

Sheaves on Subanalytic Sites

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ABSTRACT - In [7] the authors introduced the notion of ind-sheaves and defined the six Grothendieck operations in this framework. As a byproduct, they obtained subanalytic sheaves and the six Grothendieck operations on them.

The aim of this paper is to give a direct construction of the six Grothendieck operations in the framework of subanalytic sites avoiding the heavy machinery of ind-sheaves. As an application, we show how to recover the subanalytic sheaves \mathcal{O}^t and \mathcal{O}^w of temperate and Whitney holomorphic functions respectively.

Introduction.

Let X be a real analytic manifold and k a field. The spaces of functions which are not defined by local properties, such as tempered distributions, tempered and Whitney C^∞ functions, etc., are very useful in the study of systems of linear partial differential equations (Laplace transform, tempered holomorphic solutions of \mathcal{D} -modules etc.). Although these spaces do not define sheaves on X , they define sheaves on a site associated to X , the subanalytic site X_{sa} , where one just considers open subanalytic sets and locally finite coverings.

In [7], Kashiwara and Schapira, motivated by the construction of the microlocalization functor, treated a more general theory, namely that of ind-sheaves. They defined the category $I(k_X)$ of ind-sheaves on X as the category of ind-objects of the category $\text{Mod}^c(k_X)$ of sheaves with compact support and they developed the six Grothendieck operations in this framework. When restricting to \mathbb{R} -constructible sheaves, they showed the equivalence between the category $I_{\mathbb{R}\text{-c}}(k_X) = \text{Ind}(\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X))$ of ind- \mathbb{R} -constructible sheaves on X and the category $\text{Mod}(k_{X_{sa}})$ of sheaves on the subanalytic site associated to X .

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In this way, tempered distributions, tempered and Whitney C^∞ functions, etc., are obtained as a byproduct of the whole theory of ind-sheaves. It turns out to be useful to have a more straightforward introduction of these sheaves.

Our aim in this paper is to give a direct, self-contained and elementary construction of the six Grothendieck operations on $\text{Mod}(k_{X_{sa}})$, without using the more sophisticated and much more difficult theory of ind-sheaves. Indeed, contrary to the category $\text{I}(k_X)$, the category $\text{Mod}(k_{X_{sa}})$ is a Grothendieck category.

We will start by recalling some results of [7], the definition of a subanalytic site, the natural functor of sites $\rho : X \rightarrow X_{sa}$, and the functors ρ_* , ρ^{-1} and $\rho_!$ relating the categories of “classical” and subanalytic sheaves. We also recall a very useful description of subanalytic sheaves as inductive limits of \mathbb{R} -constructible sheaves.

Then we go into the study of subanalytic sheaves, without using the notion of ind-sheaf. Let $f : X \rightarrow Y$ be a morphism of real analytic manifolds. The functors $\mathcal{H}om$, \otimes , f_* and f^{-1} are well defined since X_{sa} is a site. We introduce the proper direct image functor $f_{!!}$ and we study the relations between the above operations and the functors ρ_* , ρ^{-1} and $\rho_!$. In the derived category we obtain an exceptional inverse image, denoted by $f^!$, right adjoint to $Rf_{!!}$. This is obtained via Brown representability thanks to the existence of quasi-injective objects. We study quasi-injective objects without references to ind-sheaves, we show that they are acyclic with respect to the above functors and we prove that the quasi-injective dimension of $\text{Mod}(k_{X_{sa}})$ is finite. We end this work giving some examples of subanalytic sheaves.

In more details, the contents of this paper are as follows.

In Section 1 we construct the operations in $\text{Mod}(k_{X_{sa}})$. In § 1.1, we start by recalling the definitions of the functors ρ_* , ρ^{-1} and $\rho_!$ of [7] and their properties (a more detailed study of the functor $\rho_!$ is done in § 1.6). In § 1.2, we recall the internal operations and the functors of direct and inverse image (which are well defined on any site) and their relations with ρ_* , ρ^{-1} and $\rho_!$. We also define (in § 1.4) the proper direct image functor $f_{!!}$, where the notation $f_{!!}$ follows from the fact that $f_{!!} \circ \rho_* \not\cong \rho_* \circ f_{!!}$ in general. We study its properties and the relations with the other operations. While the functors f^{-1} and \otimes are exact, the functors $\mathcal{H}om$, f_* and $f_{!!}$ are left exact, and we introduce the subcategory of quasi-injective objects which is injective with respect to these functors. This is done in § 1.5.

In Section 2 we consider the derived category of $\text{Mod}(k_{X_{sa}})$. In § 2.1, we consider the subcategory $D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}})$ consisting of bounded complexes with \mathbb{R} -constructible cohomology and we prove the equivalence of derived categories $D_{\mathbb{R}\text{-c}}^b(k_X) \simeq D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}})$. Then, in § 2.2, we study the derived functors of Hom, f_* and $f_!$ and we obtain the usual formulas (projection formula, base change formula, Künneth formula, etc.) in the framework of subanalytic sites. We also prove (in § 2.3) some vanishing theorems for subanalytic sheaves, in particular we prove that the quasi-injective dimension of $\text{Mod}(k_{X_{sa}})$ is finite. Using the Brown representability theorem we prove the existence of a right adjoint to the functor $Rf_!$, denoted by $f^!$. This is done in § 2.4. We calculate the functor $f^!$ by decomposing f as the composite of a closed embedding and a submersion.

In Section 3 we give some examples of subanalytic sheaves. We start by recalling the definition of sheaves of $\rho_! \mathcal{D}_X$ -modules, where \mathcal{D}_X denotes the sheaf of finite order differential operators on a complex analytic manifold X . Then in § 3.3 we show how to recover the sheaves of $\rho_! \mathcal{D}_X$ -modules \mathcal{O}_X^t and \mathcal{O}_X^w of temperate and Whitney holomorphic functions of [7] respectively. We prove the relations between the above sheaves and the functors of moderate and formal cohomology of [4] and [6].

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1. Sheaves on subanalytic sites.

In the following X will be a real analytic manifold and k a field. References are made to [8] and [14] for an introduction to sheaves on Grothendieck topologies, to [5] for a complete exposition on classical sheaves and \mathbb{R} -constructible sheaves and to [1] and [10] for the theory of subanalytic sets (see also the Appendix for a short overview on some properties of subanalytic subsets).

1.1 – The subanalytic site. Notations and review.

We introduce the subanalytic site. The results of § 1.1 have already been proved in [7], for sake of completeness we reproduce here the proofs.

Denote by $\text{Op}(X_{sa})$ the category of subanalytic subsets of X . One endows $\text{Op}(X_{sa})$ with the following topology: $S \subset \text{Op}(X_{sa})$ is a covering of

$U \in \text{Op}(X_{sa})$ if for any compact subset K of X there exists a finite subset S_0 of S such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site, and for $U \in \text{Op}(X_{sa})$ we denote by $U_{X_{sa}}$ the category $\text{Op}(X_{sa}) \cap U$ with the topology induced by X_{sa} . There is a natural morphism of sites $U_{sa} \rightarrow U_{X_{sa}}$.

REMARK 1.1.1. *We use the notation $U_{X_{sa}}$ to stress the difference from U_{sa} , the subanalytic site associated to U . For example, let $X = \mathbb{R}^2$ and $U = \mathbb{R}^2 \setminus \{0\}$. Let $V_n = \{x \in \mathbb{R}^2, |x| > \frac{1}{n}\}$. Then $\{V_n\}_{n \in \mathbb{N}} \in \text{Cov}(U_{sa})$ but $\{V_n\}_{n \in \mathbb{N}} \notin \text{Cov}(U_{X_{sa}})$.*

Let $\text{Mod}(k_{X_{sa}})$ denote the category of sheaves on X_{sa} . Then $\text{Mod}(k_{X_{sa}})$ is a Grothendieck category, i.e. it admits a generator and small inductive limits, and small filtrant inductive limits are exact. In particular as a Grothendieck category, $\text{Mod}(k_{X_{sa}})$ has enough injective objects.

REMARK 1.1.2. *Denote by $\text{Op}^c(X_{sa})$ the category of relatively compact subanalytic open subsets of X . One denotes by X_{sa}^c the category $\text{Op}^c(X_{sa})$ with the topology induced by X_{sa} . The forgetful functor gives an equivalence of categories $\text{Mod}(k_{X_{sa}}) \xrightarrow{\sim} \text{Mod}(k_{X_{sa}^c})$.*

PROPOSITION 1.1.3. *Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\text{Mod}(k_{X_{sa}})$ and let $U \in \text{Op}^c(X_{sa})$. Then*

$$\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i).$$

PROOF. By Remark 1.1.2 it is enough to prove the assertion in the category $\text{Mod}(k_{X_{sa}^c})$. Denote by $\varinjlim_i F_i$ the presheaf $V \mapsto \varinjlim_i \Gamma(V; F_i)$ on

X_{sa}^c . Let $U \in \text{Op}^c(X_{sa})$ and let S be a finite covering of U . Since \varinjlim_i commutes with finite projective limits we obtain the isomorphism $(\varinjlim_i F_i)(S) \xrightarrow{\sim} \varinjlim_i F_i(S)$ and $F_i(U) \xrightarrow{\sim} F_i(S)$ since $F_i \in \text{Mod}(k_{X_{sa}^c})$ for each

i . Moreover the family of finite coverings of U is cofinal in $\text{Cov}(U)$. Hence $\varinjlim_i F_i \xrightarrow{\sim} (\varinjlim_i F_i)^+$. Applying once again the functor $(\cdot)^+$ we get

$$\varinjlim_i F_i \simeq (\varinjlim_i F_i)^+ \simeq (\varinjlim_i F_i)^{++} \simeq \varinjlim_i F_i.$$

Hence applying the functor $\Gamma(U; \cdot)$ we obtain the isomorphism $\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i)$ for each $U \in \text{Op}^c(X_{sa})$. \square

There is an easy way to construct sheaves on a subanalytic site

PROPOSITION 1.1.4. *Let F be a presheaf on X_{sa}^c and assume that*

- (i) $F(\emptyset) = 0$,
- (ii) *For any $U, V \in \text{Op}^c(X_{sa})$ the sequence $0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V)$ is exact.*

Then $F \in \text{Mod}(k_{X_{sa}^c}) \simeq \text{Mod}(k_{X_{sa}})$.

PROOF. Let $U \in \text{Op}^c(X_{sa})$ and let $\{U_j\}_{j=1}^n$ be a finite covering of U . Set for short $U_{ij} = U_i \cap U_j$. We have to show the exactness of the sequence

$$0 \rightarrow F(U) \rightarrow \bigoplus_{1 \leq k \leq n} F(U_k) \rightarrow \bigoplus_{1 \leq i < j \leq n} F(U_{ij}),$$

where the second morphism sends $(s_k)_{1 \leq k \leq n}$ to $(t_{ij})_{1 \leq i < j \leq n}$ by $t_{ij} = s_i|_{U_{ij}} - s_j|_{U_{ij}}$. We shall argue by induction on n . For $n = 1$ the result is trivial, and $n = 2$ is the hypothesis. Suppose that the assertion is true for $j \leq n - 1$ and set $U' = \bigcup_{1 \leq k < n} U_k$. By the induction hypothesis the following commutative diagram is exact

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \downarrow & & & \downarrow \\
 0 & \longrightarrow & F(U) & \longrightarrow & F(U') \oplus F(U_n) & \longrightarrow & F(U' \cap U_n) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_{i < n} F(U_i) \oplus F(U_n) & \longrightarrow & \bigoplus_{i < n} F(U_{in}) \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{i < j < n} F(U_{ij}). & &
 \end{array}$$

Then the result follows. □

Let $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ be the abelian category of \mathbb{R} -constructible sheaves on X , and consider its subcategory $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ consisting of sheaves whose support is compact.

We denote by

$$\rho : X \rightarrow X_{sa}$$

the natural morphism of sites. We have functors

$$(1.1) \quad \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X) \subset \text{Mod}_{\mathbb{R}\text{-c}}(k_X) \subset \text{Mod}(k_X) \xrightleftharpoons[\rho^{-1}]{\rho_*} \text{Mod}(k_{X_{sa}}).$$

We will still denote by ρ_* the restriction of ρ_* to $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$.

REMARK 1.1.5. *By Proposition 1.1.3 for each $F \in \text{Mod}(k_X)$ and $V \in \text{Op}^c(X_{sa})$ one has*

$$(1.2) \quad \Gamma(V; \rho_* F) \simeq \varinjlim_U \Gamma(V; \rho_* F_U) \simeq \Gamma(V; \varinjlim_U \rho_* F_U),$$

where U ranges through the family of relatively compact open subanalytic subsets of X . The first isomorphism follows since V is relatively compact and $\Gamma(V, \rho_* F_U) \simeq \Gamma(V, \rho_* F)$ if $V \subset U$. The isomorphism (1.2) implies $\varinjlim_U \rho_* F_U \xrightarrow{\sim} \rho_* F$.

REMARK 1.1.6. *The functor ρ_* does not commute with filtrant inductive limits. For example consider the family $\{V_n\}_{n \in \mathbb{N}}$ of Remark 1.1.1. We have $\rho_* \varinjlim_n k_{V_n} \simeq \rho_* k_{\mathbb{R}^2 \setminus \{0\}}$, while for each $U \in \text{Op}^c(\mathbb{R}_{sa}^2)$ with $0 \in \partial U$ we have $\Gamma(U; \varinjlim_n \rho_* k_{V_n}) \simeq \varinjlim_n \Gamma(U; \rho_* k_{V_n}) = 0$.*

PROPOSITION 1.1.7. *Let U be an open subanalytic subset of X and consider the constant sheaf $k_{U_{X_{sa}}} \in \text{Mod}(k_{X_{sa}})$. We have $k_{U_{X_{sa}}} \simeq \rho_* k_U$.*

PROOF. Let F be the presheaf on X_{sa} defined by $F(V) = k$ if $V \subset U$, $F(V) = 0$ otherwise. This is a separated presheaf and $k_{U_{X_{sa}}} = F^{++}$. Moreover there is an injective arrow $F(V) \hookrightarrow \rho_* k_U(V)$ for each $V \in \text{Op}(X_{sa})$. Hence $F^{++} \hookrightarrow \rho_* k_U$ since the functor $(\cdot)^{++}$ is exact. Let \mathcal{T} be the family of $W \in \text{Op}(X_{sa})$ connected and such that W does not contain any connected component of U . Then \mathcal{T} forms a basis for the topology of X_{sa} since the connected components of U are locally finite. For each $W \in \mathcal{T}$ we have $F(W) \simeq \rho_* k_U(W) \simeq k$ if $W \subset U$ and $F(W) = 0$ otherwise. Then $F^{++} \simeq \rho_* k_U$. □

PROPOSITION 1.1.8. *The restriction of ρ_* to $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ is exact.*

PROOF. (i) Let us consider an epimorphism $G \twoheadrightarrow F$ in $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$, we have to prove that $\psi : \rho_* G \rightarrow \rho_* F$ is an epimorphism. Let

$U \in \text{Op}^c(X_{sa})$ and let $0 \neq s \in \Gamma(U; \rho_* F) \simeq \text{Hom}_{k_X}(k_U, F)$. Set $G' = G \times_F k_U$. Then $G' \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ and moreover $G' \twoheadrightarrow k_U$. There exists a finite $\{U_i\}_{i \in I} \subset \text{Op}^c(X_{sa})$ with U_i connected for each i such that $\bigoplus_i k_{U_i} \twoheadrightarrow G'$. The composition $k_{U_i} \rightarrow G' \rightarrow k_U$ is given by the multiplication by $a_i \in k$. Set $I_0 = \{k_{U_i}; a_i \neq 0\}$, we may assume $a_i = 1$. We get a diagram

$$\begin{array}{ccccc} \bigoplus_{i \in I_0} k_{U_i} & \longrightarrow & G' & \longrightarrow & G \\ & \searrow & \downarrow & & \downarrow \\ & & k_U & \xrightarrow{s} & F \end{array}$$

The composition $k_{U_i} \rightarrow G' \rightarrow G$ defines $t_i \in \text{Hom}_{k_X}(k_{U_i}, G) = \Gamma(U_i; \rho_* G)$. Hence for each $s \in \Gamma(U; \rho_* F)$ there exists a finite covering $\{U_i\}$ of U and $t_i \in \Gamma(U_i; \rho_* G)$ such that $\psi(t_i) = s|_{U_i}$. This means that ψ is surjective.

(ii) Let $F \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. By Remark 1.1.5 $\rho_* F \simeq \varinjlim_U \rho_* F_U$, where U ranges through the family $\text{Op}^c(X_{sa})$. The result follows since ρ_* is exact on $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ and filtrant \varinjlim are exact. \square

PROPOSITION 1.1.9. *The restriction of ρ_* to $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ is fully faithful.*

PROOF. It is enough to prove that $\rho^{-1}\rho_* F \simeq F$ for each $F \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Since both functors are exact on $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ we may reduce to the case $F = k_U$ with $U \in \text{Op}(X_{sa})$ and the result follows from Proposition 1.1.7. \square

NOTATIONS 1.1.10. *Since the functor ρ_* is fully faithful and exact on the category $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$, we can identify $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ with its image in $\text{Mod}(k_{X_{sa}})$. When there is no risk of confusion we will write F instead of $\rho_* F$, for $F \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$.*

The following theorem gives a fundamental characterization of subanalytic sheaves and it will be used systematically in the following Sections.

THEOREM 1.1.11. (i) *Let $G \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ and let $\{F_i\}$ be a filtrant inductive system in $\text{Mod}(k_{X_{sa}})$. Then we have an isomorphism*

$$\varinjlim_i \text{Hom}_{k_{X_{sa}}}(\rho_* G, F_i) \xrightarrow{\sim} \text{Hom}_{k_{X_{sa}}}(\rho_* G, \varinjlim_i F_i).$$

(ii) Let $F \in \text{Mod}(k_{X_{sa}})$. There exists a small filtrant inductive system $\{F_i\}_{i \in I}$ in $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ such that $F \simeq \varinjlim_i \rho_* F_i$.

