

## Fibered Cusp Versus $d$ -Index Theory.

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ABSTRACT - We prove that the indices of fibered-cusp and  $d$ -Dirac operators on a spin manifold with fibered boundary coincide if the associated family of Dirac operators on the fibers of the boundary is invertible. This answers a question raised by Piazza. Under this invertibility assumption, our method yields an index formula for the Dirac operator of horn-cone and of fibered horn metrics.

Let  $\bar{X}$  be a compact manifold whose boundary is the total space of a locally trivial fiber bundle  $\varphi : \partial X \rightarrow Y$  of closed manifolds. Let  $x : \bar{X} \rightarrow \bar{\mathbb{R}}_+$  be a defining function for  $\partial X$  and denote by  $X$  the interior of  $\bar{X}$ .

The *fibered cusp* (or  $\Phi$ -) tangent bundle  ${}^\Phi TX$  is a smooth vector bundle on  $\bar{X}$  defined in terms of the above data by its global sections:

$$\mathcal{C}^\infty(\bar{X}, {}^\Phi TX) := \{V \in \mathcal{C}^\infty(\bar{X}, T\bar{X}); V|_{\partial X} \text{ is tangent to the fibers of } \varphi, \\ \langle dx, V \rangle \in x^2 \mathcal{C}^\infty(\bar{X})\}.$$

When restricted to  $X$ , the  $\Phi$  tangent bundle is canonically isomorphic to the usual tangent bundle  $TX$ . By definition, a *fibered cusp* metric  $g_\Phi$  is the restriction to  $X$  of a Euclidean metric in the bundle  ${}^\Phi TX$ , smooth down to the boundary of  $\bar{X}$ . Let

$$g_d := x^2 g_\Phi$$

be the conformally equivalent  $d$ -metric. Such metrics appear naturally in a variety of geometric situations.

EXAMPLE 1. Let  $(X, g)$  be a complete hyperbolic manifold of finite volume. Then outside a convex set,  $X$  is isometric to the disjoint union of a

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finite number of “cusps”, i.e., cylinders  $[0, \infty) \times M$  with metric

$$dt^2 + e^{-2t}h_M,$$

where  $h_M$  is flat. Compactify  $X$  by setting  $\bar{X} := X \sqcup (\{\infty\} \times M)$ . This space becomes a smooth manifold with boundary if we impose that  $e^{-t}$  be a boundary-defining function. By the change of variables  $x = e^{-t}$ , the metric becomes a  $d$ -metric for the trivial boundary fibration  $M \rightarrow \{pt\}$ . More generally, a locally symmetric space  $X$  with  $\mathbb{Q}$ -rank 1 cusps is diffeomorphic, outside a compact set, to  $[R, \infty) \times M$  where  $M$  is the total space of a fibration  $\phi : M \rightarrow Y$  with a canonical connection; moreover,  $X$  has a natural Riemannian metric which near infinity takes the form

$$dt^2 + \phi^*g_Y + e^{-2t}g_Z$$

where  $g_Y, g_Z$  are a metric on the base, respectively a family of metrics on the fibers.

**EXAMPLE 2.** The standard metric on  $X = \mathbb{R}^n$  is an example of a  $\Phi$  metric for  $\bar{X}$  the radial compactification,  $x := 1/r$  and for the identity fibration at infinity  $1 : S^{n-1} \rightarrow S^{n-1}$  ( $\Phi$ -metrics for the identity fibration are also called *scattering metrics*).

More generally, several examples of complete Ricci-flat metrics are of  $\Phi$  type.

To understand better these metrics, choose a product decomposition of  $X$  near the boundary of  $\bar{X}$  and fix a connection in the fibration  $\phi$ .

**DEFINITION 3.** A *product  $\Phi$ -metric* (with respect to the above choices) is a  $\Phi$ -metric which near infinity takes the form

$$\frac{dx^2}{x^4} + \frac{\phi^*g^Y}{x^2} + h$$

where  $g^Y$  is a metric on  $Y$  and  $h$  a family of metrics on the fibers, both independent of  $x$ .

Thus locally symmetric spaces with  $\mathbb{Q}$ -rank 1 cusps have product  $\Phi$ -metrics.

It is straightforward to decide on the completeness of metrics conformal to a  $\Phi$ -metric.

