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On the Structure of Induced Modules and Tensor Induction for Group Representations (*).

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Dedicated to Professor Guido Zappa on the occasion of his 90th birthday

ABSTRACT - Let V be a simple module for a finite group G, over a finite field $\mathbb{F}$, and let H be a subgroup of G. Assuming that V is induced by an $\mathbb{F}H$-module, we investigate some aspects of the structure of V viewed as a module for H. This kind of analysis turns out to play a central role in a problem concerning tensor induction for representations of finite groups.

Introduction.

I. Let G be a finite group, $\mathbb{F}$ a finite field, and V a simple $\mathbb{F}G$-module; given a subgroup H having odd index in G, and an $\mathbb{F}H$-submodule W of V, assume that V is isomorphic to the *induced module* $W{\uparrow}^G$. In this setting, we are interested in exploring the structure of the $\mathbb{F}H$-module $V{\downarrow}_H$ (which is V *restricted to* H) from a particular point of view: namely, we ask whether the odd-index assumption for H implies that *the multiplicity of* W *as a composition factor in the socle of* $V{\downarrow}_H$ *is also odd*. By Lemma 1.4(b), this is equivalent to saying that $V{\downarrow}_H$ is isomorphic to the $\mathbb{F}H$-module $\left(\bigoplus_{i=1}^{s} W\right) \oplus Y$, where s is an odd positive integer, and Y is a submodule of $V{\downarrow}_H$ not containing any submodule isomorphic to W.

It follows from Clifford's Theorem ([1, 11.1]) that the answer to the above question is certainly affirmative when H is a *normal* subgroup of G. More generally, as outlined in the last paragraph of Section 2, it is not difficult to see that the same holds when W is induced from the normal core

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L of H in G, *provided L has odd index in H* (hence in G). On the other hand, if W is induced from L but $|H : L|$ is even, then the answer can be negative, as it is shown by an example ([5, 11.1]) in which G is solvable, $\mathbb{F}$ is the prime field in characteristic 3, and $|G : H|$ is 3.

In view of our original motivation for this kind of analysis (presented in Part II of this Introduction), we are actually interested in the case when W is *not* induced from L, and we can also assume that $\mathbb{F}$ has odd characteristic. The main result of this paper (which is proved in Section 2) is the following.

THEOREM A. *Let G be a finite solvable group, H a subgroup of G having odd index, $\mathbb{F}$ a (not necessarily finite) field of odd characteristic, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. Denoting by L the normal core of H in G, assume that W is not induced from L, and that G/L* is a Frobenius group with Frobenius complement H/L. *Then we have $V{\downarrow_H} \simeq \left(\bigoplus_{i=1}^{s} W\right) \oplus Y$, where s is an odd number and Y is a submodule of $V{\downarrow_H}$ such that* $\mathrm{Hom}_{\mathbb{F}H}(W, Y) = 0$ (In other words, W has odd multiplicity as a composition factor in $\mathrm{soc}(V{\downarrow_H}))$.

Note that, if H is a (not normal) subgroup of odd *prime* index in the solvable group G, then G/L does have the structure of a Frobenius group with Frobenius complement H/L (in fact, denoting by K/L a minimal normal subgroup of G/L, we have that G/L is a semidirect product of K/L and H/L; moreover, every nontrivial element of H/L acts fixed-point-freely by conjugation on K/L). Therefore Theorem A covers this case, thus providing a generalization of Theorem 9.7 in [5].

II. It may be worth mentioning the problem which led us to the question presented in Part I.

Let G be a finite group, H a subgroup of G, and D an irreducible complex representation for G. As it is easy to check, every direct summand of the restriction $D{\downarrow_H}$ must have degree at least as large as $\deg D$ divided by the index $|G : H|$, and D is induced by a representation of H if and only if $D{\downarrow_H}$ does have a direct summand of degree $(\deg D)/|G : H|$. One of the main purposes of [5] is to explore the possibility of an analogous result for *tensor induction*; more explicitly we ask whether the following holds.

CONJECTURE. *Let D be a faithful, quasi-primitive and tensor-indecomposable representation of G. Then D is tensor-induced by a projective representation of H if and only if $D{\downarrow_H}$ has a tensor factor whose degree is the $|G : H|$th root of* $\deg D$.