PROOF. (i) There exists an exact sequence $G_1 \rightarrow G_0 \rightarrow G \rightarrow 0$ with G_1, G_0 finite direct sums of constant sheaves k_U with $U \in \text{Op}^c(X_{sa})$. Since ρ_* is exact on $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and commutes with finite sums, by Proposition 1.1.7 we are reduced to prove the isomorphism $\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i)$.

Then the result follows from Proposition 1.1.3.

(ii) Let $F \in \text{Mod}(k_{X_{sa}})$, and define

$$I_0 := \{(U, s); U \in \text{Op}^c(X_{sa}), s \in \Gamma(U; F)\}$$

$$G_0 := \bigoplus_{(U,s) \in I_0} \rho_* k_U$$

The morphism $\rho_* k_U \rightarrow F$, where the section $1 \in \Gamma(U; k_U)$ is sent to $s \in \Gamma(U; F)$ defines an epimorphism $\varphi : G_0 \rightarrow F$. Replacing F by $\ker \varphi$ we construct a sheaf $G_1 = \bigoplus_{(V,t) \in I_1} \rho_* k_V$ and an epimorphism $G_1 \rightarrow \ker \varphi$. Hence we get an exact sequence $G_1 \rightarrow G_0 \rightarrow F \rightarrow 0$. For $J_0 \subset I_0$ set for short $G_{J_0} = \bigoplus_{(U,s) \in J_0} \rho_* k_U$ and define similarly G_{J_1} . Set

$$J = \{(J_1, J_0); J_k \subset I_k, J_k \text{ is finite and } \text{im } \varphi|_{G_{J_1}} \subset G_{J_0}\}.$$

The category J is filtrant and $F \simeq \varinjlim_{(J_1, J_0) \in J} \text{coker}(G_{J_1} \rightarrow G_{J_0})$. □

PROPOSITION 1.1.12. Let $F \in \text{Mod}(k_{X_{sa}})$, and let $U \in \text{Op}(X)$. Then

$$\Gamma(U; \rho^{-1}F) \simeq \varprojlim_{V \subset \subset U, V \in \text{Op}^c(X_{sa})} \Gamma(V; F)$$

PROOF. By Theorem 1.1.11 we may assume $F = \varinjlim_i \rho_* F_i$, with $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. Then $\rho^{-1}F \simeq \varinjlim_i \rho^{-1}\rho_* F_i \simeq \varinjlim_i F_i$. We have the chain of isomorphisms

$$\begin{aligned} \Gamma(U; \rho^{-1}F) &\simeq \varprojlim_{V \subset \subset U} \Gamma(\overline{V}; \rho^{-1}F) &\simeq \varprojlim_{V \subset \subset U} \Gamma(\overline{V}; \varinjlim_i F_i) \\ &\simeq \varprojlim_{V \subset \subset U} \varinjlim_i \Gamma(\overline{V}; F_i) &\simeq \varprojlim_{V \subset \subset U} \varinjlim_i \Gamma(V; F_i) \\ &\simeq \varprojlim_{V \subset \subset U} \varinjlim_i \Gamma(V; \rho_* F_i) &\simeq \varprojlim_{V \subset \subset U} \Gamma(V; F), \end{aligned}$$

where $V \in \text{Op}^c(X_{sa})$. The third isomorphism follows since \overline{V} is compact and the last isomorphism follows from Proposition 1.1.3. □

PROPOSITION 1.1.13. *One has $\rho^{-1} \circ \rho_* \xrightarrow{\sim} \text{id}$, in particular the functor ρ_* is fully faithful.*

PROOF. Let $F \in \text{Mod}(k_X)$. Every $x \in X$ has a fundamental neighborhood system consisting of open subanalytic subsets. Hence we have the chain of isomorphisms

$$(\rho^{-1}\rho_*F)_x \simeq \varinjlim_{x \in U} \rho^{-1}\rho_*F(U) \simeq \varinjlim_{x \in U} \rho_*F(U) \simeq \varinjlim_{x \in U} F(U) \simeq F_x,$$

where U ranges through the family of open subanalytic neighborhoods of x . The second isomorphism follows since, by Proposition 1.1.12

$$\varinjlim_{x \in U} \rho^{-1}\rho_*F(U) \simeq \varinjlim_{x \in U} \varprojlim_{V \subset\subset U} \rho_*F(V) \simeq \varinjlim_{x \in U} \rho_*F(U).$$

□

Now we will describe a left adjoint to the functor ρ^{-1} .

PROPOSITION 1.1.14. *The functor ρ^{-1} admits a left adjoint, denoted by $\rho_!$. It satisfies*

(i) *for $F \in \text{Mod}(k_X)$ and $U \in \text{Op}(X_{sa})$, $\rho_!F$ is the sheaf associated to the presheaf $U \mapsto \Gamma(\overline{U}; F)$,*

(ii) *for $U \in \text{Op}(X)$ one has $\rho_!k_U \simeq \varinjlim_{V \subset\subset U, V \in \text{Op}^c(X_{sa})} k_V$.*

PROOF. Let $\tilde{F} \in \text{Psh}(k_{X_{sa}})$ be the presheaf $U \mapsto \Gamma(\overline{U}; F)$, and let $G \in \text{Mod}(k_{X_{sa}})$. We will construct morphisms

$$\text{Hom}_{\text{Psh}(k_{X_{sa}})}(\tilde{F}, G) \xrightleftharpoons[\mathcal{J}]{\xi} \text{Hom}_{k_X}(F, \rho^{-1}G).$$

To define ξ , let $\varphi: \tilde{F} \rightarrow G$ and $U \in \text{Op}(X)$. Then the morphism $\xi(\varphi)(U): F(U) \rightarrow \rho^{-1}G(U)$ is defined as follows

$$F(U) \simeq \varprojlim_{V \subset\subset U, V \in \text{Op}^c(X_{sa})} F(\overline{V}) \xrightarrow{\varphi} \varprojlim_{V \subset\subset U, V \in \text{Op}^c(X_{sa})} G(V) \simeq \rho^{-1}G(U).$$

On the other hand, let $\psi: F \rightarrow \rho^{-1}G$ and $U \in \text{Op}^c(X_{sa})$. Then the morphism $\mathcal{J}(\psi)(U): \tilde{F}(U) \rightarrow G(U)$ is defined as follows

$$\tilde{F}(U) \simeq \varinjlim_{U \subset\subset V \in \text{Op}^c(X_{sa})} F(V) \xrightarrow{\psi} \varinjlim_{U \subset\subset V \in \text{Op}^c(X_{sa})} \rho^{-1}G(V) \rightarrow G(U).$$

By construction one can check that the morphisms ξ and \mathcal{A} are inverse to each others. Then (i) follows from the chain of isomorphisms

$$\mathrm{Hom}_{k_{X_{sa}}}(\widetilde{F}^{++}, G) \simeq \mathrm{Hom}_{\mathrm{Psh}(k_{X_{sa}})}(\widetilde{F}, G) \simeq \mathrm{Hom}_{k_{X_{sa}}}(F, \rho^{-1}G).$$

To show (ii), consider the following sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{k_{X_{sa}}}(\rho_!k_U, F) &\simeq \mathrm{Hom}_{k_X}(k_U, \rho^{-1}F) \\ &\simeq \varprojlim_{V \subset\subset U, V \in \mathrm{Op}^c(X_{sa})} \mathrm{Hom}_{k_{X_{sa}}}(k_V, F) \\ &\simeq \mathrm{Hom}_{k_{X_{sa}}}\left(\varinjlim_{V \subset\subset U, V \in \mathrm{Op}^c(X_{sa})} k_V, F\right), \end{aligned}$$

where the second isomorphism follows from Proposition 1.1.12. □

PROPOSITION 1.1.15. *The functor $\rho_!$ is exact and commutes with \varinjlim and \otimes .*

PROOF. It follows by adjunction that $\rho_!$ is right exact and commutes with \varinjlim , so let us show that it is also left exact. With the notations of Proposition 1.1.14, let $F \in \mathrm{Mod}(k_X)$, and let $\widetilde{F} \in \mathrm{Psh}(k_{X_{sa}})$ be the presheaf $U \mapsto \Gamma(\overline{U}; F)$. Then $\rho_!F \simeq \widetilde{F}^{++}$, and the functors $F \mapsto \widetilde{F}$ and $G \mapsto G^{++}$ are left exact.

Let us show that $\rho_!$ commutes with \otimes . Let $F, G \in \mathrm{Mod}(k_X)$, the morphism

$$F(\overline{U}) \otimes G(\overline{U}) \rightarrow (F \otimes G)(\overline{U})$$

defines a morphism in $\mathrm{Mod}(k_{X_{sa}})$

$$\rho_!F \otimes \rho_!G \rightarrow \rho_!(F \otimes G)$$

by Proposition 1.1.14 (i). Since $\rho_!$ commutes with \varinjlim we may suppose that $F = k_U$ and $G = k_V$ and the result follows from Proposition 1.1.14 (ii). □

PROPOSITION 1.1.16. *The functor $\rho_!$ is fully faithful. In particular one has $\rho^{-1} \circ \rho_! \simeq \mathrm{id}$. Moreover, for $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{X_{sa}})$ one has*

$$\rho^{-1} \mathcal{H}om(\rho_!F, G) \simeq \mathcal{H}om(F, \rho^{-1}G).$$

PROOF. For $F, G \in \mathrm{Mod}(k_X)$ we have by adjunction

$$\mathrm{Hom}_{k_X}(\rho^{-1}\rho_!F, G) \simeq \mathrm{Hom}_{k_X}(F, \rho^{-1}\rho_*G) \simeq \mathrm{Hom}_{k_X}(F, G).$$

This also implies that $\rho_!$ is fully faithful, in fact

$$\mathrm{Hom}_{k_{X_{sa}}}(\rho_!F, \rho_!G) \simeq \mathrm{Hom}_{k_X}(F, \rho^{-1}\rho_!G) \simeq \mathrm{Hom}_{k_X}(F, G).$$

Now let $K, F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{X_{sa}})$, we have

$$\begin{aligned} \mathrm{Hom}_{k_X}(K, \rho^{-1}\mathcal{H}om(\rho_!F, G)) &\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_!K, \mathcal{H}om(\rho_!F, G)) \\ &\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_!K \otimes \rho_!F, G) \\ &\simeq \mathrm{Hom}_{k_{X_{sa}}}(\rho_!(K \otimes F), G) \\ &\simeq \mathrm{Hom}_{k_X}(K \otimes F, \rho^{-1}G) \\ &\simeq \mathrm{Hom}_{k_X}(K, \mathcal{H}om(F, \rho^{-1}G)) \end{aligned}$$

and the result follows. □

1.2 – Operations on the subanalytic site.

Let X, Y be two real analytic manifolds, and let $f : X \rightarrow Y$ be a real analytic map. This defines a morphism of sites $f : X_{sa} \rightarrow Y_{sa}$. We have a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \rho & & \downarrow \rho \\ X_{sa} & \xrightarrow{f} & Y_{sa}. \end{array}$$

The following functors are always well defined on a site

$$\mathcal{H}om : \mathrm{Mod}(k_{X_{sa}})^{op} \times \mathrm{Mod}(k_{X_{sa}}) \rightarrow \mathrm{Mod}(k_{X_{sa}}),$$

$$\otimes : \mathrm{Mod}(k_{X_{sa}}^{op}) \times \mathrm{Mod}(k_{X_{sa}}) \rightarrow \mathrm{Mod}(k_{X_{sa}}),$$

$$f_* : \mathrm{Mod}(k_{X_{sa}}) \rightarrow \mathrm{Mod}(k_{Y_{sa}}),$$

$$f^{-1} : \mathrm{Mod}(k_{Y_{sa}}) \rightarrow \mathrm{Mod}(k_{X_{sa}}).$$

Let us summarize their properties:

- the functor $\mathcal{H}om$ is left exact and commutes with ρ_* ,
- the functor \otimes is exact and commutes with \varinjlim , ρ^{-1} and $\rho_!$,
- the functor f_* is left exact and commutes with ρ_* and \varinjlim ,
- the functor f^{-1} is exact and commutes with \varinjlim , \otimes and ρ^{-1} ,
- (f^{-1}, f_*) is a pair of adjoint functors.

Let Z be a subanalytic locally closed subset of X . As in classical sheaf theory we define

$$\begin{aligned} \Gamma_Z : \text{Mod}(k_{X_{sa}}) &\rightarrow \text{Mod}(k_{X_{sa}}) \\ F &\mapsto \mathcal{H}om(\rho_*k_Z, F) \\ (\cdot)_Z : \text{Mod}(k_{X_{sa}}) &\rightarrow \text{Mod}(k_{X_{sa}}) \\ F &\mapsto F \otimes \rho_*k_Z. \end{aligned}$$

We have

- the functor Γ_Z is left exact and commutes with ρ_* and $\overleftarrow{\lim}$,
- the functor $(\cdot)_Z$ is exact and commutes with $\overrightarrow{\lim}$, \otimes and ρ^{-1} ,
- $((\cdot)_Z, \Gamma_Z)$ is a pair of adjoint functors.

1.3 – \mathbb{R} -constructible sheaves on subanalytic sites.

Let us consider the category $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. We prove that the subcategory of $\text{Mod}(k_{X_{sa}})$ consisting of \mathbb{R} -constructible sheaves is stable under inverse image and tensor product.

PROPOSITION 1.3.1. *Let $F, G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Then $\rho_*(F \otimes G) \simeq \rho_*F \otimes \rho_*G$.*

PROOF. We may reduce to the case $F = k_U$, $G = k_V$ with $U, V \in \text{Op}(X_{sa})$. In this case $\rho_*k_{U \cap V} \simeq \rho_*k_U \otimes \rho_*k_V$ by Proposition 1.1.7. \square

COROLLARY 1.3.2. *Let $F \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$, and let Z be a subanalytic locally closed subset of X . Then $\rho_*F_Z \simeq (\rho_*F)_Z$.*

Let X, Y be two real analytic manifolds, and let $f : X \rightarrow Y$ be a real analytic map.

PROPOSITION 1.3.3. *Let $f : X \rightarrow Y$ be a real analytic map. Let $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_Y)$. Then $\rho_*f^{-1}G \simeq f^{-1}\rho_*G$.*

PROOF. Since the functor f^{-1} is exact, we may reduce to the case $G = k_V$, with $V \in \text{Op}(Y_{sa})$. In this case we have $\rho_*f^{-1}k_V \simeq \rho_*k_{f^{-1}(V)} \simeq f^{-1}\rho_*k_V$, where the last isomorphism follows from Proposition 1.1.7. \square

We apply the above results to calculate the functor $\mathcal{H}om$ in the category $\text{Mod}(k_{X_{sa}})$.

PROPOSITION 1.3.4. *Let $F = \varinjlim_i F_i$, with $F_i \in \text{Mod}(k_{X_{sa}})$ and let $G = \varinjlim_j \rho_* G_j$ with $G_j \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. One has*

$$\text{Hom}(G, F) \simeq \varprojlim_j \varinjlim_i \text{Hom}(\rho_* G_j, F_i).$$

PROOF. For each $U \in \text{Op}^c(X_{sa})$ one has the isomorphisms

$$\begin{aligned} \Gamma(U, \text{Hom}(G, F)) &\simeq \text{Hom}_{k_{X_{sa}}}(G_U, F) \\ &\simeq \varprojlim_j \varinjlim_i \text{Hom}_{k_{X_{sa}}}(\rho_* G_{jU}, F_i) \\ &\simeq \varprojlim_j \varinjlim_i \Gamma(U; \text{Hom}(\rho_* G_j, F_i)) \\ &\simeq \Gamma(U; \varprojlim_j \varinjlim_i \text{Hom}(\rho_* G_j, F_i)). \end{aligned}$$

In the second isomorphism we used Corollary 1.3.2, and the last isomorphism follows from Proposition 1.1.3 and because $\Gamma(U; \cdot)$ commutes with \varinjlim . □

COROLLARY 1.3.5. *Let $F = \varinjlim_i \rho_* F_i, G = \varinjlim_j \rho_* G_j$ with $F_i, G_j \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. One has*

$$\text{Hom}(G, F) \simeq \varprojlim_j \varinjlim_i \rho_* \text{Hom}(G_j, F_i).$$

PROOF. It follows from the fact that Hom commutes with ρ_* and from Proposition 1.3.4. □

COROLLARY 1.3.6. *Let $F = \varinjlim_i \rho_* F_i$, with $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}^c(X)$ be a sheaf on X_{sa} . Let Z be a subanalytic locally closed subset of X . Then $\Gamma_Z F \simeq \varinjlim_i \rho_* \Gamma_Z F_i$.*

1.4 – Proper direct image on $\text{Mod}(k_{X_{sa}})$.

In [7] the authors defined the functor $f_{!!}$ of proper direct image using ind-sheaves. Here we give a direct construction:

$$\begin{aligned} f_{!!} : \text{Mod}(k_{X_{sa}}) &\rightarrow \text{Mod}(k_{Y_{sa}}) \\ F &\mapsto \varinjlim_U f_* F_U \simeq \varinjlim_K f_* \Gamma_K F \end{aligned}$$

where U ranges through the family of relatively compact open subanalytic

subsets of X and K ranges through the family of subanalytic compact subsets of X . One shall be aware that \varinjlim is taken in the category $\text{Mod}(k_{Y_{sa}})$. Let $V \in \text{Op}^c(Y_{sa})$. Then $\Gamma(\overline{V}; f_{!!}F) = \varinjlim_U \Gamma(f^{-1}(V); F_U) \simeq \varinjlim_K \Gamma(f^{-1}(V); \Gamma_K F)$, where U ranges through the family of relatively compact open subanalytic subsets of X and K ranges through the family of subanalytic compact subsets of X . If f is proper on $\text{supp}(F)$ then $f_* \simeq f_{!!}$ and in this case $f_{!!} \circ \rho_* \simeq \rho_* \circ f_!$.