LEMMA 4. *Let  $g$  be a  $\Phi$ -metric. Then for  $p \in \mathbb{R}$ , the metric  $x^{2p}g$  is complete if and only if  $p \leq 1$ .*

PROOF. By compactness of  $\overline{X}$ , there exists a product  $\Phi$ -metric  $g'$  on  ${}^\Phi TX$  and a constant  $C > 0$  such that

$$C^{-1}g' \leq g \leq Cg'.$$

Hence,  $x^{2p}g$  is complete if and only if  $x^{2p}g'$  is. It is evident that a metric which outside a compact set has the form

$$x^{2p-4}dx^2 + h(x)$$

on  $(0, \varepsilon) \times M$  (where infinity corresponds to  $x = 0$ ), is complete if and only if the length of the segment  $(0, \varepsilon) \times \{m\}$  is infinite for all  $m \in M$ , or in other words if  $\int_0^\varepsilon x^{p-2}dx = \infty$ , which is equivalent to  $p \leq 1$ . □

DEFINITION 5. An exact  $\Phi$ -metric  $g_\Phi$  on  $X$  is a  $\Phi$ -metric which differs from a product  $\Phi$ -metric by a tensor in  $x^{C^\infty}(\overline{X}, S^2({}^\Phi TX))$ , i.e., by a symmetric bilinear form on the bundle  ${}^\Phi TX$ , smooth down to  $x = 0$  and vanishing at  $x = 0$ .

This agrees with the definition from [8]. The name “exact” is used by analogy with Melrose’s exact  $b$ -metrics. In the rest of the paper we will work with an exact  $\Phi$ -metric. These metrics were also considered by Leichtnam, Mazzeo and Piazza [2], under the additional hypothesis that the product decomposition  $[c, \infty) \times M$  is orthogonal near infinity. Note that what we call  $d$ - respectively fibered cusp (or  $\Phi$ ) metrics is called fibered cusp, respectively fibered boundary (or  $\Phi$ ) metrics in [2]. We keep our terminology for historical reasons.

### 1. – The main Theorem.

We assume that  $X$ ,  $\partial X$  and the fibers of  $\varphi$  have fixed (and compatible) spin structures. Let  $E \rightarrow \overline{X}$  be a Hermitian vector bundle with connection (smooth down to  $x = 0$ ). Since both  $g_\Phi$  and  $g_d$  are complete by Lemma 1.1, their associated Dirac operators on  $X$  (twisted by  $E$ ) are essentially self-adjoint in  $L^2$ .

The  $L^2$  index of the Dirac operator  $D^d$  was computed by Vaillant [8] by making extensive use of  $\Phi$  operators. He assumes that the dimension

of the kernel of the family of Dirac operators on the fibers of  $\varphi$  is constant. In this note we observe that in the fully elliptic case (i.e., when the above family of Dirac operators is invertible) the indices of  $D^d$  and  $D^\phi$  are the same. The possibility of such a result was conjectured by Paolo Piazza in a private communication. In general, there exists a link between the kernels, which implies that under Vaillant's hypothesis, the index of  $D^\phi$  is finite.

Our proof works for a more general metric  $x^{2p}g_\phi$  for some  $p \geq 0$ . We denote the associated Dirac operator by  $D^p$ . The metric  $g_d$  is obtained by setting  $p = 1$ , but interesting geometries are also obtained for other values of  $p$ . Using Vaillant's work, we give at the end of the paper an index formula for a manifold with various such ends.

**THEOREM 6.** *Let  $g_\phi$  be an exact  $\Phi$ -metric on  $X$ , such that the twisted Dirac operators on the fibers of  $\partial X$  with respect to  $E$ , the induced metric on the fibers of the boundary and the induced spin structure are invertible. Then the map of multiplication by  $x^{\frac{p(n-1)}{2}}$  defines an isomorphism between the  $L^2$ -kernels of the operators  $D^p$  and  $D^\phi$ . In particular the  $L^2$  indices (of the chiral operators) coincide.*

**PROOF.** The spinor bundles on  $X$  with respect to  $x^{2p}g_\phi$ , respectively  $g_\phi$  are the same (i.e., canonically identified) and inherit the same induced metric. We denote this unique spinor bundle by  $\Sigma$ . It extends naturally to a smooth vector bundle over  $\bar{X}$ . In the rest of the proof, we suppress the coefficient bundle  $\Sigma \otimes E$  from the notation.