(We refer to [5, Introduction and Section 1] for a detailed discussion about the concept of tensor induction, which motivates and explains the setting of the above Conjecture.) Following the line developed in [5], this problem can be approached by means of two subsequent reductions. First, the Conjecture appears to be deeply linked to a statement ([5, 'weak' Conjecture 4.3]) concerning *form induction* for symplectic modules over finite fields (see Section 3), and at this level it can be shown that the Conjecture is false in its full generality (Example 5.2 in [5]). Next, positive results toward the Conjecture are obtained assuming that H has odd index in G, and such results are achieved via a reduction to the question presented in Part I. In particular, the Conjecture is proved to be true when H is a normal subgroup of odd index in G (provided the Fitting subgroup of G is assumed noncentral) and also, through Theorems 9.7 and 9.10 of [5], when G is solvable and H has odd *prime* index in G.

As in this paper we generalize [5, 9.7 and 9.10] (by means of Theorem A and Theorem 3.3 respectively), we are in a position to extend the cases in which the Conjecture (together with the weak version of Conjecture 4.3 in [5]) is proved to be true. The precise statements for these results, together with the relevant definitions and notation, are formulated in Section 3.

To conclude, every abstract group considered throughout the following discussion is tacitly assumed to be finite. Also, we shall freely use (often with no reference) some basic facts in Representation Theory, such as Clifford's Theorem, Mackey's Lemma ([1, 10.13]) and Nakayama reciprocity ([3, VII, 4.5 and 4.10]).

1. Some preliminaries.

Before proving Theorem A, we recall some results and notation which will be relevant in the sequel.

LEMMA 1.1. *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel K, and let I be a subgroup of G such that $I \cap H \neq 1$. Then we have $I = (I \cap H)(I \cap K)$.*

PROOF. See [4, 4.1.8]. □

LEMMA 1.2. *Let H be a solvable Frobenius complement of even order, which does not have any subgroup of index 2. Then there exists a normal subgroup N of H such that H/N is isomorphic to the alternating group A_4.*

PROOF. Set $A := O_2(H)$, and $N := C_H(A)$; looking at the proof of Zassenhaus' Theorem 18.2 in [6], we see that our assumptions force A to be isomorphic to the quaternion group of order 8, whence H/N embeds in $\mathrm{Aut}\,(Q_8) \simeq S_4$. Moreover, a Sylow 2-subgroup of H/N must be isomorphic to $C_2 \times C_2$. As H/N can not be a 2-group, its order is necessarily divisible by 3, and the claim follows. $\square$

LEMMA 1.3. *Let H be a group, $\mathbb{F}$ a field, and M a normal subgroup of H. Also, let W be a simple $\mathbb{F}H$-module, and U a simple constituent of $W{\downarrow_M}$. If I is the inertia subgroup of U in H, and e denotes the multiplicity of U as a composition factor in $W{\downarrow_M}$, then* $|I/M| \geq e^2 \cdot (\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}M}(U))/(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}H}(W))$ *holds.*

PROOF. Let f denote the multiplicity of W as a composition factor of the largest semisimple quotient of $U{\uparrow^H}$. Since the direct sum of f copies of W is a homomorphic image of $U{\uparrow^H}$, the direct sum of ef copies of U is a homomorphic image of $U{\uparrow^H}{\downarrow_M}$. From Mackey's Lemma, it is easy to see that $U{\uparrow^H}{\downarrow_M}$ is semisimple and one of its homogeneous components is the direct sum of $|I/M|$ copies of U: therefore $|I/M| \geq ef$. By [3, VII, 4.13],

$$e \cdot \dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}M}(U) = f \cdot \dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}H}(W).$$

Thus

$$|I/M| \geq ef = e^2 \cdot (\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}M}(U))/(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}H}(W)),$$

as claimed. $\square$

LEMMA 1.4. *Let G be a group, H a subgroup of G, $\mathbb{F}$ a field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. Let T be the homogeneous component of W in the socle of $V{\downarrow_H}$. Then the following conclusions hold:*

(a) *the multiplicity of W as a composition factor in T is given by*

$$(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}G}(V))/(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}H}(W));$$

(b) *T is a direct summand in $V{\downarrow_H}$, it has a unique direct complement Y, and Y is such that* $\mathrm{Hom}_{\mathbb{F}H}(W, Y) = \mathrm{Hom}_{\mathbb{F}H}(Y, W) = 0$.