REMARK 1.4.1. *Remark that $f_{!!} \circ \rho_* \not\simeq \rho_* \circ f_!$ in general. Indeed let $V \in \text{Op}^c(Y_{sa})$, then*

$$\begin{aligned} \Gamma(V; f_{!!}\rho_*F) &= \varinjlim_K \Gamma(f^{-1}(V); \Gamma_K F), \\ \Gamma(V; \rho_*f_!F) &= \varinjlim_Z \Gamma(f^{-1}(V); \Gamma_Z F), \end{aligned}$$

where Z ranges through the family of closed subsets of $f^{-1}(V)$ such that $f|_Z : Z \rightarrow V$ is proper. Then

$$\begin{aligned} \Gamma(V; f_{!!}\rho_*F) &= \{s \in \Gamma(f^{-1}(V); F); \overline{\text{supp}(s)} \text{ is compact in } X\}, \\ \Gamma(V; \rho_*f_!F) &= \{s \in \Gamma(f^{-1}(V); F); f : \text{supp}(s) \rightarrow V \text{ is proper}\}. \end{aligned}$$

For example, let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection on the first coordinate, and let $V = (a, b) \in \text{Op}^c(\mathbb{R}_{sa})$. Suppose that $\overline{\text{supp}(s)} = \{(x, y) \in (a, b) \times \mathbb{R}, y = \frac{1}{(x-a)(b-x)}\}$. Then $f : \text{supp}(s) \rightarrow V$ is proper but $\overline{\text{supp}(s)}$ is not compact.

PROPOSITION 1.4.2. *The functor $f_{!!}$ commutes with filtrant \varinjlim . Moreover $\rho^{-1} \circ f_{!!} \simeq f_! \circ \rho^{-1}$.*

PROOF. Let us show that $f_{!!}$ commutes with filtrant \varinjlim . Let $V \in \text{Op}^c(Y_{sa})$ and let $\{F_i\}_i$ be a filtrant inductive system in $\text{Mod}(k_{X_{sa}})$. Then

$$\begin{aligned} \varinjlim_K \text{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)}, \Gamma_K \varinjlim_i F_i) &\simeq \varinjlim_K \text{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V) \cap K}, \varinjlim_i F_i) \\ &\simeq \varinjlim_{i, K} \text{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V) \cap K}, F_i) \\ &\simeq \varinjlim_{i, K} \text{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)}, \Gamma_K F_i) \\ &\simeq \varinjlim_i \text{Hom}_{k_{Y_{sa}}}(k_V, f_{!!}F_i) \\ &\simeq \text{Hom}_{k_{Y_{sa}}}(k_V, \varinjlim_i f_{!!}F_i), \end{aligned}$$

where the second isomorphism follows from the fact that $k_{f^{-1}(V) \cap K} \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$.

Let us show $\rho^{-1} \circ f_{!!} \simeq f_{!} \circ \rho^{-1}$. Let $F = \varinjlim_i \rho_* F_i$. Since $f_{!!}$ commutes with filtrant \varinjlim and F_i has compact support for each i we have $f_{!!} F = \varinjlim_i \rho_* f_{!} F_i$. We have the chain of isomorphisms

$$\begin{aligned} f_{!} \rho^{-1} \varinjlim_i \rho_* F_i &\simeq f_{!} \varinjlim_i \rho^{-1} \rho_* F_i \simeq f_{!} \varinjlim_i F_i \simeq \varinjlim_i f_{!} F_i \\ &\simeq \varinjlim_i \rho^{-1} \rho_* f_{!} F_i \simeq \rho^{-1} \varinjlim_i \rho_* f_{!} F_i. \end{aligned}$$

□

PROPOSITION 1.4.3. *The functor f_* commutes with ρ^{-1} .*

PROOF. Let $F \in \text{Mod}(k_{X_{sa}})$. Then $f_* F \simeq \varprojlim_K f_* F_K$, where K ranges through the family of subanalytic compact subsets of X . We have the chain of isomorphisms

$$\begin{aligned} f_* \rho^{-1} F &\simeq \varprojlim_K f_*(\rho^{-1} F)_K \simeq \varprojlim_K f_{!!}(\rho^{-1} F)_K \simeq \varprojlim_K f_{!!} \rho^{-1} F_K \\ &\simeq \varprojlim_K \rho^{-1} f_{!} F_K \simeq \rho^{-1} \varprojlim_K f_{!} F_K \simeq \rho^{-1} \varprojlim_K f_* F_K \simeq \rho^{-1} f_* F, \end{aligned}$$

where the second and the sixth isomorphism follow from the fact that f is proper on a compact subset of X . □

COROLLARY 1.4.4. *The functor f^{-1} commutes with $\rho_!$.*

PROOF. It follows immediately by adjunction. □

PROPOSITION 1.4.5. *Let $F \in \text{Mod}(k_{X_{sa}})$ and $G \in \text{Mod}(k_{Y_{sa}})$. Then*

$$f_{!!} F \otimes G \simeq f_{!!}(F \otimes f^{-1} G).$$

PROOF. Let $F = \varinjlim_i \rho_* F_i$, $G = \varinjlim_j \rho_* G_j$. The functors \otimes , $f_{!!}$ and f^{-1} commute with \varinjlim . Moreover $\text{supp}(F_i \otimes f^{-1} G_j)$ is compact for each i, j ,

hence f is proper on it. Then

$$\begin{aligned} f_{!!} \varinjlim_i \rho_* F_i \otimes \varinjlim_j \rho_* G_j &\simeq \varinjlim_{i,j} \rho_*(f_i F_i \otimes G_j) \\ &\simeq \varinjlim_{i,j} \rho_*(f_i(F_i \otimes f^{-1} G_j)) \\ &\simeq f_{!!}(\varinjlim_i \rho_* F_i \otimes f^{-1} \varinjlim_i \rho_* G_j). \end{aligned}$$

In the first isomorphism we used Proposition 1.3.1 and in the last one we used Propositions 1.3.1 and 1.3.3. \square

Now let us consider a cartesian square

$$\begin{array}{ccc} X'_{sa} & \xrightarrow{f'} & Y'_{sa} \\ \downarrow g' & & \downarrow g \\ X_{sa} & \xrightarrow{f} & Y_{sa} \end{array}$$

PROPOSITION 1.4.6. *Let $F \in \text{Mod}(k_{X_{sa}})$. Then $g^{-1} f_{!!} F \simeq f'_{!!} g'^{-1} F$.*

PROOF. Let $F = \varinjlim_i \rho_* F_i$. All the functors in the above formula commute with \varinjlim_i . Moreover since $\text{supp}(F_i)$ is compact, f' is proper on $\text{supp}(g'^{-1} F_i)$ for each i . Then

$$g^{-1} f_{!!} \varinjlim_i \rho_* F_i \simeq \varinjlim_i \rho_* g^{-1} f_{!!} F_i \simeq \varinjlim_i \rho_* f'_i g'^{-1} F_i \simeq f'_{!!} g'^{-1} \varinjlim_i \rho_* F_i,$$

where the first and the last isomorphisms follow from Proposition 1.3.3. \square

The following isomorphism is the analogue for subanalytic sheaves of Corollary 4.3.15 of [7].

PROPOSITION 1.4.7. *Let $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_Y)$ and let $F \in \text{Mod}(k_{X_{sa}})$. Then the natural morphism*

$$f_{!!} \mathcal{H}om(f^{-1} G, F) \rightarrow \mathcal{H}om(G, f_{!!} F)$$

is an isomorphism.

PROOF. Let us construct the morphism. By adjunction we have

$$f^{-1} G \otimes \mathcal{H}om(f^{-1} G, F) \rightarrow F,$$

hence, using the projection formula we get

$$G \otimes f_{!!} \mathcal{H}om(f^{-1}G, F) \simeq f_{!!}(f^{-1}G \otimes \mathcal{H}om(f^{-1}G, F)) \rightarrow f_{!!}F,$$

then by adjunction we obtain the desired morphism. Let us show that it is an isomorphism. We have the chain of isomorphisms

$$\begin{aligned} f_{!!} \mathcal{H}om(f^{-1}G, F) &\simeq \varinjlim_K f_* \Gamma_K \mathcal{H}om(f^{-1}G, F) \\ &\simeq \varinjlim_K f_* \mathcal{H}om(f^{-1}G, \Gamma_K F) \\ &\simeq \varinjlim_K \mathcal{H}om(G, f_* \Gamma_K F) \\ &\simeq \mathcal{H}om(G, \varinjlim_K f_* \Gamma_K F) \\ &\simeq \mathcal{H}om(G, f_{!!}F), \end{aligned}$$

where the fourth isomorphism follows from Proposition 1.3.4. □

1.5 – Quasi-injective objects.

Let us introduce a category which is useful in order to find acyclic objects with respect to the functors defined in the previous Sections. Although the definition is inspired to the definition of quasi-injective objects of [7] (or, more generally, to the definition of quasi-injective objects of a ind-category, see [8] for more details), the proofs of Theorems 1.5.4 and 1.5.16 are independent of the theory of ind-objects.

DEFINITION 1.5.1. *An object $F \in \text{Mod}(k_{X_{sa}})$ is quasi-injective if the functor $\text{Hom}_{k_{X_{sa}}}(\cdot, F)$ is exact in $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ or, equivalently (see Theorem 8.7.2 of [8]) if for each $U, V \in \text{Op}^c(X_{sa})$ with $V \subset U$ the restriction morphism $\Gamma(U; F) \rightarrow \Gamma(V; F)$ is surjective.*

It follows from the definition that injective sheaves belong to $\mathcal{J}_{X_{sa}}$. This implies that $\mathcal{J}_{X_{sa}}$ is cogenerating. Moreover the category $\mathcal{J}_{X_{sa}}$ is stable by filtrant \varinjlim and \coprod .

PROPOSITION 1.5.2. *Let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence in $\text{Mod}(k_{X_{sa}})$ and assume that F' is quasi-injective. Let $U \in \text{Op}^c(X_{sa})$.*

Then the sequence

$$0 \rightarrow \Gamma(U; F') \rightarrow \Gamma(U; F) \rightarrow \Gamma(U; F'') \rightarrow 0$$

is exact.

PROOF. Let $s'' \in \Gamma(U; F'')$, and let $\{V_i\}_{i=1}^n$ be a finite covering of U such that there exists $s_i \in \Gamma(V_i; F)$ whose image is $s''|_{V_i}$. For $n \geq 2$ on $V_1 \cap V_2$ $s_1 - s_2$ defines a section of $\Gamma(V_1 \cap V_2; F')$ which extends to $s' \in \Gamma(U; F')$. Replace s_1 with $s_1 - s'$. We may suppose that $s_1 = s_2$ on $V_1 \cap V_2$. Then there exists $t \in \Gamma(V_1 \cup V_2)$ such that $t|_{V_i} = s_i$, $i = 1, 2$. Thus the induction proceeds. \square

PROPOSITION 1.5.3. *Let $F', F, F'' \in \text{Mod}(k_{X_{sa}})$, and consider the exact sequence*

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0.$$

Suppose that $F', F \in \mathcal{J}_{X_{sa}}$. Then $F'' \in \mathcal{J}_{X_{sa}}$.

PROOF. Let $U, V \in \text{Op}^c(X_{sa})$ with $V \subset U$ and let us consider the diagram below

$$\begin{array}{ccc} \Gamma(U; F) & \longrightarrow & \Gamma(U; F'') \\ \downarrow \alpha & & \downarrow \gamma \\ \Gamma(V; F) & \xrightarrow{\beta} & \Gamma(V; F''). \end{array}$$

The morphism α is surjective since F is quasi-injective and β is surjective by Proposition 1.5.2. Then γ is surjective. \square

THEOREM 1.5.4. *The family of quasi-injective sheaves is injective with respect to the functor $\text{Hom}_{k_{X_{sa}}}(G, \cdot)$ for each $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$.*

PROOF. (i) Let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence in $\text{Mod}(k_{X_{sa}})$ and assume that $F'' \in \mathcal{J}_{X_{sa}}$. Let $G \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. We have to show that the sequence

$$0 \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F') \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F) \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F'') \rightarrow 0$$

is exact. There is an epimorphism $\varphi : \oplus_{i \in I} k_{U_i} \rightarrow G$ where I is finite and $U_i \in \text{Op}^c(X_{sa})$ for each $i \in I$.

The sequence $0 \rightarrow \ker \varphi \rightarrow \oplus_{i \in I} k_{U_i} \rightarrow G \rightarrow 0$ is exact. We set for short $G_1 = \ker \varphi$ and $G_2 = \oplus_{i \in I} k_{U_i}$. We get the following diagram where the first

column is exact

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G, F') & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G, F) & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G, F'') \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_2, F') & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_2, F) & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_2, F'') \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_1, F') & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_1, F) & \longrightarrow & \mathrm{Hom}_{k_{X_{sa}}}(G_1, F'') \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The second row is exact by Proposition 1.5.2, hence the top row is exact by the snake lemma.

(ii) Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence in $\mathrm{Mod}(k_{X_{sa}})$ with $F' \in \mathcal{J}_{X_{sa}}$. Let $\{V_n\}_{n \in \mathbb{N}} \in \mathrm{Cov}(X_{sa})$ such that $V_n \subset\subset V_{n+1}$. By (i), all the sequences

$$0 \rightarrow \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F') \rightarrow \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F) \rightarrow \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F'') \rightarrow 0$$

are exact. Moreover since $F' \in \mathcal{J}_{X_{sa}}$ the morphism $\mathrm{Hom}_{k_{X_{sa}}}(G_{V_{n+1}}, F') \rightarrow \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F')$ is surjective for all n . Then by the Mittag-Leffler property (see Proposition 1.12.3 of [5]) the sequence

$$0 \rightarrow \varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F') \rightarrow \varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F) \rightarrow \varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, F'') \rightarrow 0$$

is exact. Since $\varprojlim_n \mathrm{Hom}_{k_{X_{sa}}}(G_{V_n}, \cdot) \simeq \mathrm{Hom}_{k_{X_{sa}}}(G, \cdot)$ the result follows. \square

PROPOSITION 1.5.5. *Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Then quasi-injective sheaves are injective with respect to the functor $\mathcal{H}om(G, \cdot)$.*

PROOF. Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. It is enough to check that for each $U \in \mathrm{Op}(X_{sa})$ and each exact sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ with $F' \in \mathcal{J}_{X_{sa}}$, the sequence

$$0 \rightarrow \Gamma(U; \mathcal{H}om(G, F')) \rightarrow \Gamma(U; \mathcal{H}om(G, F)) \rightarrow \Gamma(U; \mathcal{H}om(G, F'')) \rightarrow 0$$

is exact. We have $\Gamma(U, \mathcal{H}om(G, \cdot)) \simeq \mathrm{Hom}_{k_{X_{sa}}}(G_U, \cdot)$, and quasi-injective objects are injective with respect to the functor $\mathrm{Hom}_{k_{X_{sa}}}(G_U, \cdot)$ for each $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$, and for each $U \in \mathrm{Op}(X_{sa})$. \square

COROLLARY 1.5.6. *Quasi-injective sheaves are injective with respect to the functor Γ_Z for each locally closed subanalytic subset Z of X .*

COROLLARY 1.5.7. *Let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. Then the functor $\mathcal{H}om(\cdot, F)$ is exact on $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$.*

PROOF. Let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. There is an isomorphism of functors $\Gamma(U; \mathcal{H}om(\cdot, F)) \simeq \text{Hom}_{k_{X_{sa}}}((\cdot)_U, F)$ for each $U \in \text{Op}(X_{sa})$. The functor $\text{Hom}_{k_{X_{sa}}}((\cdot)_U, F)$ is exact on $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and the result follows. \square

PROPOSITION 1.5.8. *Let $F \in \text{Mod}(k_{X_{sa}})$. Then F is quasi-injective if and only if $\mathcal{H}om(G, F)$ is quasi-injective for each $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$.*

PROOF. (i) Let F be quasi-injective, and let $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. We have $\text{Hom}_{k_{X_{sa}}}(\cdot, \mathcal{H}om(G, F)) \simeq \text{Hom}_{k_{X_{sa}}}(\cdot \otimes G, F)$, and $\text{Hom}_{k_{X_{sa}}}(\cdot \otimes G, F)$ is exact on $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$.

(ii) Suppose that $\mathcal{H}om(G, F)$ is quasi-injective for each $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. The result follows by setting $G = k_X$. \square

COROLLARY 1.5.9. *The functor Γ_Z sends quasi-injective objects to quasi-injective objects for each locally closed subanalytic subset Z of X .*

Let $f : X \rightarrow Y$ be a morphism of real analytic manifolds.

PROPOSITION 1.5.10. *Quasi-injective sheaves are injective with respect to the functor f_* . The functor f_* sends quasi-injective objects to quasi-injective objects.*

PROOF. (i) Let us consider $V \in \text{Op}(Y_{sa})$. There is an isomorphism of functors $\Gamma(V; f_*(\cdot)) \simeq \Gamma(f^{-1}(V); \cdot)$. It follows from Proposition 1.5.4 that $\mathcal{J}_{X_{sa}}$ is injective with respect to the functor $\Gamma(f^{-1}(V); \cdot) \simeq \text{Hom}_{k_{X_{sa}}}(k_{f^{-1}(V)}, \cdot)$ for any $V \in \text{Op}(Y_{sa})$.

(ii) Let $F \in \mathcal{J}_{X_{sa}}$. For each $G \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_Y)$ we have $\text{Hom}_{k_{Y_{sa}}}(G, f_*F) \simeq \text{Hom}_{k_{X_{sa}}}(f^{-1}G, F)$. Since f^{-1} is exact and sends $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_Y)$ to $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$, Proposition 1.5.4 implies that the functor $\text{Hom}_{k_{X_{sa}}}(f^{-1}(\cdot), F)$ is exact on $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_Y)$. \square

PROPOSITION 1.5.11. *The family of quasi-injective sheaves is $f_{!!}$ -injective. The functor $f_{!!}$ sends quasi-injective objects to quasi-injective objects.*

PROOF. (i) Let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence in $\text{Mod}(k_{X_{sa}})$ and assume that $F' \in \mathcal{J}_{X_{sa}}$. We have to check that the sequence $0 \rightarrow f_{!!}F' \rightarrow f_{!!}F \rightarrow f_{!!}F'' \rightarrow 0$ is exact. Since $F' \in \mathcal{J}_{X_{sa}}$, we have $\Gamma_K F' \in \mathcal{J}_{X_{sa}}$. Moreover $\mathcal{J}_{X_{sa}}$ is injective with respect to Γ_K and f_* . This implies that the sequence

$$0 \rightarrow f_*\Gamma_K F' \rightarrow f_*\Gamma_K F \rightarrow f_*\Gamma_K F'' \rightarrow 0$$

is exact. Applying the exact functor \varinjlim_K we find that the sequence

$$0 \rightarrow \varinjlim_K f_*\Gamma_K F' \rightarrow \varinjlim_K f_*\Gamma_K F \rightarrow \varinjlim_K f_*\Gamma_K F'' \rightarrow 0$$

is exact.

