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Some Fixed Point Theorems for Multi-Valued Weakly Uniform Increasing Operators.

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ABSTRACT - In the present work, we introduce the concept of weakly uniform increasing operators on an ordered spaces. We also give some common fixed point theorems for a pair of multivalued weakly uniform increasing operators in partially ordered metric space and in partially ordered Banach space.

1. Introduction.

The study on the existence of fixed points for single valued increasing operators is bounteous and successful, the results obtained are widely used to investigate the existence of solutions to the ordinary and partial differential equations (see [6], [7] and reference therein).

It is natural to extend this study to multi-valued case. Recently in [4], [7], [8], the authors have verified some fixed point theorems for multi-valued increasing operators.

Motivated by [3], [4], [9], [10] and [11] we prove some fixed point theorems for multi-valued operators satisfying some weakly uniform increasing properties in metric space and Banach space, respectively.

2. Preliminaries

In this section some notations and preliminaries are given.

Let X be a topological space and $\leq$ be a partial order endowed on X.

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In order to define the multi-valued weakly increasing and multi-valued weakly uniform increasing operators, we give the following relations between two subsets of X.

DEFINITION 1 ([4]). *Let A, B be two nonempty subsets of X, the relations between A and B are defined as follows:*

(1) *If for every $a \in A$, there exists $b \in B$ such that $a \leq b$, then $A \prec_1 B$.*

(2) *If for every $b \in B$, there exists $a \in A$ such that $a \leq b$, then $A \prec_2 B$.*

(3) *If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.*

REMARK 1. *$\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\leq$ be usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \not\prec_1 B$.*

REMARK 2. *$\prec_1$, $\prec_2$ and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\leq$ be usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders on 2^X.*

We can show that $\prec$ is a partial order on $CC(\mathbb{R})$, which denotes the family of closed and convex subsets of $\mathbb{R}$.

Let us recall some basic definitions and facts from multivalued analysis (see details [2]).

Let 2^X denote the family of all nonempty subset of X. A multivalued map $T : X \to 2^X$ is called upper semi continuous (u.s.c.) on X if for each $x_0 \in X$, the set Tx_0 is a nonempty, closed subset of X, and if for each open set V of X containing Tx_0, there exists an open neighborhood U of x_0 such that $T(U) \subseteq V$.

T is said to be completely continuous if $T(B)$ is relatively compact for every bounded subset $B \subset X$.

If the multivalued map T is completely continuous with nonempty compact values, then T is u.s.c. if and only if T has a closed graph (i.e. $u_n \to u_0, v_n \to v_0, v_n \in Tu_n$ imply $v_0 \in Tu_0$).

DEFINITION 2 ([4]). *If $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ or $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$, then $\{x_n\}$ is called a monotone sequence.*

DEFINITION 3 ([4]). *A multivalued operator $T : X \to 2^X \backslash \emptyset$ is called order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X, u_n \to u_0, v_n \to v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.*

DEFINITION 4 ([4]). *A function $f : X \to \mathbb{R}$ is called order upper (lower) semi continuous, if, for monotone sequence $\{u_n\} \subset X$, and $u_0 \in X$*

$$u_n \to u_0 \Longrightarrow \limsup_{n\to\infty} f(u_n) \leq f(u_0), \ \ (f(u_0) \leq \liminf f(u_n)).$$

REMARK 3. *An operator with closed graph must be an order closed operator and an upper (lower) semi continuous function is an order upper (lower) semi continuous. There exist some examples in* [4], *showing that the converse is not true.*

Let X be a real Banach space with a norm $\|\cdot\|_X$. A non-empty closed subset P of X is said to be an order cone in X if

(i) $P + P \subseteq P$,
(ii) $\lambda P \subseteq P$, for $\lambda \geq 0$ and
(iii) $P \cap -P = \{\theta\}$, where θ denotes the zero element of X.

Then the relation $x \leq y$ if and only if $y - x \in P$ defines the partial ordering in X. P is called regular, if every nondecreasing and bounded above in order sequence in X has a limit. i.e. if $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y(\ y \in X \ )$, then there exists $x \in X$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$. We do not require any property of the cones in the present discussion, however, the details of order cones and their properties may be found in Guo and Lakshmikantham [5].

Now we introduce the following definition.

