$ and $\mathcal{N}$ be two objects of $D^b_{\text{coh}}(\mathcal{A}_X)$. Then the object $R\text{Hom}_{\mathcal{A}_X}(\mathcal{M}, \mathcal{N})$ belongs to $D^b_I(\mathbb{C})$.

3. **Analytization of a DQ-algebroid.**

In this section, we denote by $X$ a smooth algebraic variety. Let $X_{\text{an}}$ be the corresponding complex analytic manifold of $X$ with continuous map $f : X_{\text{an}} \to X$.

Let $\mathcal{A}_X$ be a DQ-algebroid on $X$ and let $\mathcal{U} = \{U_i\}_{i=1, \ldots, n}$ be a finite affine open covering of $X$. Consider the data:

\[
\begin{aligned}
\{ & \text{a } \mathbb{C}\text{-algebroid } \mathcal{A}_X \text{ on } X \\
& \sigma_i \in \mathcal{A}_X(U_i).
\end{aligned}
\]

Then by Theorem 2.3, we have the following gluing data:

- $\mathcal{A}_i := \text{End}_{\mathcal{A}_X}(\sigma_i) = (\mathcal{O}_{U_i}[[h]], \ast_i)$,
- $f_{ij} : \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ the $\mathbb{C}^h$-algebra isomorphism,
- $a_{ijk}$: invertible elements of $\mathcal{A}_i(U_{ijk})$

which satisfies:

\[
\begin{aligned}
f_{ij} \circ f_{jk} &= \text{Ad}(a_{ijk}) \circ f_{ik} \\
a_{ijk}a_{ikl} &= f_{ij}(a_{jkl})a_{ijl}.
\end{aligned}
\]

Since $\mathcal{A}_i = (\mathcal{O}_{U_i}[[h]], \ast_i)$ is a star algebra for each $i = 1, \ldots, n$, by definition,
we have
\[ f_i \ast_i g_i = f_i g_i + \sum_{j=1}^{\infty} \beta_j(f_i, g_i) h^j \]
for \( f_i, g_i \in C_i := \Gamma(U_i, \mathcal{O}_X) \) and \( \beta_j : \mathcal{O}_{U_i} \times \mathcal{O}_{U_i} \to \mathcal{O}_{U_i} \) is a bi-differential operators for each \( j \).

From the inclusion \( C_i \hookrightarrow C_i^{an} := \Gamma(U_i, \mathcal{O}_{X_{an}}) \), we can define a star product \( \ast_i^{an} \) on the analytic sheaf \( \mathcal{O}_{U_i} \) to be
\[ f_i^{an} \ast_i^{an} g_i^{an} = f_i^{an} g_i^{an} + \sum_{j=1}^{\infty} \beta_j^{an}(f_i^{an}, g_i^{an}) h^j \]
for \( f_i^{an}, g_i^{an} \in C_i^{an} \),

where \( \beta_j^{an} : \mathcal{O}_{U_i} \times \mathcal{O}_{U_i} \to \mathcal{O}_{U_i} \) is a bi-differential operators on the analytic sheaf \( \mathcal{O}_{U_i} \) for each \( j \). Hence, we obtain the (analytic) star algebra \( \mathcal{A}_i^{an} = (\mathcal{O}_{U_i}[[h]], \ast_i^{an}) \) for each \( i \).

Therefore, we get the corresponding descent data on \( X_{an} \):

- \( \mathcal{A}_i^{an} = (\mathcal{O}_{U_i}[[h]], \ast_i^{an}) \),
- \( f_{ij}^{an} : \mathcal{A}_j^{an} |_{U_{ij}} \to \mathcal{A}_i^{an} |_{U_{ij}} \) the \( \mathbb{C}^h \)-algebra isomorphism,
- \( a_{ijk}^{an} \), the invertible element of \( \mathcal{A}_i^{an}(U_{ijk}) \)

and we obtain the DQ-algebroid \( \mathcal{A}_{X_{an}} \) on \( X_{an} \) by Theorem 2.1 (note that \( X_{an} \) is paracompact).

Hence for a DQ-algebroid \( \mathcal{A}_X \) on \( X \), we have the induced analytic DQ-algebroid \( \mathcal{A}_{X_{an}} \) on \( X_{an} \).