**LEMMA 7.** *The unbounded operator  $D^p$  acting in  $L^2(X, x^{2p}g_\phi)$  with initial domain  $C_c^\infty(X)$  is unitarily equivalent to  $x^{-p/2}D^\phi x^{-p/2}$  acting in  $L^2(X, g_\phi)$  with domain  $C_c^\infty(X)$ .*

**PROOF.** These are the full (symmetric) Dirac operators. The Dirac operators are linked by the formula of Hitchin (also used by Vaillant [8])

$$(1) \quad D^p = x^{-p(n+1)/2} D^\phi x^{p(n-1)/2}.$$

The two volume forms on  $X$  are related by

$$\text{vol}(x^{2p}g_\phi) = x^{np} \text{vol}(g_\phi).$$

Thus the map

$$C_c^\infty(X) \rightarrow C_c^\infty(X), \quad \phi \mapsto x^{-\frac{n}{2}}\phi$$

is an isometry with respect to the  $L^2$  inner products. The conjugation of  $D^p$  via this isometry is illustrated by the diagram (of unbounded operators with domain  $C_c^\infty(X)$ ):

$$(2) \quad \begin{array}{ccc} L^2(X, x^{2p}g_\phi) & \xrightarrow{D^p} & L^2(X, x^{2p}g_\phi) \\ \downarrow x^{\frac{m}{2}} & & \downarrow x^{\frac{m}{2}} \\ L^2(X, g_\phi) & \xrightarrow{x^{\frac{m}{2}}D^p x^{-\frac{m}{2}}} & L^2(X, g_\phi) \end{array}$$

Using (1), we see that  $x^{\frac{m}{2}}D^p x^{-\frac{m}{2}} = x^{-\frac{p}{2}}D^\phi x^{-\frac{p}{2}}$ . □

Unitarily equivalent operators have isomorphic kernels, hence the index of the chiral part  $D_+^p$  of  $D^p$  equals the index of  $x^{-\frac{p}{2}}D_+^\phi x^{-\frac{p}{2}}$ .

The analytic facts that we need from the general theory of  $\Phi$  operators [3] are the existence of the weighted  $\Phi$ -Sobolev spaces  $x^a H_\phi^b(X)$  on which  $\Phi$ -operators act, as well as the existence of parametrices inside the  $\Phi$  calculus [3, Proposition 8]; that is, for every  $A \in \Psi_\phi^{a,b}$  elliptic there exists  $B \in \Psi_\phi^{-a,-b}$  with

$$BA - 1 \in \Psi_\phi^{-\infty,0}.$$

By definition,  $A \in \Psi_\phi^{a,b}$  is called *fully elliptic* if the normal operator of  $x^b A$  is invertible as a family over  $Y$  of suspended operators acting on the fibers of  $\phi$ . If  $A$  is fully elliptic, then  $BA - 1$  can even be made to belong to  $\Psi_\phi^{-\infty,-\infty}$ .

LEMMA 8. *The operators  $D^\phi$  and  $x^{-\frac{p}{2}}D^\phi x^{-\frac{p}{2}}$  are simultaneously fully-elliptic.*

PROOF. Note that  $D^\phi \in \Psi_\phi^{1,0}(X)$  and  $x^{-\frac{p}{2}}D^\phi x^{-\frac{p}{2}} \in \Psi_\phi^{1,p}(X, \Sigma)$ . In cusptype calculi, it is a basic fact that commutation by a power of  $x$  decreases the total  $x$ -order (i.e., *increases* the power of  $x$ ; the filtration is defined by the negative of the power of  $x$  so that it is decreasing, like the symbol filtration). Namely, for  $A \in \Psi_\phi^{a,b}(X)$ , we have

$$[A, x^p] \in \Psi_\phi^{a-1,b+p-1}(X).$$

This implies that the normal operator satisfies

$$\mathcal{N}(x^p \cdot x^{-\frac{p}{2}}D^\phi x^{-\frac{p}{2}}) = \mathcal{N}(D^\phi). \quad \square$$

REMARK 9. This lemma fails for  $b$ -operators, see [4].

The proof of the following “elliptic regularity” lemma is standard.