PROOF. Claim (a) easily follows from the fact that, by [3, VII, 4.12b)], the multiplicity of W as a composition factor in T is given by

$$(\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}H}(W, V{\downarrow_H}))/(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}H}(W)).$$

Nakayama reciprocity yields now the conclusion.

For claim (b), note that if Z is any submodule of $V{\downarrow_H}$ with the same dimension as W, then Z is a direct summand (simply because V is the vector space direct sum of the translates of Z, the translates different from Z are permuted by H among themselves, and so their sum is an H-module complement to Z). Let Y be of minimal dimension among the submodules of $V{\downarrow_H}$ such that $T + Y = V{\downarrow_H}$. If Y contained a submodule Z isomorphic to W, then Z would lie in T and Z would be a direct summand of Y, contrary to the minimality of Y. Therefore we must have $\mathrm{Hom}_{\mathbb{F}H}(W, Y) = 0$, and $T \cap Y = 0$ (so that Y is a direct complement to T). Dually, one can use the fact that if Z' is any submodule of $V{\downarrow_H}$ with *co-dimension* equal to $\dim W$ then it is a direct summand: if Y had a nonzero homomorphism onto W, the sum of T with the kernel of that could play the role of Z' and yield a contradiction. Thus $\mathrm{Hom}_{\mathbb{F}H}(Y, W) = 0$, and from this it follows at once that there can be no direct complement to T other than Y. □

LEMMA 1.5. *Let H be a group, L a normal subgroup of H, $\mathbb{F}$ a finite field, and S a* 1*-dimensional $\mathbb{F}H$-module whose kernel contains L. Let W be a simple $\mathbb{F}H$-module. Then $W \otimes S$ and W have the same (nonzero) multiplicity as composition factors in the socle of $W{\downarrow_L}{\uparrow^H}$.*

PROOF. See [5, 9.1]. □

LEMMA 1.6. *Let H be a group, L a normal subgroup of H, $\mathbb{F}$ a finite field, and W an absolutely simple $\mathbb{F}H$-module. Assume that there exists an $\mathbb{F}H$-module S such that* $\ker S$ *contains L,* $|H : \ker S| = 2$, *and $W \otimes S$ is isomorphic to W. Then the multiplicity of W as a composition factor in the socle of $W{\downarrow_L}{\uparrow^H}$ is an even (positive) number.*

PROOF. See [5, 9.4]. □

REMARK 1.7. It is not hard to see that the ideas of the proof of [5, 9.7] can be applied more generally, and we shall need some of their consequences here. Let G be a (finite) group, H a subgroup of G, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. Then $\mathrm{End}_{\mathbb{F}G}(V)$ and $\mathrm{End}_{\mathbb{F}H}(W)$ are fields, every element of the latter arises as the restriction of one and only one element of the former, and the relevant elements of $\mathrm{End}_{\mathbb{F}G}(V)$ form a subfield: call that $\mathbb{K}$, write $V_{\mathbb{K}}$ for V regarded as $\mathbb{K}G$-module, and $W_{\mathbb{K}}$ for W regarded as $\mathbb{K}H$-module (of course $V_{\mathbb{K}}$ is simple, and it is induced by $W_{\mathbb{K}}$ from H). It is now easy to see that $\mathrm{End}_{\mathbb{K}G}(V_{\mathbb{K}}) = \mathrm{End}_{\mathbb{F}G}(V)$, and $\mathrm{End}_{\mathbb{K}H}(W_{\mathbb{K}}) = \mathrm{End}_{\mathbb{F}H}(W)$, so $W_{\mathbb{K}}$ is indeed absolutely simple, and by Lemma 1.4(a) the multiplicity of W as a composition factor in $\mathrm{soc}(V{\downarrow_H})$ is the same as the multiplicity of $W_{\mathbb{K}}$ as a composition factor in $\mathrm{soc}((V_{\mathbb{K}}){\downarrow_H})$. Moreover, if $W_{\mathbb{K}}$ is induced from some subgroup L of H, then $W_{\mathbb{K}}$ has a submodule of $\mathbb{K}$-dimension $\dim_{\mathbb{K}}(W_{\mathbb{K}})/|H:L|$; that subspace is also a submodule of W, of $\mathbb{F}$-dimension $\dim_{\mathbb{F}}(W)/|H:L|$, and so W is also induced from L.