Furthermore,
\[ \mathcal{A}_{X_{an}} \in \text{Mod}(f^{-1} \mathcal{A}_X \otimes_{\mathbb{C}^h} \mathcal{A}_{X_{an}}^{op}) \]

Hence for a DQ-algebroid \( \mathcal{A}_X \) on a smooth variety \( X \), we have the functor \( f^* := \mathcal{A}_{X_{an}} \otimes_{f^{-1} \mathcal{A}_X} f^{-1}(\cdot) : \text{Mod}(\mathcal{A}_X) \to \text{Mod}(\mathcal{A}_{X_{an}}) \) which sends \( \mathcal{M} \) to \( \mathcal{A}_{X_{an}} \otimes_{f^{-1} \mathcal{A}_X} f^{-1}(\mathcal{M}) \). Denote by \( \text{Mod}_{coh}(\mathcal{A}_X) \) and \( \text{Mod}_{coh}(\mathcal{A}_{X_{an}}) \) the categories consisting of coherent \( \mathcal{A}_X \)-modules and \( \mathcal{A}_{X_{an}} \)-modules, respectively. If \( \mathcal{M} \in \text{Mod}_{coh}(\mathcal{A}_X) \), then \( f^*(\mathcal{M}) \in \text{Mod}_{coh}(\mathcal{A}_{X_{an}}) \).

4. The main theorem.

In this section, we prove the main theorem of this paper. Let \( \mathcal{A}_X \) be a DQ-algebroid on a smooth algebraic variety \( X \).
Flatness.

Let $X_{an}$ be the corresponding complex analytic manifold of $X$ with continuous map $f : X_{an} \to X$. First, we need the following lemma. The following lemma over one point as a corollary of Theorem 1.6.5 of [8].

**Lemma 4.1.** The functor $f^* : \text{Mod}(\mathcal{A}_X) \to \text{Mod}(\mathcal{A}_{X_{an}})$ constructed above is exact.

**Proof.** We may assume that $\mathcal{A}_X$ and $\mathcal{A}_{X_{an}}$ are DQ-algebras. We need to show that $B := \mathcal{A}_{X_{an},x}$ is flat over $R := \mathcal{A}_{X,x}$ for each $x \in X$. Note that:

(a) $B$ has no $h$-torsion,
(b) $B_0 := B/hB = \mathcal{O}_{X_{an},x}$ is a flat $R_0 := R/hR = \mathcal{O}_{X,x}$-module,
(c) $B \cong \lim_{\leftarrow n} B/h^n B$.

Hence applying Theorem 1.6.5 of [8] to $X = \{\text{pt}\}$, one gets the result. □

From Lemma 4.1, one can see that the functor $f^* : \text{Mod}(\mathcal{A}_X) \to \text{Mod}(\mathcal{A}_{X_{an}})$ induces a functor (we keep the same notation):

$$f^* : D^b_{\text{coh}}(\mathcal{A}_X) \to D^b_{\text{coh}}(\mathcal{A}_{X_{an}}).$$

**Fully faithfulness.**

Now we can prove the following theorem.

**Theorem 4.2.** Let $X$ be a smooth projective variety, then the functor $f^* : D^b_{\text{coh}}(\mathcal{A}_X) \to D^b_{\text{coh}}(\mathcal{A}_{X_{an}})$ is fully faithful.

**Proof.** For any $\mathcal{M}, \mathcal{N} \in D^b_{\text{coh}}(\mathcal{A}_X)$, we need to show that the morphism

$$\text{Hom}_{D^b_{\text{coh}}(\mathcal{A}_X)}(\mathcal{M}, \mathcal{N}) \to \text{Hom}_{D^b_{\text{coh}}(\mathcal{A}_{X_{an}})}(f^* \mathcal{M}, f^* \mathcal{N})$$

is a bijection. In order to show that the morphism of (4.1) is a bijection, it is sufficient to show that the morphism

$$\text{RHom}_{D^b_{\text{coh}}(\mathcal{A}_X)}(\mathcal{M}, \mathcal{N}) \to \text{RHom}_{D^b_{\text{coh}}(\mathcal{A}_{X_{an}})}(f^* \mathcal{M}, f^* \mathcal{N})$$

is an isomorphism. Since $X$ is projective, by Theorem 2.13, the com-
plexes $\text{RHom}_{D^b_{\text{coh}}(\mathcal{M}_X)}(\mathcal{M}, \mathcal{N})$ and $\text{RHom}_{D^b_{\text{coh}}(\mathcal{O}_{X_{\text{an}}})}(f^*(\mathcal{M}), f^*(\mathcal{N}))$ belong to $D^b_f(C)$.