LEMMA 10. *Let  $A \in \Psi_\phi^{a,b}(X, \mathcal{E}, \mathcal{F})$  be fully elliptic. Then the  $L^2$  solutions of  $A\psi = 0$  belong to the ideal  $x^\infty C^\infty(\bar{X}, \mathcal{E})$ .*

PROOF. Since  $A$  is fully elliptic there exists  $B \in \Psi_\phi^{-a,-b}(X, \mathcal{F}, \mathcal{E})$  inverting  $A$  up to  $R \in x^\infty \Psi_\phi^{-\infty}(X, \mathcal{E})$ , i.e.,

$$BA = 1 + R.$$

Let  $\psi \in L^2(X, \mathcal{E})$  be a distributional solution of the pseudo-differential equation  $A\psi = 0$ . It follows

$$0 = BA\psi = (1 + R)\psi = \psi + R\psi$$

so  $\psi = -R\psi$ . But  $R \in x^\infty \Psi_\phi^{-\infty}(X, \mathcal{E})$  implies  $R\psi \in x^\infty C^\infty(\bar{X}, \mathcal{E})$ . □

We can now finish the proof of Theorem 1.1. Recall that we assumed the family of Dirac operators  $D^F$  on the fibers of  $\varphi$  to be invertible. By [8, Lemma 3.7] this implies that the normal operator of  $D^\phi$  is invertible, i.e., that  $D^\phi$  is fully elliptic. For the convenience of the reader, we include below a direct proof. Recall that an exact  $\Phi$ -metric is assumed to coincide with a product  $\Phi$ -metric up to first order terms in  $x$ . Since  $xC^\infty(\bar{X}, {}^\phi TX)$  is an ideal in the Lie algebra of fibered cusp vector fields, it follows that the Levi-Civita connection and the Dirac operator also agree, up to first order terms, with the corresponding objects for the product  $\Phi$ -metric. For that metric, it is straightforward to see (once the definition of the normal operator is recalled) that

$$\mathcal{N}(D^\phi)^2(\tau) = \|\tau\|_{g_Y}^2 + (D^F)^2.$$

Therefore  $\mathcal{N}(D^\phi)^2(\tau)$  is invertible for all  $\tau \in TY$  if and only if the family  $D^F$  is invertible.

In conclusion, the map

$$\ker(x^{-\frac{p}{2}}D^\phi x^{-\frac{p}{2}}) \rightarrow \ker D^\phi \quad f \mapsto x^{-\frac{p}{2}}f$$

is well-defined by the Lemma 2, has an obviously well-defined inverse  $f \mapsto x^{\frac{p}{2}}f$ , and preserves parity, hence it defines a graded isomorphism. We compose this isomorphism with the Hilbert space isometry (2). □

The above proof holds more generally for an arbitrary  $\Phi$ -metric whose associated Dirac operator  $D^\Phi$  is fully elliptic. This condition seems however difficult to check in practice outside the exact case. In the cusp case (i.e.,  $Y$  is a point), a small improvement was achieved in [6], where one could decide for a *closed* cusp metric whether the Dirac operator is fully-elliptic or not.

For completeness of the exposition, we describe the domains of the operators discussed here. They are certain weighted  $\Phi$ -Sobolev spaces.

LEMMA 11. *The operators  $D^\Phi$  and  $x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}$  are essentially self-adjoint in  $L^2_\Phi$ . The domain of the adjoint of  $D^\Phi$  is  $H^1_\Phi$ . If we assume moreover that  $D^\Phi$  is fully elliptic, then the domain of the adjoint of  $x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}$  coincides with  $x^p H^1_\Phi$ .*

REMARK 12. Since for  $0 \leq p \leq 1$  the metrics are complete, we know that the Dirac operators have precisely one self-adjoint extension; the extra fact here for such  $p$  is identifying the domain of the extension.

PROOF. The domains of the closures of the operators  $x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}$  and  $D^\Phi$  always contains the Sobolev spaces  $x^p H^1_\Phi$ , respectively  $H^1_\Phi$  (see [3]).