DEFINITION 5. *Let X be a topological space, $\leq$ be a partial order endowed on X and $S, T : X \to 2^X$ be two maps. The pair (S, T) is said to be weakly uniform increasing with respect to $\prec_1$ if for any $x \in X$ we have $Sx \prec_1 Sy, Sx \prec_1 Sz$ for all $y \in Sx, z \in Tx$ and $Tx \prec_1 Ty, Tx \prec_1 Tz$ for all $y \in Tx, z \in Sx$. Similarly the pair (S, T) is said to be weakly uniform increasing with respect to $\prec_2$ if for any $x \in X$ we have $Sy \prec_2 Sx, Sz \prec_2 Sx$ for all $y \in Sx, z \in Tx$ and $Ty \prec_2 Tx, Tz \prec_2 Tx$ for all $y \in Tx, z \in Sx$.*

Now we give some examples.

EXAMPLE 1. *Let $\mathbb{R}$ denote the real line with the usual order relation $\leq$ and let $X = [0, 1] \subset \mathbb{R}$. Consider two mappings $S, T : X \to 2^X$ defined* *by*

$$Sx = \begin{cases} (0,1], & if\ x = 0 \\ \{1\}, & if\ x \neq 0 \end{cases} \quad and \quad Tx = \begin{cases} \left\{\frac{1}{3}, \frac{1}{2}\right\}, & if\ x = 0 \\ \left\{\frac{2}{3}\right\}, & if\ x \neq 0 \end{cases}$$

for $x \in [0,1]$. *Then the pair of mappings* (S,T) *is weakly uniform increasing with respect to* $\prec_1$ *but not weakly uniform increasing with respect to* $\prec_2$ *on* $[0,1]$. *Indeed, since*

$$\begin{aligned} Sx &= \begin{cases} (0,1], & if\ x = 0 \\ \{1\}, & if\ x \neq 0 \end{cases} \\ &\prec_1 \{1\} \\ &= Sy,\ for\ all\ y \in Sx \end{aligned} \quad , \quad \begin{aligned} Sx &= \begin{cases} (0,1], & if\ x = 0 \\ \{1\}, & if\ x \neq 0 \end{cases} \\ &\prec_1 \{1\} \\ &= Sz,\ for\ all\ z \in Tx, \end{aligned}$$

and

$$\begin{aligned} Tx &= \begin{cases} \left\{\frac{1}{3}, \frac{1}{2}\right\}, & if\ x = 0 \\ \left\{\frac{2}{3}\right\}, & if\ x \neq 0 \end{cases} \\ &\prec_1 \left\{\frac{2}{3}\right\} \\ &= Ty,\ for\ all\ y \in Tx \end{aligned} \quad , \quad \begin{aligned} Tx &= \begin{cases} \left\{\frac{1}{3}, \frac{1}{2}\right\}, & if\ x = 0 \\ \left\{\frac{2}{3}\right\}, & if\ x \neq 0 \end{cases} \\ &\prec_1 \left\{\frac{2}{3}\right\} \\ &= Tz,\ for\ all\ z \in Sx \end{aligned}$$

so (S,T) *is weakly uniform increasing with respect to* $\prec_1$. *Note that the pair* (S,T) *is not weakly uniform increasing with respect to* $\prec_2$ *since* $S\frac{1}{2} = \{1\} \not\prec_2 (0,1] = S0$ *for* $\frac{1}{2} \in S0$.

Example 2. *Let* $X = [1,\infty)$ *and* $\leq$ *be usual order on* X. *Consider two mappings* $S,T : X \to 2^X$ *defined by* $Sx = [1,x^2]$ *and* $Tx = [1,2x]$ *for all* $x \in X$. *Then the pair of mappings* (S,T) *is weakly uniform increasing with respect to* $\prec_2$ *but not* $\prec_1$. *Indeed, since*

$$\begin{aligned} Sy = [1,y^2] \prec_2 [1,x^2] = Sx\ for\ all\ y \in Sx, \\ Sz = [1,z^2] \prec_2 [1,x^2] = Sx\ for\ all\ z \in Tx, \\ Ty = [1,2y] \prec_2 [1,2x] = Tx\ for\ all\ y \in Tx, \\ Tz = [1,2z] \prec_2 [1,2x] = Tx\ for\ all\ z \in Sx, \end{aligned}$$

then the pair (S,T) *is weakly uniform increasing with respect to* $\prec_2$. *But* $S2 = [1,4] \not\prec_1 \{1\} = S1$ *for* $1 \in S2$, *so the pair* (S,T) *is not weakly uniform increasing with respect to* $\prec_1$.