(ii) Let K be a compact subanalytic subset of X . The functors Γ_K and f_* send quasi-injective objects to quasi-injective objects, then $f_*\Gamma_K F \in \mathcal{J}_{Y_{sa}}$. Since $\mathcal{J}_{Y_{sa}}$ is stable by filtrant \varinjlim , the result follows. \square

Let S be a closed subanalytic subset of X and let $i_S : S \hookrightarrow X$ be the closed embedding. Let $F = \varinjlim_i \rho_* F_i \in \text{Mod}(k_{X_{sa}})$ with $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. We have $F_S \simeq \varinjlim_i \rho_* F_{iS} \simeq \varinjlim_i \rho_* i_{S*} i_S^{-1} F_i \simeq i_{S*} i_S^{-1} F$.

LEMMA 1.5.12. *Let S be a closed subanalytic subset of X and let $U \in \text{Op}^c(X_{sa})$. Let $F \in \text{Mod}(k_{X_{sa}})$. Then $\Gamma(U; F_S) \simeq \varinjlim_{V \supset S \cap U} \Gamma(V; F)$, with $V \in \text{Op}^c(X_{sa})$.*

PROOF. Let $F \in \text{Mod}(k_{X_{sa}})$. Then $F \simeq \varinjlim_i \rho_* F_i$ with $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. We have the chain of isomorphisms

$$\begin{aligned} \Gamma(U; F_S) &\simeq \varinjlim_i \Gamma(U; F_{iS}) \\ &\simeq \varinjlim_{i, V \supset S \cap U} \Gamma(V; F_i) \\ &\simeq \varinjlim_{V \supset S \cap U} \Gamma(V; F), \end{aligned}$$

where V ranges through the family of relatively compact open subanalytic subsets of X containing $S \cap U$. The second isomorphism follows since F_i is \mathbb{R} -constructible for each i . \square

REMARK 1.5.13. *The fact that $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ is locally constant on a subanalytic stratification of X plays an essential role. In fact the iso-*

morphism is not true in general in $\text{Mod}(k_X)$, since the family of relatively compact open subanalytic subsets of X containing $S \cap U$ is not cofinal to the family of open subsets of X containing $S \cap U$.

PROPOSITION 1.5.14. *Let S be a closed subanalytic subset of X and let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. Then F_S is quasi-injective.*

PROOF. Let $U, V \in \text{Op}^c(X_{sa})$ with $V \subset U$. Since F is quasi-injective and inductive limits are right exact, the morphism $\varinjlim_{U' \supset S \cap U} \Gamma(U'; F) \rightarrow \varinjlim_{V' \supset S \cap V} \Gamma(V'; F)$ with $V', U' \in \text{Op}^c(X_{sa})$, is surjective. Hence by Lemma 1.5.12 the morphism $\Gamma(U; F_S) \rightarrow \Gamma(V; F_S)$ is surjective and the result follows. \square

Recall that $F \in \text{Mod}(k_X)$ is c-soft if the natural morphism $\Gamma(X; F) \rightarrow \Gamma(K, F)$ is surjective for each compact $K \subset X$. If F is c-soft and Z is a locally closed subset of X , then F_Z is c-soft. Moreover c-soft sheaves are $\Gamma(U; \cdot)$ -injective for each $U \in \text{Op}(X)$.

PROPOSITION 1.5.15. *Let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. Then $\rho^{-1}F$ is c-soft.*

PROOF. Let K be a compact subset of X . Recall that if $U \in \text{Op}(X)$ then $\Gamma(U; \rho^{-1}F) \simeq \varprojlim_{V \subset\subset U} \Gamma(V; F)$, where $V \in \text{Op}(X_{sa})$. We have the chain of isomorphisms

$$\begin{aligned} \Gamma(K; \rho^{-1}F) &\simeq \varinjlim_U \Gamma(U; \rho^{-1}F) \\ &\simeq \varinjlim_U \varprojlim_{V \subset\subset U} \Gamma(V; F) \\ &\simeq \varinjlim_U \Gamma(U; F) \end{aligned}$$

where U ranges through the family of subanalytic relatively compact open subsets of X containing K and $V \in \text{Op}(X_{sa})$.

Since F is quasi-injective and filtrant inductive limits are exact, the morphism $\Gamma(X; \rho^{-1}F) \simeq \Gamma(X; F) \rightarrow \varinjlim_U \Gamma(U; F) \simeq \Gamma(K; \rho^{-1}F)$, where U ranges through the family of subanalytic open subsets of X containing K , is surjective. \square

Let us consider the following subcategory of $\text{Mod}(k_{X_{sa}})$:

$$\mathcal{P}_{X_{sa}} := \{G \in \text{Mod}(k_{X_{sa}}); G \text{ is } \text{Hom}_{k_{X_{sa}}}(\cdot, F)\text{-acyclic for each } F \in \mathcal{J}_{X_{sa}}\}.$$

This category is generating, in fact if $\{G_j\}_j$ are \mathbb{R} -constructible sheaves $\bigoplus_j \rho_* G_j \in \mathcal{P}_{X_{sa}}$ by Theorem 1.5.4. Moreover $\mathcal{P}_{X_{sa}}$ is stable by $\cdot \otimes K$, where $K \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. In fact if $G \in \mathcal{P}_{X_{sa}}$ and $F \in \mathcal{J}_{X_{sa}}$ we have

$$\text{Hom}_{k_{X_{sa}}}(G \otimes K, F) \simeq \text{Hom}_{k_{X_{sa}}}(G, \mathcal{H}om(K, F))$$

and $\mathcal{H}om(K, F) \in \mathcal{J}_{X_{sa}}$ by Proposition 1.5.8.

THEOREM 1.5.16. *The category $(\mathcal{P}_{X_{sa}}^{op}, \mathcal{J}_{X_{sa}})$ is injective with respect to the functor $\text{Hom}_{k_{X_{sa}}}(\cdot, \cdot)$.*

PROOF. (i) Let $G \in \mathcal{P}_{X_{sa}}$ and consider an exact sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ with $F' \in \mathcal{J}_{X_{sa}}$. We have to prove that the sequence

$$0 \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F') \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F) \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F'') \rightarrow 0$$

is exact. Since the functor $\text{Hom}_{k_{X_{sa}}}(G, \cdot)$ is acyclic on quasi-injective sheaves we obtain the result.

(ii) Let $F \in \mathcal{J}_{X_{sa}}$, and let $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ be an exact sequence on $\mathcal{P}_{X_{sa}}$. Since the objects of $\mathcal{P}_{X_{sa}}$ are $\text{Hom}_{k_{X_{sa}}}(\cdot, F)$ -acyclic the sequence

$$0 \rightarrow \text{Hom}_{k_{X_{sa}}}(G'', F) \rightarrow \text{Hom}_{k_{X_{sa}}}(G, F) \rightarrow \text{Hom}_{k_{X_{sa}}}(G', F) \rightarrow 0$$

is exact. □

COROLLARY 1.5.17. *The category $(\mathcal{P}_{X_{sa}}^{op}, \mathcal{J}_{X_{sa}})$ is injective with respect to the functor $\mathcal{H}om(\cdot, \cdot)$.*

PROOF. Let $G \in \mathcal{P}_{X_{sa}}$, and let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be an exact sequence with $F' \in \mathcal{J}_{X_{sa}}$. We shall show that for each $U \in \text{Op}(X_{sa})$ the sequence

$$0 \rightarrow \Gamma(U; \mathcal{H}om(G, F')) \rightarrow \Gamma(U; \mathcal{H}om(G, F)) \rightarrow \Gamma(U; \mathcal{H}om(G, F'')) \rightarrow 0$$

is exact. This is equivalent to show that for each $U \in \text{Op}(X_{sa})$ the sequence

$$0 \rightarrow \text{Hom}_{k_{X_{sa}}}(G_U, F') \rightarrow \text{Hom}_{k_{X_{sa}}}(G_U, F) \rightarrow \text{Hom}_{k_{X_{sa}}}(G_U, F'') \rightarrow 0$$

is exact. This follows since $G_U \in \mathcal{P}_{X_{sa}}$. The proof of the exactness in $\mathcal{P}_{X_{sa}}^{op}$ is similar. □

1.6 – The functor $\rho_!$.

We have seen in § 1.1 that the functor $\rho^{-1} : \text{Mod}(k_{X_{sa}}) \rightarrow \text{Mod}(k_X)$ has a left adjoint $\rho_! : \text{Mod}(k_X) \rightarrow \text{Mod}(k_{X_{sa}})$. The functor $\rho_!$ is fully faithful and exact. In particular, for $U \in \text{Op}(X)$ one has $\rho_!k_U \simeq \varinjlim_{V \subset\subset U} \rho_*k_V$, where $V \in \text{Op}(X_{sa})$.

PROPOSITION 1.6.1. *Let S be a closed subset of X . Then $\rho_!k_S \simeq \varinjlim_{W \supset S} \rho_*k_{\overline{W}}$, where $W \in \text{Op}(X_{sa})$.*

PROOF. (i) Let $U = X \setminus S$. Since $\rho_!$ is exact we have an exact sequence

$$0 \rightarrow \rho_!k_U \rightarrow \rho_!k_X \rightarrow \rho_!k_S \rightarrow 0.$$

On the other hand, let $V \in \text{Op}^c(X_{sa})$ and $V \subset\subset U$. We have an exact sequence $0 \rightarrow k_V \rightarrow k_X \rightarrow k_{X \setminus V} \rightarrow 0$. Since ρ_* is exact on $\text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ the sequence $0 \rightarrow \rho_*k_V \rightarrow \rho_*k_X \rightarrow \rho_*k_{X \setminus V} \rightarrow 0$ is exact. Applying the exact $\varinjlim_{V \subset\subset U}$ we obtain an exact sequence

$$0 \rightarrow \varinjlim_{V \subset\subset U} \rho_*k_V \rightarrow \rho_*k_X \rightarrow \varinjlim_{V \subset\subset U} \rho_*k_{X \setminus V} \rightarrow 0.$$

We have $\varinjlim_{V \subset\subset U} \rho_*k_V \simeq \rho_!k_U$ and $\rho_*k_X \simeq \rho_!k_X$. Hence $\rho_!k_S \simeq \varinjlim_{V \subset\subset U} \rho_*k_{X \setminus V}$.

(ii) We shall show that for each $U' \in \text{Op}^c(X_{sa})$ the natural morphism

$$(1.3) \quad \varinjlim_{V \subset\subset U'} \Gamma(U'; k_{X \setminus V}) \rightarrow \varinjlim_{W \supset S} \Gamma(U'; k_{\overline{W}})$$

is an isomorphism. We shall see that for each $W \in \text{Op}(X_{sa})$ with $W \supset S$ there exists $W' \in \text{Op}(X_{sa})$ such that $X \setminus \overline{W'} \subset\subset U$ and $\overline{W} \cap U' = \overline{W'} \cap U'$. Set $W' = W \cup (X \setminus \overline{U'})$. Since U' is relatively compact, $X \setminus \overline{W'} \subset\subset U$, and $\overline{W} \cap U' = \overline{W'} \cap U'$ by construction. Then

$$\varinjlim_{V \subset\subset U'} \Gamma(U'; k_{X \setminus V}) \simeq \varinjlim_{(\alpha \setminus \overline{W}) \subset\subset U'} \Gamma(U'; k_{\overline{W}}) \simeq \varinjlim_{W \supset S} \Gamma(U'; k_{\overline{W}}).$$

□

NOTATIONS 1.6.2. *Let $Z = U \cap S$, where $U \in \text{Op}(X)$ and let S be a closed subset of X . Let $F \in \text{Mod}(k_{X_{sa}})$. We set for short ${}_Z F = F \otimes \rho_!k_Z$*

LEMMA 1.6.3. *Let $F \in \text{Mod}(k_{X_{sa}})$. Let $U \in \text{Op}(X)$ and let S be a closed subset of X .*

- (i) One has ${}_U F \simeq \varinjlim_{V \subset\subset U} F_V \simeq \varinjlim_{V \subset\subset U} \Gamma_{\overline{V}} F$, $V \in \text{Op}^c(X_{sa})$.
- (ii) One has ${}_S F \simeq \varinjlim_{W \supset S} F_{\overline{W}} \simeq \varinjlim_{W \supset S} \Gamma_W F$, $W \in \text{Op}(X_{sa})$.

PROOF. (i) The first isomorphism is obvious. Let us show the second isomorphism. We have the chain of isomorphisms

$$\varinjlim_{V \subset\subset U} F_V \xleftarrow{\sim} \varinjlim_{V, V' \subset\subset U} (\Gamma_{\overline{V'}} F)_V \xrightarrow{\sim} \varinjlim_{V' \subset\subset U} \Gamma_{\overline{V'}} F,$$

where V, V' range through the family of subanalytic open subsets of X .

The proof of (ii) is similar. □

PROPOSITION 1.6.4. *Let Z be a locally closed subset of X . Let $G \in \text{Mod}_{\mathbb{R}\text{-}c}(k_X)$ and $F \in \text{Mod}(k_{X_{sa}})$. Then ${}_Z \mathcal{H}om(G, F) \simeq \mathcal{H}om(G, {}_Z F)$.*

PROOF. (i) Let $U \in \text{Op}(X)$. We have the chain of isomorphisms

$$\begin{aligned} {}_U \mathcal{H}om(G, F) &\simeq \varinjlim_{V \subset\subset U} \Gamma_{\overline{V}} \mathcal{H}om(G, F) \\ &\simeq \varinjlim_{V \subset\subset U} \mathcal{H}om(G, \Gamma_{\overline{V}} F) \\ &\simeq \mathcal{H}om(G, \varinjlim_{V \subset\subset U} \Gamma_{\overline{V}} F) \\ &\simeq \mathcal{H}om(G, {}_U F), \end{aligned}$$

where $V \in \text{Op}(X_{sa})$. The third isomorphism follows from Proposition 1.3.4.

(ii) If S is a closed subset of X the proof is similar. □

PROPOSITION 1.6.5. *Let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. Then $\rho_! K \otimes F$ is quasi-injective for each $K \in \text{Mod}(k_X)$.*

PROOF. (i) Let us show the result when $K = k_Z$, for a locally closed subset Z of X . Let $G \in \text{Mod}_{\mathbb{R}\text{-}c}^c(k_X)$. We have

$$\begin{aligned} \text{Hom}_{k_{X_{sa}}}(G, {}_Z F) &\simeq \Gamma(X; \mathcal{H}om(G, {}_Z F)) \\ &\simeq \Gamma(X; {}_Z \mathcal{H}om(G, F)) \\ &\simeq \Gamma(X; \rho^{-1} {}_Z \mathcal{H}om(G, F)) \\ &\simeq \Gamma(X; (\rho^{-1} \mathcal{H}om(G, F))_Z). \end{aligned}$$

Since F is quasi-injective, $\mathcal{H}om(G, F)$ is quasi-injective. Then by Pro-

position 1.5.15 the sheaf $(\rho^{-1}\mathcal{H}om(G, F))_Z$ is c-soft and it is injective with respect to the functor $\Gamma(X, \cdot)$. Hence the functor $\Gamma(X; \rho^{-1}\mathcal{H}om(\cdot, F)_Z)$ is exact on $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$.

(ii) Let $K \in \text{Mod}(k_X)$. There exists an epimorphism $\bigoplus_{i \in J} k_{U_i} \rightarrow K$ with $U_i \in \text{Op}(X_{sa})$ for each i . Let K_J be the image of $\bigoplus_{i \in J} k_{U_i}$, with $J \subset I$ finite. We have $K \simeq \varinjlim_J K_J$, hence $\rho_1 K \simeq \varinjlim_J \rho_1 K_J$ since ρ_1 commutes with \varinjlim . It is enough to prove the result for K_J . We argue by induction on the cardinal of J . Set $K = K_J$. If $|J| = 1$ then $K \simeq k_Z$ with Z locally closed subset of X and the result follows from (i).

Let us show $n - 1 \Rightarrow n$. There is an epimorphism $\bigoplus_{i=1}^n k_{U_i} \rightarrow K$. Let K_1 be the image of $k_{U_1} \rightarrow K$ and let $K_2 = K/K_1$. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & k_{U_1} & \longrightarrow & \bigoplus_{i=1}^n k_{U_i} & \longrightarrow & \bigoplus_{i=2}^n k_{U_i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_1 & \longrightarrow & K & \longrightarrow & K_2 & \longrightarrow & 0, \end{array}$$

where the vertical arrows are surjective, and the rows are exact. By the exactness of ρ_1 and \otimes we obtain the exact sequence $0 \rightarrow \rho_1 K_1 \otimes F \rightarrow \rho_1 K \otimes F \rightarrow \rho_1 K_2 \otimes F \rightarrow 0$. By the inductive hypothesis $\rho_1 K_1 \otimes F$ and $\rho_1 K_2 \otimes F$ are quasi-injective, then $\rho_1 K \otimes F$ is quasi-injective. \square

PROPOSITION 1.6.6. *Let $F \in \text{Mod}(k_{X_{sa}})$, $G \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$ and let $K \in \text{Mod}(k_X)$. One has the isomorphism $\mathcal{H}om(G, F) \otimes \rho_1 K \simeq \mathcal{H}om(G, F \otimes \rho_1 K)$.*

PROOF. Both sides are left exact with respect to F . Hence we may assume that F is quasi-injective. Since quasi-injective sheaves are $\mathcal{H}om(G, \cdot)$ -injective, both sides are exact with respect to K . Moreover as a consequence of Proposition 1.3.4 both sides commute with filtrant \varinjlim with respect to K . We may reduce to the case $K = k_U$, with $U \in \text{Op}(X_{sa})$. Then the result follows from Proposition 1.6.4. \square

2. Derived category.

As usual, we denote $D(k_{X_{sa}})$ the derived category of $\text{Mod}(k_{X_{sa}})$ and its full subcategory consisting of bounded (resp. bounded below, resp. bounded above) complexes is denoted by $D^b(k_{X_{sa}})$ (resp. $D^+(k_{X_{sa}})$, resp. $D^-(k_{X_{sa}})$).