Since  $D^\Phi$  is elliptic in the  $\Phi$ -sense, there exists  $B \in \Psi_\Phi^{-1}$  with  $R := BD^\Phi - 1 \in \Psi_\Phi^{-\infty}$ . Let  $\phi \in L^2_\Phi$  be in the domain of the adjoint of  $D^\Phi$ . Then

$$\phi = BD^\Phi \phi - R\phi.$$

Since by hypothesis  $D^\Phi \phi \in L^2_\Phi$ , we get  $BD^\Phi \phi \in H^1_\Phi$ . At the same time,  $R\phi \in H^\infty_\Phi$ . Thus  $\phi \in H^1_\Phi$ . Therefore

$$\text{Dom}(\overline{D^\Phi}) \supset H^1_\Phi \supset \text{Dom}(D^{\Phi*}).$$

If moreover we assume that  $D^\Phi$  is fully elliptic, there exists  $B \in \Psi_\Phi^{-1,-p}$  with  $R := Bx^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}} - 1 \in \Psi_\Phi^{-\infty,-\infty}$ . Let  $\phi \in L^2_\Phi$  be in the domain of the adjoint of  $x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}$ , so  $\phi = Bx^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}\phi - R\phi$ . By hypothesis  $x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}\phi \in L^2_\Phi$ , so  $Bx^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}\phi \in x^p H^1_\Phi$ . Moreover,  $R\phi \in x^\infty H^\infty_\Phi$ , so in conclusion  $\phi \in x^p H^1_\Phi$ . Thus

$$\text{Dom}\left(\overline{x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}}\right) \supset x^p H^1_\Phi \supset \text{Dom}\left(x^{-\frac{p}{2}}D^\Phi x^{-\frac{p}{2}}\right)^*.$$

But for every symmetric operator  $D$  we have  $\text{Dom}(\overline{D}) \subset \text{Dom}(D^*)$ . □

Since the embedding  $x^a H_\Phi^b \hookrightarrow L_\Phi^2$  is compact for  $a, b > 0$ , it follows that fully elliptic  $\Phi$ -operators are Fredholm [3]. Hence the kernels discussed here are all finite-dimensional.

In a recent work, Leichtnam, Mazzeo and Piazza [2] computed the index of the Dirac operator  $D^\Phi$  corresponding to an exact  $\Phi$  metric for which the decomposition  $[c, \infty) \times M$  is orthogonal near infinity. They deform the metric to a  $b$ -metric using results of Melrose and Rochon [5], apply Melrose's  $b$ -index formula [4], and use the adiabatic limit formula of Bismut and Cheeger [1]. Together with Theorem 1.1 and a careful analysis of the index density their argument gives a short proof of Vaillant's index formula [8] in the exact, orthogonal, fully elliptic case. Although the orthogonality hypothesis is used in the proof, the authors informed us in a private communication that the argument goes through under a weaker hypothesis of orthogonality only up to second order at the boundary.

## 2. – Horn-cone and fibered horn metrics.

The case  $p = 1$  corresponds to the  $d$ -metric, our original motivation, and  $p = 0$  corresponds to the initial  $\Phi$ -metric. For  $p > 1$  the metric  $x^{2p} g_\Phi$  is incomplete, nevertheless with the invertibility assumption along the fibers, the associated Dirac operator is essentially self-adjoint, by Lemma 2. In this case, if we start with a product  $\Phi$  metric, the change of variables  $y = x^{p-1}$  gives (up to a constant) a metric depending on the parameter  $a = \frac{p}{p-1} > 1$ , with singularity at  $y = 0$ :

$$(3) \quad x^{2p} g_\Phi = dy^2 + y^2 \varphi^* g^Y + y^{2a} h$$

where  $g^Y$  is a metric on the base of the boundary fibration, and  $h$  is a family of metrics on the fibers. We call (3) a *horn-cone metric*. Similarly, for  $0 < p < 1$ , the change of variables  $y = x^{1-p}$  gives for  $a = \frac{p}{p-1} < 0$ :

$$(4) \quad x^{2p} g_\Phi = \frac{dy^2}{y^4} + \frac{\varphi^* g^Y}{y^2} + y^{-2a} h$$

We call this a *fibered horn metric*. It is singular at  $y = 0$ . The metric (3) is incomplete, while (4) is complete, expands with time in the base directions and shrinks in the fibers. In the result below, we could allow some weaker asymptotics for these metrics, such that after reverting back to the variable  $x$  the resulting metric is smooth in  $x$  and exact as a conformal  $\Phi$ -metric.