3. Fixed point theorems.

In this section, we prove some fixed point theorems for multivalued weakly uniform increasing operators with respect to $\prec_1$and $\prec_2$ in partial order metric space and partial order Banach space.

We will use the following lemma in our theorems.

LEMMA 1 ([1]). *Let* (X,d) *be a metric space,* $\varphi : X \to \mathbb{R}$. *Define the relation* $\leq$ *on* X *as follows:*

$$x \leq y \Longleftrightarrow d(x,y) \leq \varphi(x) - \varphi(y).$$

Then $\leq$ *is a partial order on* X *and* X *is called a partial ordered metric space.*

Now we give a fixed point theorem.

THEOREM 1. *Let* (X,d) *be a complete metric space,* $\varphi : X \to \mathbb{R}$ *be a bounded below function,* $\leq$ *be a partial order induced by* φ *and* $S,T : X \to 2^X \backslash \emptyset$ *be two order closed with respect to* $\leq$ *and weakly uniform increasing operator with respect to* $\prec_1$. *Then there exist* $z_1, z_2 \in X$ *such that* $z_1 \in Sz_2$ *and* $z_2 \in Tz_1$. *Further if* $Sz_2 \prec_1 \{z_2\}$ *and* $Tz_1 \prec_1 \{z_1\}$, *then* S *and* T *have a common fixed point* ($z_1 = z_2$) *in* X.

PROOF. Let $x_0 \in X$ be arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now since (S,T) is weakly uniform increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Sy$ for all $y \in Sx_0$ and so $x_1 \in Sx_0 \prec_1 Sx_1$. Again, since (S,T) is weakly uniform increasing with respect to $\prec_1$ we have $Sx_1 \prec_1 Sz$ for all $z \in Tx_1$. Now we choose $x_2 \in Tx_1$, then $Sx_1 \prec_1 Sx_2$ and so $x_1 \in Sx_0 \prec_1 Sx_1 \prec_1 Sx_2$. Now using the definition of $\prec_1$, we can choose $x_3 \in Sx_2$ such that $x_1 \leq x_3$. Similarly, since (S,T) is weakly uniform increasing with respect to $\prec_1$, we have $x_2 \in Tx_1 \prec_1 Ty$ for all $y \in Tx_1$ and so $x_2 \in Tx_1 \prec_1 Tx_2$. Again since (S,T) is weakly uniform increasing with respect to $\prec_1$, we have $Tx_2 \prec_1 Tz$ for all $z \in Sx_2$. Now since $x_3 \in Sx_2$, we have $Tx_2 \prec_1 Tx_3$ and so $x_2 \in Tx_1 \prec_1 Tx_2 \prec_1 Tx_3$. Now using the definition of

$\prec_1$we can choose $x_4 \in Tx_3$ such that $x_2 \leq x_4$. Continue this process, we will get a sequence $\{x_n\}_{n=1}^{\infty}$ which its subsequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ are increasing and satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \ldots$.

By the definition $\leq$ we have

$$\varphi(x_1) \geq \varphi(x_3) \geq \cdots \geq \varphi(x_{2n+1}) \geq \cdots$$

and

$$\varphi(x_2) \geq \varphi(x_4) \geq \cdots \geq \varphi(x_{2n+2}) \geq \cdots.$$

Since φ is bounded from below, then $\{\varphi(x_{2n+1})\}_{n=0}^{\infty}$ and $\{\varphi(x_{2n+2})\}_{n=0}^{\infty}$ are convergent. Thus for $\varepsilon > 0$, there exists N, if $n > m > N$ then

$$d(x_{2m+1}, x_{2n+1}) \leq \varphi(x_{2m+1}) - \varphi(x_{2n+1}) < \varepsilon$$

and

$$d(x_{2m+2}, x_{2n+2}) \leq \varphi(x_{2m+2}) - \varphi(x_{2n+2}) < \varepsilon$$

which indicates $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ are Cauchy sequence in X. Due to completeness of X, we know there exist $z_1, z_2 \in X$, $x_{2n+1} \to z_1$ and $x_{2n+2} \to z_2$. Since S and T are order closed, $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we deduce that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Now if $Sz_2 \prec_1 \{z_2\}$, then $z_1 \leq z_2$ and if $Tz_1 \prec_1 \{z_1\}$, then $z_2 \leq z_1$ and so $z_1 = z_2$ is a common fixed point of S and T. □

The proof of the following theorem carries over in the same manner.