2. A proof of the main theorem.

We present next a proof of Theorem A, which was stated in the Introduction. We are interested in this result when the field $\mathbb{F}$ is finite (and this will be our assumption), but in Remark 2.1 we shall take the opportunity to explain that the theorem is in fact true also if $\mathbb{F}$ is infinite. It may be worth stressing that in the special case when $\mathbb{F}$ is a splitting field for G (for instance, when $\mathbb{F}$ is algebraically closed), Theorem A is an immediate consequence of Lemma 1.4(a).

PROOF OF THEOREM A. In what follows, we shall assume the statement true for all groups having order strictly smaller than $|G|$, and our aim will be to show that the statement is true for G as well. As the first step, we shall prove that W can be assumed absolutely simple.

In fact, let us suppose that Theorem A is true when the relevant H-module is absolutely simple. Taking in account Remark 1.7 and its set-up, we can apply Theorem A with $\mathbb{K}$, $V_{\mathbb{K}}$ and $W_{\mathbb{K}}$ in place of $\mathbb{F}$, V and W respectively. Then we get that the multiplicity of $W_{\mathbb{K}}$ as a composition factor in $\mathrm{soc}((V_{\mathbb{K}}){\downarrow_H})$ is odd. But, as explained in 1.7, that multiplicity equals the multiplicity of W as a composition factor in $\mathrm{soc}(V{\downarrow_H})$, and we achieve the desired conclusion.

In view of the previous step, we henceforth assume that W is absolutely simple. Let X be a simple constituent of $W{\downarrow_L}$, and let I denote the inertia subgroup $I_G(X)$ (recall that this is the subgroup of all the elements g of G such that X^g is isomorphic to X as an $\mathbb{F}L$-module). Also, denote by K/L the Frobenius kernel of G/L. We shall proceed by discussing the various situations which may occur, depending on I.

(1). *Case* $I \cap H = L$. This can not happen, as otherwise we would get $I_H(X) = L$, and Clifford's Theorem would yield that W is induced by X from L, against the hypothesis.

(2). *Case* $L < I \cap H < H$. Set $J := IK = (I \cap H)K$ (see Lemma 1.1), and let U be the (unique) submodule of $W{\downarrow_{I\cap H}}$ with the property that $U{\downarrow_L}$ is the homogeneous component of $W{\downarrow_L}$ containing X. Since $U{\uparrow^H} \simeq W$, this U must be absolutely simple. Moreover, we get $(U{\uparrow^J}){\uparrow^G} \simeq U{\uparrow^G} \simeq$ $\simeq (U{\uparrow^H}){\uparrow^G} \simeq W{\uparrow^G} \simeq V$, so that $U{\uparrow^J}$ is a simple $\mathbb{F}J$-module. Of course J is a solvable group, $I \cap H$ is a subgroup of it having odd index, L is the normal core of $I \cap H$ in J, and J/L is a Frobenius group with Frobenius complement $(I \cap H)/L$. Moreover, U is not induced from L. By our inductive hypothesis, we can conclude that U has odd multiplicity as a composition factor in the socle of $(U{\uparrow^J}){\downarrow_{I\cap H}}$. By Lemma 1.4(a), this is equivalent to saying that $\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}J}(U{\uparrow^J})$ is an odd number. Now, we have

$$\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}G}(V) = \dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}J}(U{\uparrow^J}) \cdot (\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}G}(V))/(\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}J}(U{\uparrow^J})),$$

and it suffices to show that $U{\uparrow^J}$ has odd multiplicity (as a composition factor) in $\mathrm{soc}(V{\downarrow_J})$. We shall see that this multiplicity is in fact 1.