Moreover, since $X$ is a smooth variety and $\text{gr}(\mathcal{A}_X)$ is an invertible $\mathcal{O}_X$-algebroid, $\text{gr}(\mathcal{A}_X)$ is equivalent to $\mathcal{O}_X$ by Remark 2.5 and hence $\text{gr}(\mathcal{A}_{X_{\text{an}}})$ is equivalent to $\mathcal{O}_{X_{\text{an}}}$. Thus, the equivalence $D^b_{\text{coh}}(\mathcal{O}_X) \cong D^b_{\text{coh}}(\mathcal{O}_{X_{\text{an}}})$ (Corollary 1.4) implies that the following morphism

$$\text{(4.3) } \text{RHom}_{D_{\text{coh}}(\mathcal{O}_X)}(\text{gr}(\mathcal{M}), \text{gr}(\mathcal{N})) \to \text{RHom}_{D_{\text{coh}}(\mathcal{O}_{X_{\text{an}}})}(f^*(\mathcal{M}), f^*(\mathcal{N}))$$

is an isomorphism in $D^b_f(C)$. Applying the functor $C \otimes_{\text{coh}} \cdot$ to (4.2) and using Proposition 2.12, we get (4.3). Since the functor $C \otimes_{\text{coh}} \cdot$ is conservative by Corollary 2.11, the morphism of (4.2) is an isomorphism and the result follows.

**COROLLARY 4.3.** Let $X$ be a smooth projective variety, then the natural functor $f^* : \text{Mod}_{\text{coh}}(\mathcal{A}_X) \to \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}})$ is exact and fully faithful.

For each $n > 0$, we denote by $\text{Mod}(\mathcal{A}_X/h^n\mathcal{A}_X)$ (resp. $\text{Mod}(\mathcal{A}_{X_{\text{an}}}/h^n\mathcal{A}_{X_{\text{an}}})$) the full subcategory of $\text{Mod}(\mathcal{A}_X)$ (resp. $\text{Mod}(\mathcal{A}_{X_{\text{an}}})$) consisting of objects $\mathcal{M}$ such that $h^n : \mathcal{M} \to \mathcal{M}$ is the zero morphism.

Similarly, we denote by $\text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X)$ (resp. $\text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}}/h^n\mathcal{A}_{X_{\text{an}}})$) the full subcategory of $\text{Mod}_{\text{coh}}(\mathcal{A}_X)$ (resp. $\text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}})$) consisting of objects $\mathcal{M}$ such that $h^n : \mathcal{M} \to \mathcal{M}$ is the zero morphism for each $n > 0$. Therefore, we have a functor $f^*_n = f^*|_{\text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X)} : \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X) \to \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}}/h^n\mathcal{A}_{X_{\text{an}}})$ for each $n > 0$.

Note that for $n = 1$, the category $\text{Mod}_{\text{coh}}(\mathcal{A}_X/h^1\mathcal{A}_X) \simeq \text{Mod}_{\text{coh}}(\mathcal{O}_X)$ is equivalent to the category $\text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}}/h^1\mathcal{A}_{X_{\text{an}}}) \simeq \text{Mod}_{\text{coh}}(\mathcal{O}_{X_{\text{an}}})$ by Theorem 1.1.

**COROLLARY 4.4.** Let $X$ be a smooth projective variety, then the functor $f^*_n : \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X) \to \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}}/h^n\mathcal{A}_{X_{\text{an}}})$ is exact and fully faithful for each $n > 0$.

**Essential surjectivity.**

Denote by $X$ a smooth projective variety. Next, we shall prove that the functor $f^* : D^b_{\text{coh}}(\mathcal{A}_X) \to D^b_{\text{coh}}(\mathcal{A}_{X_{\text{an}}})$ is essentially surjective.