THEOREM 2. *Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}$ be a bounded above function, $\leq$ be a partial order induced by φ and $S, T : X \to 2^X \backslash \emptyset$ be two order closed with respect to $\leq$ and weakly uniform increasing operator with respect to $\prec_2$. Then there exist $z_1, z_2 \in X$ such that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Further if $\{z_2\} \prec_2 Sz_2$ and $\{z_1\} \prec_2 Tz_1$, then S and T have a common fixed point ($z_1 = z_2$) in X.*

PROOF. Let $x_0 \in X$ be arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now since (S, T) is weakly uniform increasing with respect to $\prec_2$, we have $Sx_1 \prec_1 Sx_0$ for $x_1 \in Sx_0$. Again, since (S, T) is weakly uniform increasing with respect to $\prec_2$ we have $Sz \prec_2 Sx_1$ for all $z \in Tx_1$. Now we choose $x_2 \in Tx_1$, then $Sx_2 \prec_2 Sx_1$ and so $Sx_2 \prec_2 Sx_1 \prec_2 Sx_0$. Now using the definition of $\prec_2$, we can choose $x_3 \in Sx_2$ such that $x_3 \leq x_1$. Similarly, since (S, T) is weakly uniform increasing with respect to $\prec_2$, we have $Tx_2 \prec_1 Tx_1$ $x_2 \in Tx_1$. Again since (S, T) is weakly uniform increasing with respect to $\prec_2$, we have $Tz \prec_2 Tx_2$ for all $z \in Sx_2$. Now since $x_3 \in Sx_2$, we have $Tx_3 \prec_2 Tx_2$ and so $Tx_3 \prec_2 Tx_2 \prec_2 Tx_1$. Now using the definition of $\prec_2$we can choose $x_4 \in Tx_3$ such that $x_4 \leq x_2$. Continue this process, we will get a sequence $\{x_n\}_{n=1}^{\infty}$ which its subsequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ are decreasing and satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$.

By the definition $\leq$ we have

$$\varphi(x_1) \leq \varphi(x_3) \leq \cdots \leq \varphi(x_{2n+1}) \leq \cdots$$

and

$$\varphi(x_2) \leq \varphi(x_4) \leq \cdots \leq \varphi(x_{2n+2}) \leq \cdots.$$

Since φ is bounded from above, then $\{\varphi(x_{2n+1})\}_{n=0}^{\infty}$ and $\{\varphi(x_{2n+2})\}_{n=0}^{\infty}$ are convergent. Thus for $\varepsilon > 0$, there exists N, if $n > m > N$ then

$$d(x_{2m+1}, x_{2n+1}) \leq \varphi(x_{2n+1}) - \varphi(x_{2m+1}) < \varepsilon$$

and

$$d(x_{2m+2}, x_{2n+2}) \leq \varphi(x_{2n+2}) - \varphi(x_{2m+2}) < \varepsilon$$

which indicates $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ are Cauchy sequence in X. Due to completeness of X, we know there exist $z_1, z_2 \in X$, $x_{2n+1} \to z_1$ and $x_{2n+2} \to z_2$. Since S and T are order closed, $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we deduce that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Now if $\{z_2\} \prec_2 Sz_2$, then $z_2 \leq z_1$ and if $\{z_1\} \prec_2 Tz_1$, then $z_1 \leq z_2$ and so $z_1 = z_2$ is a common fixed point of S and T. □

Now we give some fixed point theorem in ordered Banach space.

THEOREM 3. *Let X be a Banach space, $P \subset X$ be a regular cone, $\leq$ be a partial order induced by P and $S, T : X \to 2^X \backslash \emptyset$ be two order closed with respect to $\leq$ and weakly uniform increasing operator with respect to $\prec_1$. If the following condition hold:*

(i) *there exists some $u \in X$ such that $Sx \prec_1 \{u\}$ and $Tx \prec_1 \{u\}$ for all $x \in X$.*

Then there exist $z_1, z_2 \in X$ such that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Further if $Sz_2 \prec_1 \{z_2\}$ and $Tz_1 \prec_1 \{z_1\}$, then S and T have a common fixed point ($z_1 = z_2$) in X.