2.1 – The category $D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}})$.

As usual we denote by $D_{\mathbb{R}\text{-c}}^b(k_X)$ (resp. $D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}})$) the full subcategory of $D^b(k_X)$ (resp. $D^b(k_{X_{sa}})$) consisting of objects with \mathbb{R} -constructible cohomology.

Recall that $\rho : X \rightarrow X_{sa}$ is the natural morphism of sites. It induces the functor $\rho_* : \text{Mod}(k_X) \rightarrow \text{Mod}(k_{X_{sa}})$.

LEMMA 2.1.1. *Let $F \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Then $R^j\rho_*F = 0$ for each $j \neq 0$.*

PROOF. The sheaf $R^j\rho_*F$ is the sheaf associated to the presheaf $V \rightarrow R^j\Gamma(V; F)$. We have to show that $R^j\Gamma(V; F) = 0$ for $j \neq 0$ on a family of generators of the topology of X_{sa} . This means that for each $V \in \text{Op}^c(X_{sa})$ and for each $j \neq 0$, there exists I finite and $\{V_i\}_{i \in I} \in \text{Cov}(V_{sa})$ such that $R^j\Gamma(V_i; R\rho_*F) \simeq R^j\Gamma(V_i; F) = 0$.

We use the notation of [5]. There exists a locally finite stratification $\{X_i\}_{i \in I}$ of X consisting of subanalytic subsets such that for all $j \in \mathbb{Z}$ and all $i \in I$ the sheaf $F|_{X_i}$ is locally constant. By the triangulation theorem there exist a simplicial complex (S, \mathcal{A}) and a subanalytic homeomorphism $\psi : |S| \xrightarrow{\sim} X$ compatible with the stratification and such that V is a finite union of the images by ψ of open subsets $V(\sigma)$ of $|S|$, where $V(\sigma) = \bigcup_{\tau \in \mathcal{A}, \tau \supset \sigma} |\tau|$. By Proposition 8.1.4 of [5] we have $R^j\Gamma(\psi(V(\sigma)); F) = 0$ for each σ and for each $j \neq 0$. The result follows because $V = \bigcup_{\psi(|\sigma|) \subset V} \psi(V(\sigma))$. \square

Since \mathbb{R} -constructible sheaves are injective with respect to the functor ρ_* , the following diagram of derived categories is quasi-commutative.

$$(2.1) \quad \begin{array}{ccc} D_{\mathbb{R}\text{-c}}^b(k_X) & \begin{array}{c} \xrightarrow{R\rho_*} \\ \xleftarrow{\rho^{-1}} \end{array} & D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}}) \\ \uparrow \wr & \nearrow \rho_* & \\ D^b(\text{Mod}_{\mathbb{R}\text{-c}}(k_X)) & & \end{array}$$

THEOREM 2.1.2. *One has the equivalence of categories*

$$D_{\mathbb{R}\text{-c}}^b(k_X) \simeq D^b(\text{Mod}_{\mathbb{R}\text{-c}}(k_X)) \simeq D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}}).$$

PROOF. By dévissage, to prove the equivalence between $D^b(\text{Mod}_{\mathbb{R}\text{-c}}(k_X))$ and $D_{\mathbb{R}\text{-c}}^b(k_{X_{sa}})$ it is enough to check that the functor ρ_* in (2.1) is fully faithful. We have $\rho^{-1} \circ \rho_* \simeq \text{id}$ and the result follows.

The equivalence between $D^b(\text{Mod}_{\mathbb{R}\text{-c}}(k_X))$ and $D_{\mathbb{R}\text{-c}}^b(k_X)$ was shown by Kashiwara in [4]. \square

2.2 – Operations in the derived category.

Let us study the operations in the derived category of $\text{Mod}(k_{X_{sa}})$. Let $f : X \rightarrow Y$ be an analytic map. We will use quasi-injective objects and the results of § 1.5 to derive the formulas of Section 1.

Since $\text{Mod}(k_{X_{sa}})$ has enough injectives, then the derived functors

$$R\mathcal{H}om : D^-(k_{X_{sa}})^{op} \times D^+(k_{X_{sa}}) \rightarrow D^+(k_{X_{sa}}),$$

$$Rf_* : D^+(k_{X_{sa}}) \rightarrow D^+(k_{Y_{sa}}),$$

$$Rf_{!!} : D^+(k_{X_{sa}}) \rightarrow D^+(k_{Y_{sa}}),$$

are well defined.

PROPOSITION 2.2.1. *Let $f : X \rightarrow Y$ be an analytic map. Then*

- (i) *The functors Rf_* and $R\mathcal{H}om$ commute with $R\rho_*$.*
- (ii) *The functors Rf_* and $Rf_{!!}$ commute with ρ^{-1} .*
- (iii) *We have $R(g \circ f)_* \simeq Rg_* \circ Rf_*$ and $R(g \circ f)_{!!} \simeq Rg_{!!} \circ Rf_{!!}$.*
- (iv) *The functor $R^k f_{!!} : \text{Mod}(k_{X_{sa}}) \rightarrow \text{Mod}(k_{Y_{sa}})$ commutes with filtrant inductive limits for each $k \in \mathbb{Z}$.*
- (v) *If $F \in D^+(k_{X_{sa}})$ and f is proper on $\text{supp}(F)$, then $Rf_{!!} \simeq Rf_*$.*

PROOF. (i) The functor ρ_* sends injective sheaves to injective sheaves, then Rf_* and $R\mathcal{H}om$ commute with $R\rho_*$.

(ii) Since ρ^{-1} has an exact left adjoint it sends injective sheaves to injective sheaves. Then Rf_* and $Rf_{!!}$ commute with ρ^{-1} .

(iii) The functor f_* (resp. $f_{!!}$) sends injective sheaves to injective (resp. quasi-injective) sheaves. Then $R(g \circ f)_* \simeq Rg_* \circ Rf_*$ and $R(g \circ f)_{!!} \simeq Rg_{!!} \circ Rf_{!!}$.

(iv) Quasi-injective objects of $\text{Mod}(k_{X_{sa}})$ are stable by filtrant \varinjlim , and the functor $f_{!!}$ commutes with such limits. Then $R^k f_{!!}$ commutes with filtrant \varinjlim for each $k \in \mathbb{Z}$.

(v) We can find a representative F' of F in $K^+(\mathcal{J}_{X_{sa}})$ with f proper on $\text{supp}(F')$. Then the result follows from the non derived case. \square

PROPOSITION 2.2.2. *Let $F = \varinjlim_i F_i$ with $F_i \in \text{Mod}(k_{X_{sa}})$ and let $G \in D_{\mathbb{R}\text{-c}}^b(k_X)$. One has $R^k\mathcal{H}om(G, F) \simeq \varinjlim_i R^k\mathcal{H}om(G, F_i)$ for each $k \in \mathbb{Z}$.*

PROOF. There exists (see [8], Corollary 9.6.7) an inductive system of injective resolutions I_i^\bullet of F_i . Then $\varinjlim_i I_i^\bullet$ is a complex of quasi-injective objects quasi-isomorphic to F . Each object of $(\text{Mod}_{\mathbb{R}\text{-c}}(k_X)^{op}, \mathcal{J}_{X_{sa}})$ is $\mathcal{H}om(\cdot, \cdot)$ -acyclic. Proposition 1.3.4 implies the isomorphism

$$\mathcal{H}om(G, \varinjlim_i I_i^\bullet) \simeq \varinjlim_i \mathcal{H}om(G, I_i^\bullet)$$

and the result follows. \square

PROPOSITION 2.2.3. *Let $F \in D^+(k_{X_{sa}})$, $G \in D_{\mathbb{R}\text{-c}}^b(k_X)$ and let $K \in D^+(k_X)$. One has the isomorphism $R\mathcal{H}om(G, F) \otimes_{\rho_1} K \simeq R\mathcal{H}om(G, F \otimes_{\rho_1} K)$.*

PROOF. Let I^\bullet be a quasi-injective resolution of F . By Proposition 1.6.5 we have that $I^\bullet \otimes_{\rho_1} K$ is a complex of quasi-injective objects. Each object of $(\text{Mod}_{\mathbb{R}\text{-c}}(k_X)^{op}, \mathcal{J}_{X_{sa}})$ is $\mathcal{H}om(\cdot, \cdot)$ -acyclic. Hence we are reduced to prove the isomorphism $\mathcal{H}om(G, I^\bullet) \otimes_{\rho_1} K \simeq \mathcal{H}om(G, I^\bullet \otimes_{\rho_1} K)$. The result follows from Proposition 1.6.6. \square

PROPOSITION 2.2.4. *Let $U \in \text{Op}^c(X_{sa})$. Let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. Then F_U is $\Gamma(V, \cdot)$ -acyclic for each $V \in \text{Op}(X_{sa})$.*

PROOF. Since F_U has compact support, we may suppose that V is relatively compact. Let $S = X \setminus U$. Since F is quasi-injective and filtrant \varinjlim are exact, the morphism $\Gamma(V; F) \rightarrow \varinjlim_{W \supset S \cap V} \Gamma(W; F) \xrightarrow{\sim} \Gamma(V; F_S)$ is surjective.

Consider the exact sequence $0 \rightarrow F_U \rightarrow F \rightarrow F_S \rightarrow 0$. We get the exact sequence

$$0 \rightarrow \Gamma(V; F_U) \rightarrow \Gamma(V; F) \rightarrow \Gamma(V; F_S) \rightarrow 0.$$

By Proposition 1.5.14 F and F_S are quasi-injective, hence $\Gamma(V; \cdot)$ -acyclic. This implies that F_U is $\Gamma(V; \cdot)$ -acyclic. \square

COROLLARY 2.2.5. *Let $f: X \rightarrow Y$ be a real analytic map and*

let $U \in \text{Op}^c(X_{sa})$ Let $F \in \text{Mod}(k_{X_{sa}})$ be quasi-injective. Then F_U is $f_{!!}$ -acyclic.

PROOF. Since F_U has compact support, $Rf_{!!}F_U \simeq Rf_*F_U$. The result follows because F_U is $\Gamma(f^{-1}(V); \cdot)$ -acyclic for each $V \in \text{Op}(Y_{sa})$. \square

LEMMA 2.2.6. Let F be quasi-injective object of $\text{Mod}(k_{X_{sa}})$ and let $G \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. Then $F \otimes \rho_*G$ is $f_{!!}$ -acyclic.

PROOF. Let $G \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. Then G has a resolution

$$0 \rightarrow \bigoplus_{i_1 \in I_1} k_{U_{i_1}} \rightarrow \dots \rightarrow \bigoplus_{i_n \in I_n} k_{U_{i_n}} \xrightarrow{\varphi} G \rightarrow 0,$$

where I_j is finite and $U_{i_j} \in \text{Op}^c(X_{sa})$ for each $i_j \in I_j, j \in \{1, \dots, n\}$. Let us argue by induction on the length n of the resolution.

If $n = 1$, then G is isomorphic to a finite sum $\bigoplus_i k_{U_i}$, with $U_i \in \text{Op}^c(X_{sa})$, and the result follows from Corollary 2.2.5.

Let us show $n - 1 \Rightarrow n$. The sequence $0 \rightarrow \ker \varphi \rightarrow \bigoplus_{i_n \in I_n} k_{U_{i_n}} \rightarrow G \rightarrow 0$ is exact. The sheaf $\ker \varphi$ belongs to $\text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ and it has a resolution of length $n - 1$. Applying $F \otimes \rho_*(\cdot)$ we get the exact sequence

$$0 \rightarrow F \otimes \rho_* \ker \varphi \rightarrow \bigoplus_{i_n \in I_n} F_{U_{i_n}} \rightarrow F \otimes \rho_* G \rightarrow 0.$$

By the induction hypothesis $F \otimes \rho_* \ker \varphi$ is $f_{!!}$ -acyclic. Moreover $\bigoplus_{i_n \in I_n} F_{U_{i_n}}$ is $f_{!!}$ -acyclic, then $F \otimes \rho_* G$ is $f_{!!}$ -acyclic. \square

PROPOSITION 2.2.7. Let F be quasi-injective object of $\text{Mod}(k_{X_{sa}})$ and let $G \in \text{Mod}(k_{X_{sa}})$. Then $F \otimes G$ is $f_{!!}$ -acyclic.

PROOF. Let $G \simeq \varinjlim_i \rho_* G_i$ with $G_i \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$ for each i . Since the functors \otimes and $R^k f_{!!}$ commute with filtrant \varinjlim we have

$$R^k f_{!!}(F \otimes \varinjlim_i \rho_* G_i) \simeq \varinjlim_i R^k f_{!!}(F \otimes \rho_* G_i) = 0$$

if $k \neq 0$ by Lemma 2.2.6. \square

PROPOSITION 2.2.8. Let $F \in D^+(k_{X_{sa}})$ and $G \in D^+(k_{Y_{sa}})$. Then

$$Rf_{!!}F \otimes G \simeq Rf_{!!}(F \otimes f^{-1}G).$$

PROOF. First assume that $F \in \text{Mod}(k_{X_{sa}})$ is injective. By Proposition 2.2.7 $F \otimes f^{-1}G$ is $f_{!!}$ -acyclic.

Now let $F \in D^+(k_{X_{sa}})$ and $G \in D^+(k_{Y_{sa}})$. Let F' be a complex of injective sheaves quasi-isomorphic to F . Then

$$Rf_{!!}F \otimes G \simeq f_{!!}F' \otimes G \simeq f_{!!}(F' \otimes f^{-1}G) \simeq Rf_{!!}(F \otimes f^{-1}G),$$

where the second isomorphism follows from Proposition 1.4.5. □

Now let us consider a cartesian square

$$\begin{array}{ccc} X'_{sa} & \xrightarrow{f'} & Y'_{sa} \\ \downarrow g' & & \downarrow g \\ X_{sa} & \xrightarrow{f} & Y_{sa} \end{array}$$

PROPOSITION 2.2.9. *Let $F \in D^+(k_{X_{sa}})$. Then $g^{-1}Rf_{!!}F \simeq Rf'_{!!}g'^{-1}F$.*

PROOF. We have an isomorphism $f'_{!!}g'^{-1} \simeq g^{-1}f_{!!}$, and $R(g^{-1}f_{!!}) \simeq g^{-1}Rf_{!!}$ since g^{-1} is exact. Then we obtain a morphism $g^{-1}Rf_{!!} \rightarrow Rf'_{!!}g'^{-1}$. It is enough to prove that for any $k \in \mathbb{Z}$ and for any $F \in \text{Mod}(k_{X_{sa}})$ we have $g^{-1}R^k f_{!!}F \xrightarrow{\sim} R^k f'_{!!}g'^{-1}F$. Since both sides commute with filtrant \lim , we may assume $F = \rho_*G$ with $G \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. Moreover since $\text{supp}(G)$ is compact, f' is proper on $\text{supp}(g'^{-1}G)$. Then both sides commute with ρ_* and the result follows from the corresponding one for classical sheaves. □

As in classical sheaf theory, the Künneth formula follows from the projection formula and the base change formula.

PROPOSITION 2.2.10. *Consider a cartesian square*

$$\begin{array}{ccc} X'_{sa} & \xrightarrow{f'} & Y'_{sa} \\ \downarrow g' & \searrow \delta & \downarrow g \\ X_{sa} & \xrightarrow{f} & Y_{sa} \end{array}$$

where $\delta = fg' = g'f$. There is a natural isomorphism

$$R\delta_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rf_{!!}F \otimes Rg_{!!}G$$

for $F \in D^+(k_{X_{sa}})$ and $G \in D^+(k_{Y'_{sa}})$.

PROOF. Using the projection formula and the base change formula we deduce

$$Rf'_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rf'_{!!}g'^{-1}F \otimes G \simeq g^{-1}Rf_{!!}F \otimes G.$$

Using the projection formula once again we find

$$Rg_{!!}Rf'_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rg_{!!}(g^{-1}Rf_{!!}F \otimes G) \simeq Rf_{!!}F \otimes Rg_{!!}G$$

and the result follows since $R\delta_{!!} \simeq Rg_{!!} \circ Rf'_{!!}$. □

The following isomorphism is the analogue for subanalytic sheaves of Lemma 5.2.8 of [7]

PROPOSITION 2.2.11. *Let $G \in D_{\mathbb{R}\text{-c}}^b(k_Y)$ and let $F \in D^+(k_{X_{sa}})$. Then the natural morphism*

$$Rf_{!!}R\mathcal{H}om(f^{-1}G, F) \rightarrow R\mathcal{H}om(G, Rf_{!!}F)$$

is an isomorphism.

PROOF. The morphism is obtained as in the non derived case. Let us show that it is an isomorphism. Let F' be a complex of injective sheaves quasi-isomorphic to F . Then

$$\begin{aligned} Rf_{!!}\mathcal{H}om(f^{-1}G, F) &\simeq f_{!!}\mathcal{H}om(f^{-1}G, F') \\ &\simeq \mathcal{H}om(G, f_{!!}F') \\ &\simeq R\mathcal{H}om(G, Rf_{!!}F), \end{aligned}$$

where the second isomorphism follows from Proposition 1.4.7. □

2.3 – Vanishing theorems on $\text{Mod}(k_{X_{sa}})$.

In § 2.3 we prove some results on the vanishing of the cohomology of sheaves on a subanalytic site.