Denoting by $\mathcal{T}$ a transversal for $I \cap H$ in H, we get

$$V{\downarrow_J} \simeq W{\downarrow_{I\cap H}}{\uparrow^J} \simeq \Big(\bigoplus_{t\in\mathcal{T}} U^t\Big){\uparrow^J} \simeq U{\uparrow^J} \oplus \Big(\bigoplus_{t\in\mathcal{T}\setminus(I\cap H)} U^t\Big){\uparrow^J}$$

(Observe that, although the individual U^t are not necessarily $\mathbb{F}[I \cap H]$-modules, certainly $\bigoplus_{t\in\mathcal{T}\setminus(I\cap H)} U^t$ is invariant under the action of $I \cap H$). For a proof by contradiction, suppose that the multiplicity of $U{\uparrow^J}$ in $\mathrm{soc}(V{\downarrow_J})$ is greater than 1; this means that $\mathrm{Hom}_{\mathbb{F}J}\Big(U{\uparrow^J}, \big(\bigoplus_{t\in\mathcal{T}\setminus(I\cap H)} U^t\big){\uparrow^J}\Big)$ $\Big($which is isomorphic, as a vector space, to $\mathrm{Hom}_{\mathbb{F}[I\cap H]}\Big(U, \big(\bigoplus_{t\in\mathcal{T}\setminus(I\cap H)} U^t\big){\uparrow^J}{\downarrow_{I\cap H}}\Big)\Big)$ is not the zero space. Therefore, X is a constituent of $\big(\bigoplus_{t\in\mathcal{T}\setminus(I\cap H)} U^t\big){\uparrow^J}{\downarrow_L}$ and, finally, there exist t in $\mathcal{T} \setminus (I \cap H)$ and j in J such that X is a constituent of $(U^{tj}){\downarrow_L}$. Now, $X^{(tj)^{-1}}$ is a constituent of $U{\downarrow_L}$, so that tj lies in I. Writing j as hk, where h is in $I \cap H$ and k in K, we have that $thk = tj$ is in $I = (I \cap H)(I \cap K)$ (see Lemma 1.1). This implies that t lies in $I \cap H$, which is not the case.

(3). *Case* $I = H$. We see that in this situation W has multiplicity 1 in $\mathrm{soc}(V{\downarrow_H})$. In fact, $W{\downarrow_L}$ is now a homogeneous component of $V{\downarrow_L}$. If $V{\downarrow_H}$ contained another isomorphic copy of W, the restriction of that to L would be isomorphic to $W{\downarrow_L}$; but this is a contradiction, as a homogeneous component can never be isomorphic to any submodule distinct from it.

(4). *Case* $H < I < G$. We have that I is a solvable group, H is a subgroup of I having odd index, L is the normal core of H in I, and I/L is a Frobenius group with Frobenius complement H/L. Also, let R be the submodule of $V{\downarrow_I}$ generated by the subspace W (so that R is isomorphic to $W{\uparrow^I}$, and it is certainly a simple $\mathbb{F}I$-module). By the inductive hypothesis we deduce that $\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}I}(R)$ is an odd number and, as $\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}G}(V)$ is given by that number times the multiplicity of R in $\mathrm{soc}(V{\downarrow_I})$, it is enough to show that the latter multiplicity is odd. But, similarly to what happens in Case (3), $R{\downarrow_L}$ is a homogeneous component of $V{\downarrow_L}$ and, as above, there can not be any other copy of R in $V{\downarrow_I}$: therefore the multiplicity of R in $\mathrm{soc}(V{\downarrow_I})$ is 1, and the argument for this case is complete.

(5). *Case* $I = G$. Our assumption that G/L is a Frobenius group with Frobenius complement H/L implies that, considering the action of H on the set of its right cosets in G (given by right multiplication), the orbits not containing the trivial coset H have a common length, namely $|H : L|$. Therefore, Mackey's Lemma applied to the present situation yields

$$V{\downarrow_H} \simeq W \oplus \Big(\bigoplus_{i=1}^{n} W{\downarrow_L}{\uparrow^H} \Big),$$

where n is the number of nontrivial double cosets of H in G. This number is given by $(|G : H| - 1)/|H : L|$, so there is nothing to prove if $|H : L|$ is odd (in that case n is even). From now on we shall then assume $|H : L|$ even, and most of the time our aim will be to show that W has even multiplicity as a composition factor in $\mathrm{soc}(W{\downarrow_L}{\uparrow^H})$.