We first prove that the functor $f^*_n : \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X) \to \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{\text{an}}}/h^n\mathcal{A}_{X_{\text{an}}})$ is essentially surjective.
\[ \rightarrow \text{Mod}_{\text{coh}}(\mathcal{A}_{\text{cm}}/h^n\mathcal{A}_{\text{cm}}) \] is essentially surjective for each \( n > 0 \). We need the following lemma.

**Lemma 4.5.** Let \( \mathcal{A}' \) and \( \mathcal{B}' \) be thick subcategories of abelian categories \( \mathcal{A} \) and \( \mathcal{B} \), respectively. Let \( \Phi : \mathcal{A} \rightarrow \mathcal{B} \) be an exact functor which takes \( \mathcal{A}' \) to \( \mathcal{B}' \) and such that the natural functor (we keep the same notation) \( \Phi : D^b_{\mathcal{A}}(\mathcal{A}) \rightarrow D^b_{\mathcal{B}}(\mathcal{B}) \) induced by \( \Phi \) is fully faithful. Consider an exact sequence in \( \mathcal{B} \)

\[
(*) \quad 0 \rightarrow \Phi(M') \rightarrow N \rightarrow \Phi(M'') \rightarrow 0,
\]

with \( M', M'' \in \mathcal{A}' \) and \( N \in \mathcal{B}' \).

Then there exists a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \Phi(M') & \rightarrow & \Phi(M) & \rightarrow & \Phi(M'') & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Phi(M') & \rightarrow & N & \rightarrow & \Phi(M'') & \rightarrow & 0
\end{array}
\]

for some \( M \in \mathcal{A}' \) (note that the middle arrow is an isomorphism).

**Proof.** Since \( (*) \) is an exact sequence, we get the morphism \( v : \Phi(M'') \rightarrow \Phi(M')[1] = \Phi(M'[1]) \) in \( D^b_{\mathcal{B}}(\mathcal{B}) \). Since \( \Phi \) is fully faithful, there exists a morphism \( u : M'' \rightarrow M'[1] \) in \( D^b_{\mathcal{A}}(\mathcal{A}) \) such that \( v = \Phi(u) \). Consider the distinguished triangle

\[
M'' \xrightarrow{u} M'[1] \rightarrow L \xrightarrow{+1}
\]

in \( D^b_{\mathcal{A}}(\mathcal{A}) \) induced by \( u \) with \( L \in D^b_{\mathcal{A}}(\mathcal{A}) \). Then from the long exact sequence

\[
\cdots \rightarrow H^i(M'') \rightarrow H^i(M'[1]) \rightarrow H^i(L) \rightarrow H^i(M''[1]) \rightarrow \cdots,
\]

we get \( H^i(L) = 0 \) for \( i \neq -1 \). Hence \( L[-1] \) is isomorphic to \( H^0(L[-1]) \in \mathcal{A}' \) in \( D^b_{\mathcal{A}}(\mathcal{A}) \). Denote by \( M = H^0(L[-1]) \), then from the morphism of distinguished triangles

\[
\begin{array}{cccccc}
\Phi(M') & \rightarrow & \Phi(M) & \rightarrow & \Phi(M'') & \rightarrow & \Phi(M'[1]) \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\Phi(M') & \rightarrow & N & \rightarrow & \Phi(M'') & \rightarrow & \Phi(M'[1])
\end{array}
\]

we obtain \( N \cong \Phi(M) \) and the result follows. \( \square \)
Set \( \mathcal{A} := \text{Mod}(\mathcal{A}_X/h^n\mathcal{A}_X) \), \( \mathcal{A}' := \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X) \), \( \mathcal{B} := \text{Mod}(\mathcal{A}_{X_{an}}/h^n\mathcal{A}_{X_{an}}) \) and \( \mathcal{B}' = \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{an}}/h^n\mathcal{A}_{X_{an}}) \). We shall apply Lemma 4.5.

**Theorem 4.6.** The functor

\[ f_n^* : \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X) \to \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{an}}/h^n\mathcal{A}_{X_{an}}) \]

is essentially surjective for each \( n > 0 \).