**COROLLARY 13.** *Let  $(X, g)$  be a spin Riemannian manifold with a boundary fibration  $\varphi : \partial X \rightarrow Y$ , isometric outside a compact set to a disjoint union of ends of the following types:*

- *horn-cone type (3);*
- *fibered horn (4);*
- *exact  $d$ -metric;*
- *exact  $\Phi$ -metric.*

*The conformal weights  $p$  may be different for each end. Let  $E \rightarrow X$  be an auxiliary vector bundle endowed with a Hermitian connection smooth in  $x$  down to  $x = 0$ , where  $x = y^{\frac{1}{p-1}}$  for ends of type (3), respectively  $x = y^{\frac{1}{1-p}}$  for ends of type (4). If the family of (twisted) Dirac operators on the fibers on each end with respect to the induced vertical family of metrics is invertible, then the Dirac operator  $D^g$  on  $X$  is essentially self-adjoint and Fredholm. If the dimension of  $X$  is even, the index of  $D^g$  is given by*

$$\text{index}(D_+^g) = \int_X \hat{A}(g) \text{ch}(E) + \int_Y \hat{A}(g_Y) \hat{\eta}(D^h).$$

*Moreover,  $D^g$  has compact resolvent (thus, purely discrete spectrum) if and only if  $X$  has no  $\Phi$ -type end. In that case a generalized Weyl law holds as in [6].*

The right-hand side involves the eta-form defined in [1].

**PROOF.** We compactify  $X$  using the variable  $x = y^{a-1}$  for each end of horn-cone type, and  $x = y^{1-a}$  for ends of fibered horn type, thus our metric becomes of the form  $x^{2p} g_\phi$ , where  $p$  is now a function from the set of ends into  $[0, \infty)$ . The metric  $g_\phi$  is exact for all the ends. Then by [8, Chapter 2], the Pontrjagin forms of  $g$  are smooth down to  $x = 0$ , so there is no need to regularize the local index integral. The operator  $D^g$  is fully elliptic by the invertibility assumption. By Lemma 2, it follows that it is essentially self-adjoint. The resolvent lives in  $\Psi^{-1,-p}$  so it is compact if and only if  $p$  takes values in  $(0, \infty)$ , i.e., if there is no  $\Phi$ -type end. By Theorem 1.1, the index is the same as that of the associated conformal  $d$ -metric  $g_d$ , which is computed by Vaillant by the above formula, only that it involves the index density for  $g_d$ , not for  $g$ . But of course, the Pontrjagin forms are conformal invariants so the formula follows. Alternately, instead of Vaillant’s result we may use [2] (see the discussion on orthogonality at the end of the previous Section).

The Weyl law follows entirely as in [6], based on the fact that the zeta function of  $D^g$  has a meromorphic extension to the complex plane, with a possibly double first pole.  $\square$

### 3. – The non-fully elliptic case.

Let us mention also a partial result in the non-fully elliptic case. Assume that the hypothesis of Vaillant [8] holds, namely that the kernels of the family of Dirac operators along the fibers of the boundary form a vector bundle. Then Vaillant [8] showed that the dimension of  $\ker(D^d)$  is finite.

PROPOSITION 14. *Under the above hypothesis, for every  $0 \leq p \leq 1$ , the kernel of the Dirac operator on  $X$  associated to  $x^{2p}g_\phi$  is finite-dimensional.*

PROOF. Let  $D(p)$  denote the Dirac operator of  $x^{2p}g_\phi$ . Following the proof of Theorem 1.1, we see that the map of multiplication by  $x^{\frac{1-p}{2}}$  inside  $L^2(X, g_d)$  maps the kernel of an operator in  $L^2(X, g_d)$  unitarily equivalent to  $D(p)$ , into the kernel of  $D^d$ . This multiplication map is injective but not surjective. Thus  $\ker D(p)$  injects into  $\ker(D^d)$ , whose dimension is finite by [7, Chapter 3].  $\square$

For  $p = 0$ , under Vaillant's hypothesis, the index of  $D^\phi$  is finite, although from Melrose's Fredholm criterion we see that in the non-fully elliptic case,  $D^\phi$  is not Fredholm on any Sobolev space with exponential weight  $e^{\alpha x^b}L^2(X, g_\phi)$ .

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