DEFINITION 2.3.1. *The quasi-injective dimension of the category $\text{Mod}(k_{X_{sa}})$ is the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that for any $F \in \text{Mod}(k_{X_{sa}})$ there exists an exact sequence*

$$0 \rightarrow F \rightarrow I^0 \rightarrow \dots \rightarrow I^n \rightarrow 0$$

with I^j quasi-injective for $0 \leq j \leq n$.

PROPOSITION 2.3.2. *The category $\text{Mod}(k_{X_{sa}})$ has finite quasi-injective dimension.*

PROOF. Let $F \in \text{Mod}(k_{X_{sa}})$. Then $F = \varinjlim_i \rho_* F_i$, with $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}^c(k_X)$. There exists (see [8], Corollary 9.6.7) an inductive system of injective resolutions I_i^\bullet of F_i . By Proposition 3.3.11 of [5], the category $\text{Mod}(k_X)$ has finite homological dimension. Then we may assume that I_i^\bullet has length $N_0 < \infty$ for each i . Since F_i is ρ_* -injective for each i , $\rho_* I_i^\bullet$ is an injective resolution of $\rho_* F_i$ of length N_0 . Taking the inductive limit we find that $\varinjlim_i \rho_* I_i^\bullet$ is a resolution of F of length N_0 , and $\varinjlim_i \rho_* I_i^j \in \mathcal{J}_{X_{sa}}$ for each j . □

COROLLARY 2.3.3. *Let $f : X \rightarrow Y$ be a real analytic map, and let $F \in \text{Mod}_{\mathbb{R}\text{-c}}(k_X)$. The functors f_* , $f_!$ and $\mathcal{H}om(F, \cdot)$ have finite cohomological dimension.*

PROPOSITION 2.3.4. *Let $F \in \text{Mod}(k_X)$ and let $G \in \text{Mod}(k_{X_{sa}})$. There exists a finite $j_0 \in \mathbb{N}$ such that*

$$R^j \mathcal{H}om(\rho_! F, G) = 0 \text{ for } j > j_0.$$

PROOF. Let $U \in \text{Op}(X_{sa})$. We have the chain of isomorphisms

$$\begin{aligned} R\Gamma(U; R\mathcal{H}om(\rho_! F, G)) &\simeq \text{RHom}_{k_{X_{sa}}}(\rho_! F, R\Gamma_U G) \\ &\simeq \text{RHom}_{k_X}(F, \rho^{-1} R\Gamma_U G). \end{aligned}$$

The functor Γ_U has finite cohomological dimension, and the homological dimension of the category $\text{Mod}(k_X)$ is finite. Hence we can find a finite $j_0 \in \mathbb{N}$ such that $R^j \Gamma(U; R\mathcal{H}om(\rho_! F, G))$ vanishes for $j > j_0$ and for each $U \in \text{Op}(X_{sa})$. This shows the result. □

REMARK 2.3.5. *We have seen that the functor $\mathcal{H}om(F, \cdot)$ has finite cohomological dimension when F is \mathbb{R} -constructible and when $F = \rho_! G$ with $G \in \text{Mod}(k_X)$. We do not know if the cohomological dimension is finite for any $F \in \text{Mod}(k_{X_{sa}})$. Indeed we do not know if the homological dimension of $\text{Mod}(k_{X_{sa}})$ is finite or not.*

2.4 – Duality.

In the following we find a right adjoint to the functor $Rf_{!!}$, denoted by $f^!$. The construction is not inspired to the classical one as in [7], but it is done via Brown representability (we refer to [8] for an exposition on this subject). We calculate $f^!$ by decomposing f as the composite of a closed embedding and a submersion.

The subcategory $\mathcal{J}_{X_{sa}}$ of quasi-injective objects and the functor $f_{!!}$ have the following properties:

$$(2.2) \quad \left\{ \begin{array}{l} \text{(i) } \mathcal{J}_{X_{sa}} \text{ is cogenerating,} \\ \text{(ii) } \text{Mod}(k_{X_{sa}}) \text{ has finite quasi-injective dimension,} \\ \text{(iii) } \mathcal{J}_{X_{sa}} \text{ is } f_{!!}\text{-injective,} \\ \text{(iv) } \mathcal{J}_{X_{sa}} \text{ is closed by small } \oplus, \\ \text{(v) } f_{!!} \text{ commutes with small } \oplus. \end{array} \right.$$

As a consequence of the Brown representability theorem (see [8], Corollary 14.3.7 for details) we find a right adjoint to the functor $Rf_{!!}$. Remark that the functor $Rf_{!!}$ extends to a functor $Rf_{!!} : D(k_{X_{sa}}) \rightarrow D(k_{Y_{sa}})$.

THEOREM 2.4.1. *The functor $Rf_{!!} : D(k_{X_{sa}}) \rightarrow D(k_{Y_{sa}})$ admits a right adjoint. We denote by $f^! : D(k_{Y_{sa}}) \rightarrow D(k_{X_{sa}})$ the adjoint functor.*

COROLLARY 2.4.2. *Let $G \in D^+(k_{Y_{sa}})$. Then $f^!G \in D^+(k_{X_{sa}})$.*

PROOF. We may reduce to the case that $G \in D^{\geq 0}(k_{Y_{sa}})$. Let N_0 be the quasi-injective dimension of $\text{Mod}(k_{X_{sa}})$ and let $F \in D^{\leq -N_0-1}(k_{X_{sa}})$. Then $Rf_{!!}F \in D^{\leq -1}(k_{Y_{sa}})$ and

$$\text{Hom}_{D(k_{Y_{sa}})}(Rf_{!!}F, G) \simeq \text{Hom}_{D(k_{X_{sa}})}(F, f^!G) = 0.$$

Hence for each $F \in D^{\leq -N_0-1}(k_{X_{sa}})$ we have $\text{Hom}_{D(k_{X_{sa}})}(F, f^!G) = 0$. Set for short $a = -N_0 - 1$, if $F = \tau^{\leq a}f^!G$ we have

$$\text{Hom}_{D(k_{X_{sa}})}(\tau^{\leq a}f^!G, f^!G) \simeq \text{Hom}_{D^{\leq a}(k_{X_{sa}})}(\tau^{\leq a}f^!G, \tau^{\leq a}f^!G) = 0.$$

This implies $\tau^{\leq a}f^!G = 0$, hence $f^!G \in D^+(k_{X_{sa}})$. □

REMARK 2.4.3. *As in classical sheaf theory one can prove that for $F \in D^b(k_{X_{sa}})$ and $G \in D^+(k_{Y_{sa}})$ one has $Rf_*R\mathcal{H}om(F, f^!G) \simeq R\mathcal{H}om(Rf_{!!}F, G)$.*

REMARK 2.4.4. *As in classical sheaf theory, one can prove by adjunction the dual projection formula and the dual base change formula.*

PROPOSITION 2.4.5. *The functor $f^!$ commutes with $R\rho_*$, and the functor $H^k f^! : \text{Mod}(k_{Y_{sa}}) \rightarrow \text{Mod}(k_{X_{sa}})$ commutes with filtrant \varinjlim .*

PROOF. Since $Rf_{!!}$ commutes with ρ^{-1} , then $f^!$ commutes with $R\rho_*$ by adjunction.

Let us show that $H^k f^!$ commutes with \varinjlim . Let $\{F_i\}_i$ be a filtrant inductive system in $\text{Mod}(k_{Y_{sa}})$. Remark that $\varinjlim_i H^k f^! F_i$ (resp. $H^k f^! \varinjlim_i F_i$) is the sheaf associated to the presheaf $U \mapsto \varinjlim_i R^k \Gamma(U; f^! F_i)$ (resp. $U \mapsto R^k \Gamma(U; f^! \varinjlim_i F_i)$), for $U \in \text{Op}^c(X_{sa})$.

We will show the isomorphism $R^k \Gamma(U; f^! \varinjlim_i F_i) \xrightarrow{\sim} \varinjlim_i R^k \Gamma(U; f^! F_i)$ for each $U \in \text{Op}^c(X_{sa})$. By adjunction it is enough to prove the isomorphism $R^k \text{Hom}_{k_{Y_{sa}}}(Rf_! k_U, \varinjlim_i F_i) \simeq \varinjlim_i R^k \text{Hom}_{k_{Y_{sa}}}(Rf_! k_U, F_i)$ (remark that we have $Rf_{!!} k_U \simeq Rf_! k_U \simeq Rf_* k_U$ since $U \in \text{Op}^c(X_{sa})$).

Let $\mathcal{J}_{Y_{sa}}$ be the family of quasi-injective objects of $\text{Mod}(k_{Y_{sa}})$. Each object of $(\text{Mod}_{\mathbb{R}\text{-c}}^c(k_Y)^{op}, \mathcal{J}_{Y_{sa}})$ is $\text{Hom}_{k_{Y_{sa}}}(\cdot, \cdot)$ -acyclic. Moreover $\mathcal{J}_{Y_{sa}}$ is stable by filtrant inductive limits. There exists (see [8], Corollary 9.6.7) an inductive system of injective resolutions I_i^\bullet of F_i . Then $\varinjlim_i I_i^\bullet$ is a quasi-injective resolution of $\varinjlim_i F_i$. We have

$$\text{Hom}_{K^+(k_{Y_{sa}})}(Rf_! k_U, \varinjlim_i I_i^\bullet) \simeq \varinjlim_i \text{Hom}_{K^+(k_{Y_{sa}})}(Rf_! k_U, I_i^\bullet)$$

and the result follows. □

COROLLARY 2.4.6. *Let $F \in D^b(k_{Y_{sa}})$. Then $f^! F \in D^b(k_{X_{sa}})$.*

PROOF. We may reduce to the case $F \in \text{Mod}(k_{Y_{sa}})$. Then $F \simeq \varinjlim_i \rho_* F_i$ with $F_i \in \text{Mod}_{\mathbb{R}\text{-c}}(k_Y)$ for each i . By Proposition 2.4.5 we have

$$H^k f^! F \simeq H^k f^! \varinjlim_i \rho_* F_i \simeq \varinjlim_i \rho_* H^k f^! F_i,$$

and $H^k f^! F_i = 0$ if $k > j_0$ for a fixed $j_0 \in \mathbb{N}$ and for each i . □

The following result is the analogue for subanalytic sheaves of Proposition 5.3.9 of [7].

PROPOSITION 2.4.7. *Let $F \in D^+(k_{Y_{sa}})$ and let $G \in D^+(k_Y)$. Then one has the isomorphism $f^!(F \otimes \rho_! G) \simeq f^! F \otimes \rho_! f^{-1} G$.*

PROOF. We have the chain of morphisms

$$Rf_{!!}(f^! F \otimes \rho_! f^{-1} G) \simeq Rf_{!!} f^! F \otimes \rho_! G \rightarrow F \otimes \rho_! G,$$

by adjunction we obtain the desired morphism. To prove that it is an isomorphism it is enough to show $R^k \Gamma(U; f^!(F \otimes \rho_! G)) \simeq R^k \Gamma(U; f^! F \otimes \rho_! f^{-1} G)$ for each $U \in \text{Op}^c(X_{sa})$ and each $k \in \mathbb{Z}$. We have the chain of isomorphisms

$$\begin{aligned} R^k \Gamma(U; f^!(F \otimes \rho_! G)) &\simeq R^k \text{Hom}_{k_{Y_{sa}}}(Rf_{!!} k_U, F \otimes \rho_! G) \\ &\simeq R^k \text{Hom}_{k_{Y_{sa}}}(k_Y, R\mathcal{H}om(Rf_{!!} k_U, F \otimes \rho_! G)) \\ &\simeq R^k \text{Hom}_{k_{Y_{sa}}}(k_Y, R\mathcal{H}om(Rf_{!!} k_U, F) \otimes \rho_! G) \\ &\simeq R^k \text{Hom}_{k_{Y_{sa}}}(k_Y, Rf_{!!} R\mathcal{H}om(k_U, f^! F) \otimes \rho_! G) \\ &\simeq R^k \text{Hom}_{k_{Y_{sa}}}(k_Y, Rf_{!!}(R\mathcal{H}om(k_U, f^! F) \otimes f^{-1} \rho_! G)) \\ &\simeq R^k \text{Hom}_{k_{Y_{sa}}}(k_Y, Rf_{!!}(R\mathcal{H}om(k_U, f^! F \otimes \rho_! f^{-1} G))) \\ &\simeq R^k \text{Hom}_{k_{X_{sa}}}(k_U, f^! F \otimes \rho_! f^{-1} G). \end{aligned}$$

Here the fourth and the last isomorphism follow from the fact that since k_U has compact support, then $R\mathcal{H}om(k_U, K)$ has compact support for any $K \in D^+(k_{X_{sa}})$ and $Rf_{!!} R\mathcal{H}om(k_U, K) \simeq Rf_* R\mathcal{H}om(k_U, K)$. \square

PROPOSITION 2.4.8. *Let $F \in D^+(k_{Y_{sa}})$, and let $f : X \rightarrow Y$ be a closed embedding. Then $f^! F \simeq f^{-1} R\mathcal{H}om(k_X, F)$ and $\text{id} \xrightarrow{\sim} f^! Rf_{!!}$.*

PROOF. Since f is proper, then $Rf_* \simeq Rf_{!!}$. We have the isomorphisms $Rf_* f^! F \simeq Rf_* R\mathcal{H}om(k_X, f^! F) \simeq R\mathcal{H}om(k_X, F)$. Since $f^{-1} Rf_* f^! F \simeq f^! F$, then $f^! F \simeq f^{-1} R\mathcal{H}om(k_X, F)$.

Let $F' \in D^+(k_{X_{sa}})$. We have the isomorphisms

$$f^! Rf_* F' \simeq f^{-1} R\mathcal{H}om(k_X, Rf_* F') \simeq f^{-1} Rf_* R\mathcal{H}om(k_X, F') \simeq f^{-1} Rf_* F',$$

and $f^{-1} Rf_* F' \simeq F'$ since f is a closed embedding. \square

Recall that f is a topological submersion (of fiber dimension n) if locally on X , f is isomorphic to the projection $Y \times \mathbb{R}^n \rightarrow Y$.

PROPOSITION 2.4.9. *Assume that f is a topological submersion. Then for $F \in D^+(k_{Y_{sa}})$ one has the isomorphism $f^{-1}F \otimes f^!k_Y \xrightarrow{\sim} f^!F$.*

PROOF. We have the chain of morphisms

$$Rf_{!!}(f^{-1}F \otimes f^!k_Y) \simeq F \otimes Rf_{!!}f^!k_Y \rightarrow F \otimes k_Y \simeq F,$$

by adjunction we obtain the desired morphism.

Let us show that it is an isomorphism. We may reduce to the case $F \simeq \varinjlim_i \rho_* F_i \in \text{Mod}(k_{Y_{sa}})$. We have the chain of isomorphisms

$$\begin{aligned} H^k(f^{-1} \varinjlim_i \rho_* F_i \otimes f^! \rho_* k_Y) &\simeq \varinjlim_i \rho_* H^k(f^{-1} F_i \otimes f^! k_Y) \\ &\simeq \varinjlim_i \rho_* H^k f^! F_i \\ &\simeq H^k f^! \varinjlim_i \rho_* F_i. \end{aligned}$$

□

Using these results we can calculate explicitly the functor $f^!$. Let $f : X \rightarrow Y$ be an analytic map. We decompose it as the composite of a closed embedding and a submersion. In fact

$$f : X \xrightarrow{j} X \times Y \xrightarrow{p} Y$$

where p is the projection and j is the graph embedding $j(x) = (x, f(x))$. Let $F \in D^+(k_{Y_{sa}})$. Applying Propositions 2.4.8 and 2.4.9 we get

$$f^!F \simeq j^{-1}R\mathcal{H}om(k_{j(X)}, p^{-1}F \otimes p^!k_Y).$$

COROLLARY 2.4.10. *Assume that f is a topological submersion. Then:*

- (i) *the functor $f^!$ commutes with ρ^{-1} ,*
- (ii) *the functor $Rf_{!!}$ commutes with ρ_* .*

PROOF. (i) One has the chain of isomorphisms

$$\begin{aligned} \rho^{-1}(f^{-1}F \otimes f^! \rho_* k_Y) &\simeq \rho^{-1}f^{-1}F \otimes \rho^{-1}f^! \rho_* k_Y \\ &\simeq f^{-1}\rho^{-1}F \otimes \rho^{-1}f^! \rho_* k_Y \\ &\simeq f^{-1}\rho^{-1}F \otimes \rho^{-1}\rho_* f^! k_Y \\ &\simeq f^{-1}\rho^{-1}F \otimes f^! k_Y. \end{aligned}$$

The result follows from Proposition 2.4.9.

(ii) The result follows by adjunction. \square

PROPOSITION 2.4.11. *Assume that f is a topological submersion and moreover that $Rf_!f^!k_Y \simeq k_Y$. Then for $F \in D^+(k_{Y_{sa}})$ the morphism $Rf_*f^{-1}F \rightarrow F$ is an isomorphism.*

PROOF. First let us show that $Rf_!f^!k_Y \simeq k_Y$. We have the chain of isomorphisms

$$\begin{aligned} Rf_!f^!\rho_*k_Y &\simeq Rf_!\rho_*f^!k_Y \\ &\simeq Rf_!\rho_!f^!k_Y \\ &\simeq \rho_!Rf_!f^!k_Y \\ &\simeq \rho_!k_Y \\ &\simeq \rho_*k_Y, \end{aligned}$$

where the second isomorphism follows because $f^!k_Y$ is locally constant and the third from Corollary 2.4.10 (ii). It follows from Proposition 2.4.9 that $f^{-1}F \simeq R\mathcal{H}om(f^!k_Y, f^!F)$. Then we have the chain of isomorphisms

$$\begin{aligned} Rf_*f^{-1}F &\simeq Rf_*R\mathcal{H}om(f^!k_Y, f^!F) \\ &\simeq R\mathcal{H}om(Rf_!f^!k_Y, F) \\ &\simeq F. \end{aligned}$$

\square

3. Examples of applications.

In this Section we give some example of subanalytic sheaves. Let X be a real analytic manifold, and let X_{sa} be the associated subanalytic site. We first introduce sheaves of \mathcal{R} -modules, where \mathcal{R} is a sheaf of k -algebras on X_{sa} . Let \mathcal{D}_X be the sheaf of finite order differential operators on X . We define the $\rho_!\mathcal{D}_X$ -modules \mathcal{O}_X^t and \mathcal{O}_X^w of tempered and Whitney holomorphic functions respectively. References are made to [8] for an exposition on sheaves of rings on a Grothendieck topology.