**Proof.** We shall prove this by induction. 
When \( n = 1 \), it is Theorem 1.1.
We shall prove the theorem for \( n > 1 \).
For any \( \mathcal{M}_{an} \in \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{an}}/h^n\mathcal{A}_{X_{an}}) \), consider the exact sequence

\[ 0 \to h\mathcal{M}_{an} \to \mathcal{M}_{an} \to \mathcal{M}_{an}/h\mathcal{M}_{an} \to 0 \]

where \( h\mathcal{M}_{an} \in \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{an}}/h^{n-1}\mathcal{A}_{X_{an}}) \) and \( \mathcal{M}_{an}/h\mathcal{M}_{an} \in \text{Mod}_{\text{coh}}(C_{X_{an}}) \).
Denote by \( \mathcal{M}_{1an}^an = h\mathcal{M}_{an} \) and by \( \mathcal{M}_{2an}^an = \mathcal{M}_{an}/h\mathcal{M}_{an} \). By induction hypothesis, there exists \( \mathcal{M}_1 \in \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^{n-1}\mathcal{A}_X) \) such that \( f_{n-1}^*(\mathcal{M}_1) \cong \mathcal{M}_{1an}^an \). On the other hand, by Theorem 1.1, there exists \( \mathcal{M}_2 \in \text{Mod}_{\text{coh}}(C_X) \) such that \( f_n^*(\mathcal{M}_2) \cong \mathcal{M}_{2an}^an \). Since \( f_n^*|_A : \mathcal{A} \to \mathcal{B} \) is exact by Lemma 4.1 and the functor \( D^b_A(\mathcal{A}) \to D^b_B(\mathcal{B}) \) induced by \( f_n^*|_A \) is fully faithful by the proof of Theorem 4.2, applying Lemma 4.5, we obtain the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & f_n^*(\mathcal{M}_1) & \longrightarrow & f_n^*(\mathcal{M}) & \longrightarrow & f_n^*(\mathcal{M}_2) & \longrightarrow & 0 \\
\downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} & & \\
0 & \longrightarrow & \mathcal{M}_1^an & \longrightarrow & \mathcal{M}_{an} & \longrightarrow & \mathcal{M}_2^an & \longrightarrow & 0
\end{array}
\]

for some \( \mathcal{M} \in \mathcal{A}' \). Hence \( f_n^* \) is essentially surjective. \( \square \)

From Corollary 4.4 and Theorem 4.6, we obtain the following.

**Theorem 4.7.** The functor

\[ f_n^* : \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X) \to \text{Mod}_{\text{coh}}(\mathcal{A}_{X_{an}}/h^n\mathcal{A}_{X_{an}}) \]

is an equivalence for each \( n > 0 \).

In order to prove that the functor \( f^*: D^b_{\text{coh}}(\mathcal{A}_X) \to D^b_{\text{coh}}(\mathcal{A}_{X_{an}}) \) is essentially surjective, we need the notion of projective limit in the 2-category \( \text{Cat} \). For its definition, we refer to [9] Definition 19.1.6.
Recall that a presite $X$ is nothing but a category which we denote by $C_X$. If $\Xi$ is a prestack on $X$, then we have the morphism $u : U_1 \to U_2$ in $C_X$, and the functor $r_u : \Xi(U_2) \to \Xi(U_1)$ for any $U_1, U_2 \in C_X$.

Denote by $\mathbb{N}$ the set of positive integers, viewed as a category defined by

$$\text{Ob}(\mathbb{N}) = \mathbb{N}$$

$$\text{Hom}_\mathbb{N}(i,j) = \begin{cases} \{\text{pt}\} & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

We define prestacks $\Xi$ and $\Xi_{\text{an}}$ on $\mathbb{N}$ as follows:

- $\Xi(n) := \text{Mod}_{\text{coh}}(\mathcal{A}_X/h^n\mathcal{A}_X)$ for any $n \in \mathbb{N}$,
- $r_u : \Xi(j) \to \Xi(i)$ is the functor for any $i \leq j$ and $u \in \text{Hom}_\mathbb{N}(i,j)$,

and

- $\Xi_{\text{an}}(n) := \text{Mod}_{\text{coh}}(\mathcal{A}_{\text{an}}/h^n\mathcal{A}_{\text{an}})$ for any $n \in \mathbb{N}$,
- $r_u : \Xi_{\text{an}}(j) \to \Xi_{\text{an}}(i)$ is the functor for any $i \leq j$ and $u \in \text{Hom}_\mathbb{N}(i,j)$.