PROOF. We can construct a sequence $\{x_n\}_{n=1}^{\infty}$ in X as in the proof of Theorem 1. By (i), and the definition of $\prec_1$, we have $x_n \leq u, n = 1, 2, \dots$.

Since P is regular, $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ are nondecreasing and bounded above in order, there exist $z_1, z_2 \in X$ such that $x_{2n+1} \to z_1$ and $x_{2n+2} \to z_2$. Now the proof can complete as in the proof of Theorem 1.u □

The following theorem can be similarly verified.

THEOREM 4. *Let X be a Banach space, $P \subset X$ be a regular cone, $\leq$ be a partial order induced by P and $S, T : X \to 2^X \backslash \emptyset$ be two order closed with respect to $\leq$ and weakly uniform increasing operator with respect to $\prec_2$. If the following condition hold:*

(ii) *there exists some $u \in X$ such that $\{u\} \prec_2 Sx$ and $\{u\} \prec_2 Tx$ for all $x \in X$.*

Then there exist $z_1, z_2 \in X$ such that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Further if $\{z_2\} \prec_2 Sz_2$ and $\{z_1\} \prec_2 Tz_1$, then S and T have a common fixed point ($z_1 = z_2$) in X.

REMARK 4. *By Theorems* 1, 2, 3 *and* 4, *we give some improved versions of Theorems* 3.2, 3.3, 4.2 *and* 4.3 *of* [4], *respectively.*

Now we introduce the following condition.

CONDITION 1. *Let X be a Banach space. Two maps $S, T : X \to 2^X$ are said to satisfy Condition* 1 *if for any countable subset A of X and for any fixed $a \in X$ the condition*

$$A \subseteq \{a\} \cup S(A) \cup T(A) \text{ implies } \overline{A} \text{ is compact};$$

here $T(A) = \cup_{x \in A} Tx$.

Using the above condition, we can give the following theorems.

THEOREM 5. *Let X be a Banach space, $P \subset X$ be a cone (not necessary regular), $\leq$ be a partial order induced by P and $S, T : X \to 2^X \backslash \emptyset$ be two order closed with respect to $\leq$ and weakly uniform increasing operator with respect to $\prec_1$. Suppose that S and T are satisfy the Condition* 1, *then there exist $z_1, z_2 \in X$ such that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Further if $Sz_2 \prec_1 \{z_2\}$ and $Tz_1 \prec_1 \{z_1\}$, then S and T have a common fixed point ($z_1 = z_2$) in X.*

PROOF. We can construct a sequence $\{x_n\}_{n=1}^{\infty}$ in X as in the proof of Theorem 1.

Now let $A = \{x_0, x_1, ...\}$. Since A is countable,

$$\begin{aligned} A &= \{x_0\} \cup \{x_1, x_3, ...\} \cup \{x_2, x_4, ...\} \\ &\subseteq \{x_0\} \cup S(A) \cup T(A), \end{aligned}$$

and S and T are satisfy the Condition 1, then $\overline{A}$ is compact. Thus $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ have convergent subsequences which converges to say $z_1, z_2 \in X$, respectively. However $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ are increasing, so $x_{2n+1} \to z_1$ and $x_{2n+2} \to z_2$ for $n \to \infty$. Now the proof can complete as in the proof of Theorem 1. □

The following theorem can be similarly verified.

THEOREM 6. *Let X be a Banach space, $P \subset X$ be a cone (not necessary regular), $\leq$ be a partial order induced by P and $S, T : X \to 2^X \backslash \emptyset$ be two order closed with respect to $\leq$ and weakly uniform increasing operator with respect to $\prec_2$. Suppose that S and T are satisfy the Condition* 1, *then there exist $z_1, z_2 \in X$ such that $z_1 \in Sz_2$ and $z_2 \in Tz_1$. Further if $\{z_2\} \prec_2 Sz_2$ and $\{z_1\} \prec_2 Tz_1$, then S and T have a common fixed point ($z_1 = z_2$) in X.*

REMARK 5. *By Theorems* 5 *and* 6, *we give some improved versions of Theorem* 3.1 *of* [3] *and Theorem* 4 *of* [12].

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