3.1 – Modules over a $k_{X_{sa}}$ -algebra.

A sheaf of $k_{X_{sa}}$ -algebras (or a $k_{X_{sa}}$ -algebra, for short) is an object $\mathcal{R} \in \text{Mod}(k_{X_{sa}})$ such that $\Gamma(U; \mathcal{R})$ is a k -algebra for each $U \in \text{Op}(X_{sa})$ and the restriction maps are morphisms of algebras. Let us denote by \mathcal{R}^{op} the opposite $k_{X_{sa}}$ -algebra.

Let \mathcal{R} be a $k_{X_{sa}}$ -algebra and denote by $\text{Mod}(\mathcal{R})$ the category of sheaves of (left) \mathcal{R} -modules. The category $\text{Mod}(\mathcal{R})$ is a Grothendieck category and the family $\{\mathcal{R}_U\}_{U \in \text{Op}^c(X_{sa})}$ is a small system of generators. Moreover the forgetful functor $\text{for} : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(k_{X_{sa}})$ is exact.

In this Section we shall extend some results on $k_{X_{sa}}$ -modules, by replacing $k_{X_{sa}}$ with \mathcal{R} . Since the formalism is similar to that we developed previously we shall not give proofs. The functors

$$\begin{aligned} \text{Hom}_{\mathcal{R}} : \text{Mod}(\mathcal{R})^{op} \times \text{Mod}(\mathcal{R}) &\rightarrow \text{Mod}(k_{X_{sa}}), \\ \otimes_{\mathcal{R}} : \text{Mod}(\mathcal{R}^{op}) \times \text{Mod}(\mathcal{R}) &\rightarrow \text{Mod}(k_{X_{sa}}) \end{aligned}$$

are well defined. Let us summarize their properties:

- the functor $\text{Hom}_{\mathcal{R}}$ is left exact,
- the functor $\otimes_{\mathcal{R}}$ is right exact and commutes with \varinjlim .

Let X, Y be two real analytic manifolds, and let $f : X \rightarrow Y$ be a morphism of real analytic manifolds. Let \mathcal{R} be a $k_{Y_{sa}}$ -algebra. The functors f^{-1}, f_* and $f_{!!}$ induce functors

$$\begin{aligned} f^{-1} : \text{Mod}(\mathcal{R}) &\rightarrow \text{Mod}(f^{-1}\mathcal{R}), \\ f_* : \text{Mod}(f^{-1}\mathcal{R}) &\rightarrow \text{Mod}(\mathcal{R}), \\ f_{!!} : \text{Mod}(f^{-1}\mathcal{R}) &\rightarrow \text{Mod}(\mathcal{R}). \end{aligned}$$

Let us summarize their properties:

- the functor f^{-1} is exact and commutes with \varinjlim and $\otimes_{\mathcal{R}}$,
- the functor f_* is left exact and commutes with \varprojlim ,
- (f^{-1}, f_*) is a pair of adjoint functors,
- the functor $f_{!!}$ is left exact and commutes with filtrant \varinjlim .

Now we consider the derived category of sheaves of \mathcal{R} -modules.

DEFINITION 3.1.1. *An object $F \in \text{Mod}(\mathcal{R})$ is flat if the functor $\text{Mod}(\mathcal{R}^{op}) \ni G \rightarrow G \otimes_{\mathcal{R}} F$ is exact.*

Small direct sums and filtrant inductive limits of flat \mathcal{R} -modules are flat. Since the generators of $\text{Mod}(\mathcal{R})$ are flat, then the subcategory of $\text{Mod}(\mathcal{R})$ consisting of flat modules is generating. Thanks to flat objects we can find a left derived functor $\otimes_{\mathcal{R}}^L$ of the tensor product $\otimes_{\mathcal{R}}$.

DEFINITION 3.1.2. *An object $F \in \text{Mod}(\mathcal{R})$ is quasi-injective if its image via the forgetful functor $\text{for} : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(k_{X_{\text{sa}}})$ is quasi-injective.*

Let X, Y be two real analytic manifolds, and let $f : X \rightarrow Y$ be a real analytic map. Let \mathcal{R} be a $k_{Y_{\text{sa}}}$ -algebra. As in § 1.5 one can prove that quasi-injective objects are injective with respect to the functors f_* and $f_{!!}$. The functors Rf_* and $Rf_{!!}$ are well defined and projection formula, base change formula and Künneth formula remain valid for \mathcal{R} -modules. Moreover hypothesis (2.2) are satisfied and we have

THEOREM 3.1.3. *The functor $Rf_{!!} : D^+(f^{-1}\mathcal{R}) \rightarrow D^+(\mathcal{R})$ admits a right adjoint. We denote by $f^! : D^+(\mathcal{R}) \rightarrow D^+(f^{-1}\mathcal{R})$ the adjoint functor.*

3.2 – Sheaves of $\rho_! \mathcal{R}$ -modules.

We will consider the case where the ring is $\rho_! \mathcal{R}$, where \mathcal{R} is a sheaf of k_X -algebras. We will also assume the following hypothesis:

\mathcal{R} has finite flat dimension.

The functor $\rho_!$ induces an exact functor $\text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(\rho_! \mathcal{R})$ which is left adjoint to $\rho^{-1} : \text{Mod}(\rho_! \mathcal{R}) \rightarrow \text{Mod}(\mathcal{R})$. We will still denote by $\rho_!$ that functor. The functor $\rho_* : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(\rho_! \mathcal{R})$ is well defined too, in fact the morphism $\zeta_F \in \text{Hom}_{k_X}(\mathcal{R}, \mathcal{E}nd(F))$ defines a morphism in $\text{Hom}_{k_{X_{\text{sa}}}}(\rho_! \mathcal{R}, \mathcal{E}nd(\rho_* F))$. That follows from the chain of isomorphism

$$\begin{aligned} \text{Hom}_{k_{X_{\text{sa}}}}(\rho_! \mathcal{R}, \mathcal{E}nd(\rho_* F)) &\simeq \text{Hom}_{k_{X_{\text{sa}}}}(\rho_! \mathcal{R}, \rho_* \mathcal{E}nd(F)) \\ &\simeq \text{Hom}_{k_X}(\rho^{-1} \rho_! \mathcal{R}, \mathcal{E}nd(F)) \\ &\simeq \text{Hom}_{k_X}(\mathcal{R}, \mathcal{E}nd(F)). \end{aligned}$$

We briefly summarize the properties of these functors:

- ρ^{-1} commutes with $\otimes_{\mathcal{R}}^L, f^{-1}$ and $Rf_{!!}$,
- $R\rho_*$ commutes with $R\mathcal{H}om_{\mathcal{R}}$ and Rf_* ,
- $\rho_!$ commutes with $\otimes_{\mathcal{R}}^L$ and f^{-1} .

Finally we recall the following result (which has been proved in [7])

PROPOSITION 3.2.1. *Denote by $\tilde{\mathcal{R}}$ the presheaf $U \rightarrow \Gamma(\bar{U}; \mathcal{R})$, where $U \in \text{Op}^c(X_{sa})$. Suppose that F is a presheaf of $\tilde{\mathcal{R}}$ -modules and denote by F^{++} the sheaf associated to F . Then $F^{++} \in \text{Mod}(\rho_! \mathcal{R})$.*

PROOF. Let $U \in \text{Op}(X_{sa})$, and let $r \in \Gamma(\bar{U}; \mathcal{R})$. Then r defines a morphism $\Gamma(\bar{V}; \mathcal{R}) \otimes \Gamma(V; F) \rightarrow \Gamma(V; F)$ for each subanalytic $V \subset U$, hence an endomorphism of $(F^{++})|_{U_{X_{sa}}} \simeq (F|_{U_{X_{sa}}})^{++}$. This morphism defines a morphism of presheaves $\tilde{\mathcal{R}} \rightarrow \mathcal{E}nd(F^{++})$ and $\tilde{\mathcal{R}}^{++} \simeq \rho_! \mathcal{R}$ by Proposition 1.1.14. Then $F^{++} \in \text{Mod}(\rho_! \mathcal{R})$. □

3.3 – Some examples of subanalytic sheaves.

From now on, the base field is \mathbb{C} . Let M be a real analytic manifold. One denotes by \mathcal{C}_M^∞ and $\mathcal{D}b_M$ the sheaves of \mathcal{C}^∞ functions and Schwartz’s distributions respectively, and by \mathcal{D}_M the sheaf of finite order differential operators with analytic coefficients. As usual, given a sheaf F on M , we set $D'F = R\mathcal{H}om(F, \mathbb{C}_M)$.

In [4] the author defined the functor

$$T\mathcal{H}om(\cdot, \mathcal{D}b_M) : \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)^{op} \rightarrow \text{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic open subset of M and $Z = M \setminus U$. Then the sheaf $T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)$ is defined by the exact sequence

$$0 \rightarrow \Gamma_Z \mathcal{D}b_M \rightarrow \mathcal{D}b_M \rightarrow T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M) \rightarrow 0.$$

This functor is exact and extends as a functor in the derived category, from $D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$ to $D^b(\mathcal{D}_M)$. Moreover the sheaf $T\mathcal{H}om(F, \mathcal{D}b_M)$ is soft for any \mathbb{R} -constructible sheaf F .

DEFINITION 3.3.1. *One denotes by $\mathcal{D}b_M^t$ the presheaf of tempered distributions on M_{sa} defined as follows:*

$$U \mapsto \Gamma(M; \mathcal{D}b_M) / \Gamma_{M \setminus U}(M; \mathcal{D}b_M).$$

As a consequence of the Łojasiewicz’s inequalities [11], for $U, V \in \text{Op}(M_{sa})$ the sequence

$$0 \rightarrow \mathcal{D}b_M^t(U \cup V) \rightarrow \mathcal{D}b_M^t(U) \oplus \mathcal{D}b_M^t(V) \rightarrow \mathcal{D}b_M^t(U \cap V) \rightarrow 0$$

is exact. Then $\mathcal{D}b_M^t$ is a sheaf on M_{sa} . Moreover it follows by definition that $\mathcal{D}b_M^t$ is quasi-injective.

DEFINITION 3.3.2. *Let Z be a closed subset of M . We denote by $\mathcal{I}_{M,Z}^\infty$ the sheaf of C^∞ functions on M vanishing up to infinite order on Z .*

DEFINITION 3.3.3. *A Whitney function on a closed subset Z of M is an indexed family $F = (F^k)_{k \in \mathbb{N}^n}$ consisting of continuous functions on Z such that $\forall m \in \mathbb{N}, \forall k \in \mathbb{N}^n, |k| \leq m, \forall x \in Z, \forall \varepsilon > 0$ there exists a neighborhood U of x such that $\forall y, z \in U \cap Z$*

$$\left| F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leq \varepsilon d(y, z)^{m-|k|}.$$

We denote by $W_{M,Z}^\infty$ the space of Whitney C^∞ functions on Z . We denote by $\mathcal{W}_{M,Z}^\infty$ the sheaf $U \mapsto W_{U, U \cap Z}^\infty$.

In [6] the authors defined the functor

$$\cdot \otimes^w C_M^\infty : \text{Mod}_{\mathbb{R}\text{-c}}(C_M) \rightarrow \text{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic open subset of M and $Z = M \setminus U$. Then $C_U \otimes^w C_M^\infty = \mathcal{I}_{M,Z}^\infty$, and $C_Z \otimes^w C_M^\infty = \mathcal{W}_{M,Z}^\infty$. This functor is exact and extends as a functor in the derived category, from $D_{\mathbb{R}\text{-c}}^b(C_M)$ to $D^b(\mathcal{D}_M)$. Moreover the sheaf $F \otimes^w C_M^\infty$ is soft for any \mathbb{R} -constructible sheaf F .

DEFINITION 3.3.4. *One denotes by $C_M^{\infty,w}$ the presheaf of Whitney C^∞ functions on M_{sa} defined as follows:*

$$U \mapsto \Gamma(M; \text{Hom}(C_U, C_M) \otimes^w C_M^\infty).$$

As a consequence of a result of [12], for $U, V \in \text{Op}(M_{sa})$ the sequence

$$0 \rightarrow C_M^{\infty,w}(U \cup V) \rightarrow C_M^{\infty,w}(U) \oplus C_M^{\infty,w}(V) \rightarrow C_M^{\infty,w}(U \cap V)$$

is exact. Then $C_M^{\infty,w}$ is a sheaf on M_{sa} .

REMARK 3.3.5. *Let us consider a locally cohomologically trivial (l.c.t.) subanalytic open subset, i.e. $U \in \text{Op}(M_{sa})$ satisfying $D^i C_U \simeq C_{\overline{U}}$ and $D^i C_{\overline{U}} \simeq C_U$. Thanks to the triangulation theorem one can prove that l.c.t. open subanalytic subsets form a basis for the topology of M_{sa} , and given a*

l.c.t. $U \in \text{Op}(M_{sa})$ we have

$$\begin{aligned} \Gamma(U; \mathcal{C}_M^{\infty, w}) &= \Gamma(M; \mathcal{H}om(\mathbb{C}_U, \mathbb{C}_M) \overset{w}{\otimes} \mathcal{C}_M^{\infty}) \\ &\simeq \Gamma(M; \mathbb{C}_{\overline{U}} \overset{w}{\otimes} \mathcal{C}_M^{\infty}) \\ &= W_{M, \overline{U}}^{\infty}. \end{aligned}$$

Remark that $\Gamma(U; \mathcal{D}b_M^t)$ and $\Gamma(U, \mathcal{C}_M^{\infty, w})$ are $\Gamma(\overline{U}; \mathcal{D}_M)$ -modules for each $U \in \text{Op}(M_{sa})$, hence applying Proposition 3.2.1 the sheaves $\mathcal{D}b_M^t$ and $\mathcal{C}_M^{\infty, w}$ belong to $\text{Mod}(\rho; \mathcal{D}_M)$.

We have the following result

PROPOSITION 3.3.6. *For each $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$ one has the isomorphisms*

$$\begin{aligned} \rho^{-1} \mathcal{H}om(F, \mathcal{D}b_M^t) &\simeq T\mathcal{H}om(F, \mathcal{D}b_M), \\ \rho^{-1} R\mathcal{H}om(F, \mathcal{C}_M^{\infty, w}) &\simeq D'F \overset{w}{\otimes} \mathcal{C}_M^{\infty}. \end{aligned}$$

PROOF. We may reduce to the case $F = \mathbb{C}_U$ with $U \in \text{Op}^c(M_{sa})$. Let $V \in \text{Op}^c(M_{sa})$.

By definition of $T\mathcal{H}om$ we have $\Gamma(V; T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)) \simeq \Gamma(U \cap V; \mathcal{D}b_V^t)$. Let us consider a subanalytic $W \subset\subset V$. The natural morphism $\Gamma(U \cap V; \mathcal{D}b_V^t) \rightarrow \Gamma(U \cap W; \mathcal{D}b_M^t)$ defines the morphism

$$\varphi : \Gamma(U \cap V; \mathcal{D}b_V^t) \rightarrow \varprojlim_{W \subset\subset V} \Gamma(U \cap W; \mathcal{D}b_M^t) \simeq \Gamma(V; \rho^{-1} \Gamma_U \mathcal{D}b_M^t).$$

Since the family $\{W \in \text{Op}^c(M_{sa}); W \subset\subset V\}$ is a covering of V and $T\mathcal{H}om(\mathbb{C}_U, \mathcal{D}b_M)$ is a sheaf φ is an isomorphism.

To prove the second isomorphism we shall first prove the isomorphism

$$(3.1) \quad \rho^{-1} \mathcal{H}om(F, \mathcal{C}_M^{\infty, w}) \simeq \mathcal{H}om(F, \mathbb{C}_M) \overset{w}{\otimes} \mathcal{C}_M^{\infty}$$

for $F \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. We may reduce to the case $F = \mathbb{C}_U$ with U l.c.t. and subanalytic. Let $V \in \text{Op}^c(M_{sa})$ such that V and $U \cap V$ are l.c.t. and let us consider the family $\mathcal{T} = \{W \in \text{Op}^c(M_{sa}) \text{ l.c.t.}; W \subset\subset V, W \cap U \text{ l.c.t.}\}$. The natural morphism $\psi : \Gamma(V; \mathbb{C}_{\overline{U}} \overset{w}{\otimes} \mathcal{C}_M^{\infty}) \simeq W_{V, V \cap \overline{U}}^{\infty} \rightarrow W_{M, U \cap \overline{W}}^{\infty} \simeq \Gamma(M; \mathbb{C}_{\overline{U \cap W}} \overset{w}{\otimes} \mathcal{C}_M^{\infty})$ defines the morphism

$$\psi : \Gamma(V; \mathbb{C}_{\overline{U}} \overset{w}{\otimes} \mathcal{C}_M^{\infty}) \rightarrow \varprojlim_{W \in \mathcal{T}} \Gamma(M; \mathbb{C}_{\overline{U \cap W}} \overset{w}{\otimes} \mathcal{C}_M^{\infty}) \simeq \Gamma(V; \rho^{-1} \Gamma_U \mathcal{C}_M^{\infty, w}),$$

where the second isomorphism follows since the family \mathcal{T} is cofinal in $\{W \in \text{Op}^c(M_{sa}); W \subset \subset V\}$. Moreover we have the isomorphism

$$\varprojlim_{W \in \mathcal{T}} \Gamma(M; \mathbb{C}_{U \cap W}^{\text{w}} \otimes \mathcal{C}_M^\infty) \simeq \varprojlim_{W \in \mathcal{T}} \Gamma(W; \mathbb{C}_{\overline{U}}^{\text{w}} \otimes \mathcal{C}_M^\infty).$$

Since the family \mathcal{T} is a covering of V and $\mathbb{C}_{\overline{U}}^{\text{w}} \otimes \mathcal{C}_M^\infty$ is a sheaf ψ is an isomorphism. Hence we get the desired isomorphism.