The following lemma shows that coherent $\mathcal{A}$-modules are $h$-complete.

**Lemma 4.8 ([8]).** Let $(X, \mathcal{A}_X)$ be a complex manifold or a smooth variety endowed with a DQ-algebroid $\mathcal{A}_X$. Let $\{\mathcal{M}_n\}_{n \geq 0}$ be a projective system of coherent $\mathcal{A}_X$-modules. Assume that $h^{n+1}\mathcal{M}_n = 0$ and the induced morphism $\mathcal{M}_{n+1}/h^{n+1}\mathcal{M}_{n+1} \to \mathcal{M}_n$ is an isomorphism for any $n \geq 0$. Then $\mathcal{M} := \lim_{\longrightarrow} \mathcal{M}_n$ is a coherent $\mathcal{A}_X$-module and $\mathcal{M}/h^{n+1}\mathcal{M} \to \mathcal{M}_n$ is an isomorphism for any $n \geq 0$.

We need the following theorem.

**Theorem 4.9.** We have the following equivalences:

1. $\lim_{\longrightarrow} \Xi(n) \xrightarrow{\sim} \text{Mod}_{\text{coh}}(\mathcal{A}_X)$
2. $\lim_{\longrightarrow} \Xi_{\text{an}}(n) \xrightarrow{\sim} \text{Mod}_{\text{coh}}(\mathcal{A}_{\text{an}})$.

**Proof.** We only need to prove (1) and (2) can be proved similarly. Let $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_X)$, then we obtain the family $\{(F_n, \varphi_n)\}$ where:

(i) $F_n := \mathcal{M}/h^n\mathcal{M} \in \Xi(n)$ for any $n \in \mathbb{N}$,

(ii) $\varphi_n : r_n F_j \xrightarrow{\sim} F_i$ for any $i \leq j$ and $u \in \text{Hom}_\mathbb{N}(i,j)$ and $r_u : \Xi(j) \to \Xi(i)$ is defined by sending $\mathcal{M}$ to $\mathcal{M}/h^i\mathcal{M}$.
It is easy to check that \( \{(F_n, \varphi_n)\} \) satisfies the cocycle condition (a) of [9] 19.1.6 and hence \( \{(F_n, \varphi_u)\} \in \lim_{\mathfrak{U}} \Xi(n) \).

Let \( \mathcal{M}, \mathcal{M}' \in \text{Mod} \text{coh}(\mathcal{A}_X) \), then these define two objects \( F = \{(F_n, \varphi_u)\} \) and \( F' = \{(F_n', \varphi'_u)\} \in \lim_{\mathfrak{U}} \Xi(n) \). Let \( f : \mathcal{M} \to \mathcal{M}' \in \text{Mod} \text{coh}(\mathcal{A}_X) \), then we have the set of families \( \{f_n\}_{n \in \mathbb{N}} \) where \( f_n : \mathcal{M}/h^n \mathcal{M} \to \mathcal{M}'/h^n \mathcal{M}' \in \text{Hom}_{\mathfrak{U}}(\mathcal{M}/h^n \mathcal{M}, \mathcal{M}'/h^n \mathcal{M}') \). It is easy to check that \( \{f_n\} \) satisfies the commutative diagram of definition (b) of [9] 19.1.6 and hence \( \{f_n\}_{n \in \mathbb{N}} \in \text{Hom}_{\mathfrak{U}} \lim_{\mathfrak{U}} \Xi(n)(F, F') \). Hence we can define a functor \( \Phi : \text{Mod} \text{coh}(\mathcal{A}_X) \to \lim_{\mathfrak{U}} \Xi(n) \) by sending \( \mathcal{M} \) to \( \{(F_n, \varphi_u)\} \) and \( f \in \text{Hom}_{\mathcal{A}_X}(\mathcal{M}, \mathcal{M}') \) to \( \{f_n\}_{n \in \mathbb{N}} \).