Now let $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$. We have the chain of morphisms

$$\begin{aligned} D'F \otimes \mathbb{C}_M^{\text{w}} &\simeq \varinjlim_{F' \rightarrow F} \mathcal{H}om(F', \mathbb{C}_M) \otimes \mathbb{C}_M^{\text{w}} \\ &\simeq \varinjlim_{F' \rightarrow F} \rho^{-1} \mathcal{H}om(F', \mathcal{C}_M^{\infty, \text{w}}) \\ &\simeq \rho^{-1} R\mathcal{H}om(F, \mathcal{C}_M^{\infty, \text{w}}), \end{aligned}$$

where $F' \rightarrow F$ ranges to the family of qis. By Theorem 2.1.2 we may suppose $F' \in K^b(\text{Mod}_{\mathbb{R}\text{-c}}(k_M))$, then we can restrict to $F' = \mathbb{C}_U$, $U \in \text{Op}^c(M_{sa})$ l.c.t. and the second isomorphism follows from (3.1). The third isomorphism follows since ρ^{-1} is exact. \square

REMARK 3.3.7. *As a consequence of Proposition 3.3.6, given $U \in \text{Op}(M_{sa})$ we have $R\Gamma(U; \mathcal{C}_M^{\infty, \text{w}}) \simeq R\Gamma(M; D'\mathbb{C}_U \otimes \mathbb{C}_M^{\text{w}})$. In particular when U is l.c.t. $R\Gamma(U; \mathcal{C}_M^{\infty, \text{w}})$ is concentrated in degree zero since $\mathbb{C}_{\overline{U}}^{\text{w}} \otimes \mathbb{C}_M^{\text{w}}$ is soft.*

Now let X be a complex manifold, $X_{\mathbb{R}}$ the underlying real analytic manifold and \overline{X} the complex conjugate manifold. One denotes by \mathcal{O}_X^t and \mathcal{O}_X^{w} the sheaves of tempered and Whitney holomorphic functions respectively which are defined as follows:

$$\begin{aligned} \mathcal{O}_X^t &:= R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{O}_{\overline{X}}, \mathcal{D}b_{X_{\mathbb{R}}}^t) \\ \mathcal{O}_X^{\text{w}} &:= R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{O}_{\overline{X}}, \mathcal{C}_{X_{\mathbb{R}}}^{\infty, \text{w}}). \end{aligned}$$

By definition, \mathcal{O}_X^t and \mathcal{O}_X^{w} belong to $D^b(\rho_! \mathcal{D}_X)$. The relation with the functors of temperate and formal cohomology is given by the following result

PROPOSITION 3.3.8. *For each $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X)$ one has the isomorphisms*

$$\begin{aligned} \rho^{-1} R\mathcal{H}om(F, \mathcal{O}_X^t) &\simeq T\mathcal{H}om(F, \mathcal{O}_X), \\ \rho^{-1} R\mathcal{H}om(F, \mathcal{O}_X^{\text{w}}) &\simeq D'F \otimes \mathbb{C}_X^{\text{w}}. \end{aligned}$$

PROOF. We have the chain of isomorphisms

$$\begin{aligned}
 \rho^{-1}R\mathcal{H}om(F, \mathcal{O}_X^t) &\simeq \rho^{-1}R\mathcal{H}om(F, R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{O}_{\overline{X}}, \mathcal{D}b_{X_{\mathbb{R}}}^t)) \\
 &\simeq \rho^{-1}R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{O}_{\overline{X}}, R\mathcal{H}om(F, \mathcal{D}b_{X_{\mathbb{R}}}^t)) \\
 &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{\overline{X}}, \rho^{-1}R\mathcal{H}om(F, \mathcal{D}b_{X_{\mathbb{R}}}^t)) \\
 &\simeq R\mathcal{H}om(\mathcal{O}_{\overline{X}}, T\mathcal{H}om(F, \mathcal{D}b_{X_{\mathbb{R}}}^t)) \\
 &\simeq T\mathcal{H}om(F, \mathcal{O}_X).
 \end{aligned}$$

The proof of $\rho^{-1}R\mathcal{H}om(F, \mathcal{O}_X^w) \simeq D'F \overset{w}{\otimes} \mathcal{O}_X$ is similar. □

A. Appendix.

A.1 – Review on subanalytic sets.

We recall briefly some properties of subanalytic subsets. References are made to [1] and [10]. Let X be a real analytic manifold.

DEFINITION A.1.1. *Let A be a subset of X .*

(i) *A is said to be semi-analytic if it is locally analytic, i.e. each $x \in A$ has a neighborhood U_x such that $X \cap U_x = \cup_{i \in I} \cap_{j \in J} X_{ij}$, where I, J are finite sets and either $X_{ij} = \{y \in U_x; f_{ij} > 0\}$ or $X_{ij} = \{y \in U_x; f_{ij} = 0\}$ for some analytic function f_{ij} .*

(ii) *A is said to be subanalytic if it is locally a projection of a relatively compact semi-analytic subset, i.e. each $x \in A$ has a neighborhood U such that there exists a real analytic manifold Y and a relatively compact semi-analytic subset $A' \subset X \times Y$ satisfying $X \cap U = \pi(A')$, where $\pi : X \times Y \rightarrow X$ denotes the projection.*

(iii) *Let Y be a real analytic manifold. A continuous map $f : X \rightarrow Y$ is subanalytic if its graph is subanalytic in $X \times Y$.*

Let us recall some result on subanalytic subsets.

PROPOSITION A.1.2. *Let A, B be subanalytic subsets of X . Then $A \cup B, A \cap B, \overline{A}, \partial A$ and $A \setminus B$ are subanalytic.*

PROPOSITION A.1.3. *Let A be a subanalytic subsets of X . Then the connected components of A are locally finite.*

PROPOSITION A.1.4. *Let $f : X \rightarrow Y$ be a subanalytic map. Let A be a relatively compact subanalytic subset of X . Then $f(A)$ is subanalytic.*

DEFINITION A.1.5. *A simplicial complex (K, Δ) is the data consisting of a set K and a set Δ of subsets of K satisfying the following axioms:*

- S1 *any $\sigma \in \Delta$ is a finite and non-empty subset of K ,*
- S2 *if τ is a non-empty subset of an element σ of Δ , then τ belongs to Δ ,*
- S3 *for any $p \in K$, $\{p\}$ belongs to Δ ,*
- S4 *for any $p \in K$, the set $\{\sigma \in \Delta; p \in \sigma\}$ is finite.*

If (K, Δ) is a simplicial complex, an element of K is called a vertex. Let \mathbb{R}^K be the set of maps from K to \mathbb{R} equipped with the product topology. To $\sigma \in \Delta$ one associate $|\sigma| \subset \mathbb{R}^K$ as follows:

$$|\sigma| = \left\{ x \in \mathbb{R}^K; x(p) = 0 \text{ for } p \notin \sigma, x(p) > 0 \text{ for } p \in \sigma \text{ and } \sum_p x(p) = 1 \right\}.$$

As usual we set:

$$|K| = \bigcup_{\sigma \in \Delta} |\sigma|,$$

$$U(\sigma) = \bigcup_{\tau \in \Delta, \tau \supset \sigma} |\tau|,$$

and for $x \in |K|$:

$$U(x) = U(\sigma(x)),$$

where $\sigma(x)$ is the unique simplex such that $x \in |\sigma|$.

THEOREM A.1.6. *Let $X = \bigsqcup_{i \in I} X_i$ be a locally finite partition of X consisting of subanalytic subsets. Then there exists a simplicial complex (K, Δ) and a subanalytic homeomorphism $\psi : |K| \xrightarrow{\sim} X$ such that*

- (i) *for any $\sigma \in \Delta$, $\psi(|\sigma|)$ is a subanalytic submanifold of X ,*
- (ii) *for any $\sigma \in \Delta$ there exists $i \in I$ such that $\psi(|\sigma|) \subset X_i$.*

A.2 – Sheaves on Grothendieck topologies.

We recall the definition of a Grothendieck topology. We will not treat the most general case, for which we refer to [8]. We will follow the presentation of [14] and [7].

Let \mathcal{C} be a category admitting finite products and fiber products, and given $U \in \mathcal{C}$, denote by \mathcal{C}_U the category of arrows $V \rightarrow U$. Given a morphism $V \rightarrow U$ and $S \subset \text{Ob}(\mathcal{C}_U)$, one denotes by $V \times_U S \subset \text{Ob}(\mathcal{C}_V)$ the subset defined by $\{V \times_U W \rightarrow V; W \in S\}$.

DEFINITION A.2.1. *If, $S_1, S_2 \subset \text{Ob}(\mathcal{C}_U)$, one says that S_1 is a refinement of S_2 ($S_1 \preceq S_2$ for short) if any $V_1 \rightarrow U$ in S_1 factorizes as $V_1 \rightarrow V_2 \rightarrow U$ with $V_2 \rightarrow U \in S_2$.*

DEFINITION A.2.2. *A Grothendieck topology on \mathcal{C} associates to each $U \in \mathcal{C}$ a family $\text{Cov}(U) \subset \text{Ob}(\mathcal{C}_U)$ satisfying the following axioms:*

GT1 $\{U \xrightarrow{\text{id}} U\} \in \text{Cov}(U)$,

GT2 if $\text{Cov}(U) \ni S_1 \preceq S_2 \subset \text{Ob}(\mathcal{C}_U)$, then $S_2 \in \text{Cov}(U)$,

GT3 if $S \in \text{Cov}(U)$, then for each $V \rightarrow U$, $V \times_U S \in \text{Cov}(V)$,

GT4 if $S_1, S_2 \subset \text{Ob}(\mathcal{C}_U)$, $S_1 \in \text{Cov}(U)$ and $V \times_U S_2 \in \text{Cov}(V)$, then $S_2 \in \text{Cov}(U)$.

An object $S \in \text{Cov}(U)$ is called a covering of U .

DEFINITION A.2.3. *A site X is a category \mathcal{C}_X endowed with a Grothendieck topology.*

Let \mathcal{C}_X and \mathcal{C}_Y be two categories admitting finite products and fiber products. A functor of sites $f : X \rightarrow Y$ is a functor $f^t : \mathcal{C}_Y \rightarrow \mathcal{C}_X$ which commutes with fiber products and such that if $U \in \mathcal{C}_Y$ and $S \in \text{Cov}(U)$, then $f^t(S) \in \text{Cov}(f^t(U))$.

Now let k be a field.

DEFINITION A.2.4. *Let X be a site. A presheaf of k -modules on X is a functor $\mathcal{C}_X^{op} \rightarrow \text{Mod}(k)$.*

One denotes by $\text{Psh}(k_X)$ the abelian category of presheaves of k -modules on X . Let $F \in \text{Psh}(k_X)$, let $U \in \mathcal{C}_X$ and consider $V \rightarrow U \in \mathcal{C}_U$. The restriction morphism $F(U) \rightarrow F(V)$ is denoted by $s \mapsto s|_U$.

Let F be a presheaf of k -modules on X and let $S \subset \text{Ob}(\mathcal{C}_U)$. One defines

$$F(S) = \ker \left(\prod_{V \in S} F(V) \rightrightarrows \prod_{V', V'' \in S} F(V' \times_U V'') \right)$$

DEFINITION A.2.5. *A presheaf F of k -modules on X is a separated presheaf (resp. a sheaf) if for each $U \in \mathcal{C}_X$ and each $S \in \text{Cov}(U)$ the morphism $F(U) \rightarrow F(S)$ is a monomorphism (resp. an isomorphism).*

One denotes by $\text{Mod}(k_X)$ the category of sheaves of k -modules on X . We set for short Hom_{k_X} instead of $\text{Hom}_{\text{Mod}(k_X)}$.

We recall the construction of a sheaf associated to a presheaf. The relation “ \preceq ” defines a preorder on $\text{Cov}(U)$, $U \in \mathcal{C}_X$. Let $F \in \text{Psh}(k_X)$, one defines the functor $(\cdot)^+ : \text{Psh}(k_X) \rightarrow \text{Psh}(k_X)$ in the following way. For each $U \in \mathcal{C}_X$

$$F^+(U) = \varinjlim_{S \in \text{Cov}(U)} F(S)$$

THEOREM A.2.6. (i) *The functor $(\cdot)^+ : \text{Psh}(k_X) \rightarrow \text{Psh}(k_X)$ is left exact,*

(ii) *if $F \in \text{Psh}(k_X)$, then F^+ is separated,*

(iii) *if $F \in \text{Psh}(k_X)$ is separated, then $F^+ \in \text{Mod}(k_X)$,*

(iv) *the functor $(\cdot)^{++} : \text{Psh}(k_X) \rightarrow \text{Mod}(k_X)$ is exact,*

(v) *let $F \in \text{Psh}(k_X)$ and $G \in \text{Mod}(k_X)$, one has the adjunction formula:*

$$\text{Hom}_{\text{Psh}(k_X)}(F, iG) \simeq \text{Hom}_{k_X}(F^{++}, G),$$

where i denotes the embedding functor.

Let $F \in \text{Psh}(k_X)$, the sheaf F^{++} is called the sheaf associated to F .

PROPOSITION A.2.7. *Let $F, G \in \text{Mod}(k_X)$. A morphism $\varphi \in \text{Hom}_{k_X}(F, G)$ is an epimorphism if and only if for each $U \in \mathcal{C}_X$ there exists $\{U_i\}_{i \in I} \in \text{Cov}(U)$ such that for each $s \in G(U)$ there exists $t_i \in F(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for each i .*

Let $f : X \rightarrow Y$ be a morphism of sites. Let $F \in \text{Psh}(k_X)$ and $G \in \text{Psh}(k_Y)$. One defines the functors

$$(A.1) \quad f_* : \text{Psh}(k_X) \rightarrow \text{Psh}(k_Y)$$

$$(A.2) \quad f^{\leftarrow} : \text{Psh}(k_Y) \rightarrow \text{Psh}(k_X)$$

in the following way: let $U \in \mathcal{C}_X$ and $V \in \mathcal{C}_Y$, then

$$f_*F(V) = F(f^t(V))$$

$$f^{-1}F(U) = \varinjlim_{U \rightarrow f^t(W)} G(W),$$

where $W \in \mathcal{C}_Y$.

DEFINITION A.2.8. *Let $f : X \rightarrow Y$ be a functor of sites*

- (i) *the functor of direct image $f_* : \text{Mod}(k_X) \rightarrow \text{Mod}(k_Y)$ is the functor induced by (A.1),*
- (ii) *the functor of inverse image $f^{-1} : \text{Mod}(k_Y) \rightarrow \text{Mod}(k_X)$ is defined by $f^{-1} = (f^t(\cdot))^{++}$.*

PROPOSITION A.2.9. (i) *The functor f_* is left exact and commutes with \varprojlim ,*

- (ii) *the functor f^{-1} is exact and commutes with \varinjlim ,*
- (iii) *(f^{-1}, f_*) is a pair of adjoint functors.*

DEFINITION A.2.10. *Let X be a site and let $F, G \in \text{Mod}(k_X)$.*

- (i) *One denotes by $\mathcal{H}om(F, G)$ the sheaf $U \mapsto \text{Hom}_{k_U}(F|_U, G|_U)$,*
- (ii) *one denotes by $F \otimes G$ the sheaf associated to the presheaf $U \mapsto F(U) \otimes G(U)$.*

PROPOSITION A.2.11. *Let $F \in \text{Mod}(k_X)$, $G, G' \in \text{Mod}(k_Y)$.*

- (i) *$\mathcal{H}om_{k_Y}(G, f_*F) \simeq f_*\mathcal{H}om_{k_X}(f^{-1}G, F)$,*
- (ii) *$f^{-1}(G \otimes G') \simeq f^{-1}G \otimes f^{-1}G'$.*

REFERENCES

- [1] E. BIERSTORNE - D. MILMANN, *Semianalytic and subanalytic sets*, Publ. I.H.E.S., **67** (1988), pp. 5–42.
- [2] S. I. GELFAND - YU. I. MANIN, *Methods of homological algebra*, Springer-Verlag, Berlin (1996).
- [3] R. GODEMENT, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris (1958).

- [4] M. KASHIWARA, *The Riemann-Hilbert problem for holonomic systems*, Publ. RIMS, Kyoto Univ., **20** (1984), pp. 319–365.
- [5] M. KASHIWARA - P. SCHAPIRA, *Sheaves on manifolds*, Grundlehren der Math. **292**, Springer-Verlag, Berlin (1990).
- [6] M. KASHIWARA - P. SCHAPIRA, *Moderate and formal cohomology associated with constructible sheaves*, Mémoires Soc. Math. France, **64** (1996).
- [7] M. KASHIWARA - P. SCHAPIRA, *Ind-sheaves*, Astérisque, **271** (2001).
- [8] M. KASHIWARA - P. SCHAPIRA, *Categories and sheaves*, Grundlehren der Math., **332**, Springer-Verlag, Berlin (2005).
- [9] B. KELLER, *Derived categories and their uses*, Handbook of algebra vol. **1** pp. 671–701 North Holland, Amsterdam (1996).
- [10] S. ŁOJACIEWICZ, *Sur la géométrie semi- et sous-analytique*, Ann. Inst. Fourier, **43** (1993), pp. 1575–1595.
- [11] S. ŁOJACIEWICZ, *Sur le problème de la division*, Studia Mathematica, **8** (1959) pp. 87–136.
- [12] B. MALGRANGE, *Ideals of differentiable functions*, Tata Institute, Oxford University Press (1967).
- [13] L. PRELLI, *Sheaves on subanalytic sites*, Phd Thesis, Universities of Padova and Paris 6 (2006).
- [14] G. TAMME, *Introduction to étale cohomology*, Universitext Springer-Verlag, Berlin (1994).
- [15] J. L. VERDIER, *Catégories dérivées, état 0*, SGA 4₃, Lecture Notes in Math., **569** Springer-Verlag, Berlin (1977).
- [16] SGA4: Sém. Géom. Algébrique du Bois-Marie by M. Artin, A. Grothendieck, J. L. Verdier, Lecture Notes in Math. **269** Springer-Verlag, Berlin (1972).

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