On the other hand, if \( \{(F_n, \varphi_u)\} \in \lim_{\mathfrak{U}} \Xi(n) \), then by definition (a) of [9] 19.1.6, we have

(i) \( F_n \in \Xi(n) \) for any \( n \in \mathbb{N} \),
(ii) \( \varphi_u : r_u F_j \sim F_i \) for any \( i \leq j \) and \( u \in \text{Hom}_{\mathfrak{U}}(i, j) \) and \( r_u : \Xi(j) \to \Xi(i) \).

Hence the system \( \{F_n\}_{n \in \mathbb{N}} \) is a projective system and \( \lim_{\mathfrak{U}} F_n \in \text{Mod} \text{coh}(\mathcal{A}_X) \) by Lemma 4.8.

For two objects \( F = \{(F_n, \varphi_u)\} \) and \( F' = \{(F_n', \varphi'_u)\} \in \lim_{\mathfrak{U}} \Xi(n) \). By definition (b) of [9] 19.1.6, \( \text{Hom}_{\mathfrak{U}} \lim_{\mathfrak{U}} \Xi(n)(F, F') \) is the set of families \( f = \{f_n\}_{n \in \mathbb{N}} \) such that \( f_n \in \text{Hom}_{\mathfrak{U}}(F_n, F_n') \) and the following diagram commutes for any \( u : i \to j \) and \( i \leq j \),

\[
\begin{array}{ccc}
r_u F_j & \xrightarrow{\varphi_u} & F_i \\
\downarrow_{r_u(f_j)} & & \downarrow_{f_i} \\
r_u F'_j & \xrightarrow{\varphi'_u} & F'_i 
\end{array}
\]

Hence the system \( f = \{f_n\}_{n \in \mathbb{N}} \) is a projective system and we get that the morphism \( \lim_{\mathfrak{U}} f_n \) belongs to \( \text{Mod} \text{coh}(\mathcal{A}_X) \). Therefore, we can define a functor \( \Psi : \lim_{\mathfrak{U}} \Xi(n) \to \text{Mod} \text{coh}(\mathcal{A}_X) \) by \( \Psi(\{(F_n, \varphi_u)\}) = \lim_{\mathfrak{U}} F_n \) and \( \Psi(f = \{f_n\}_{n \in \mathbb{N}}) = \lim_{\mathfrak{U}} f_n \). Now it is easy to check that \( \Phi \circ \Psi \simeq \text{id}_{\lim_{\mathfrak{U}} \Xi(n)} \) and \( \Psi \circ \Phi \simeq \text{id}_{\text{Mod} \text{coh}(\mathcal{A}_X)} \). Therefore, the result follows. \( \square \)
Corollary 4.10. The functor $f^* : \text{Mod}_{\text{coh}}(\mathcal{X}) \rightarrow \text{Mod}_{\text{coh}}(\mathcal{X}_{\text{an}})$ is an equivalence.

Proof. This follows from Theorem 4.7 and Theorem 4.9. □

From Theorem 4.2, Corollary 4.10 and the proof of Lemma 1.2 for essentially surjectivity, we obtain what we want mentioned above.

Corollary 4.11. The natural functor $f^* : D_{\text{coh}}^b(\mathcal{X}) \rightarrow D_{\text{coh}}^b(\mathcal{X}_{\text{an}})$ is essentially surjective.

Equivalence.

Therefore, we obtain the main theorem of this paper.

Main Theorem 4.12. The natural functor $f^* : D_{\text{coh}}^b(\mathcal{X}) \rightarrow D_{\text{coh}}^b(\mathcal{X}_{\text{an}})$ is an equivalence.

Proof. This follows from Theorem 4.2 and Corollary 4.11. □

Acknowledgments. I would like to thank Pierre Schapira for getting me interested in this problem and helping me with suggestions. Also, I like to express my thanks to Jungkai Alfred Chen for teaching me algebraic geometry and supporting me when I studied in Taiwan. Finally, I thank the support of FIT and NTU when I studied in France.

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Manoscritto pervenuto in redazione il 18 maggio 2009.