Let us start by assuming that H has a subgroup Q which contains L and is such that $|H : Q| = 2$; then we can consider the representation of H/Q which maps the generator to -1 in $\mathbb{F}$ (and view it as a representation for H). We claim that, if S denotes an $\mathbb{F}H$-module associated to this representation, then $W \otimes S$ is isomorphic to W. In fact, by Lemma 1.5, $W \otimes S$ and W have the same multiplicity (call it r) as composition factors in the socle of $W{\downarrow_L}{\uparrow^H}$. If they are assumed to be nonisomorphic, then Lemma 1.4(a) yields

$$nr + 1 = |\,\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(W)| = |\,\mathrm{End}_{\mathbb{F}G}(V) : \mathrm{End}_{\mathbb{F}H}(W \otimes S)| = nr$$

(here we used that $\mathrm{End}_{\mathbb{F}H}(W)$ and $\mathrm{End}_{\mathbb{F}H}(W \otimes S)$ are isomorphic vector spaces), a clear contradiction. We are now in a position to apply Lemma 1.6 (as of course the kernel of S has index 2 in H), and we are done in this case.

If H/L does not have a subgroup of index 2, then (by Lemma 1.2) there exists a normal subgroup N of H, containing L, such that H/N is isomorphic to A_4. In what follows, we denote by M the subgroup of H which contains N and such that M/N is the Sylow 2-subgroup of H/N.

Let us assume that $W{\downarrow_M}$ is not homogeneous. Then we get $W{\downarrow_M} = U \oplus U^h \oplus U^{h^2}$, where h is in $H \setminus M$ and the three summands are simple homogeneous components. Now, M is a subgroup of KM having odd index, L is the normal core of M in KM, and KM/L is a Frobenius group with Frobenius complement M/L. Since W is induced by U from M, we see that U is absolutely simple; moreover, $U{\uparrow^{KM}}$ is simple, as it induces V. Now, M/L does have a subgroup of index 2 and, since $U{\uparrow^{KM}}{\downarrow_L}$ is homogeneous, we can apply the same argument as in the first two paragraphs of Case (5) (with KM, M, $U{\uparrow^{KM}}$ and U in place of G, H, V and W respectively) concluding that the multiplicity of U in $\mathrm{soc}(U{\downarrow_L}{\uparrow^M})$ is even: say, $2k$. Since $I = G$, the restriction of V to L is homogeneous; thus $U{\downarrow_L}$ and $U^h{\downarrow_L}$, which are submodules of equal dimension in $V{\downarrow_L}$, must be isomorphic. From this, we get

$$\begin{aligned}\mathrm{Hom}_{\mathbb{F}M}(U^h, U{\downarrow_L}{\uparrow^M}) \simeq \mathrm{Hom}_{\mathbb{F}L}(U^h{\downarrow_L}, U{\downarrow_L}) \simeq \mathrm{Hom}_{\mathbb{F}L}(U{\downarrow_L}, U{\downarrow_L}) \simeq \\ \simeq \mathrm{Hom}_{\mathbb{F}M}(U, U{\downarrow_L}{\uparrow^M}),\end{aligned}$$

whence U^h and (similarly) U^{h^2} have the same multiplicity as U in $\mathrm{soc}(U{\downarrow_L}{\uparrow^M})$. Now we have

$$\begin{aligned}W{\downarrow_L}{\uparrow^M} \simeq (U{\downarrow_L} \oplus U^h{\downarrow_L} \oplus U^{h^2}{\downarrow_L}){\uparrow^M} \simeq U{\downarrow_L}{\uparrow^M} \oplus U{\downarrow_L}{\uparrow^M} \oplus U{\downarrow_L}{\uparrow^M} \simeq \\ \simeq \Big(\bigoplus_{i=1}^{6k} U\Big) \oplus \Big(\bigoplus_{i=1}^{6k} U^h\Big) \oplus \Big(\bigoplus_{i=1}^{6k} U^{h^2}\Big) \oplus Z,\end{aligned}$$

where the socle of Z does not contain any of the U^{h^i} as a submodule (from each of the three copies of $U{\downarrow_L}{\uparrow^M}$ we "extracted" all the copies of each U^{h^i}; note that every submodule of $U{\downarrow_L}{\uparrow^M}$ isomorphic to one of the U^{h^i} is certainly a direct summand of $U{\downarrow_L}{\uparrow^M}$). Finally,

$$W{\downarrow_L}{\uparrow^H} \simeq \Big(\bigoplus_{i=1}^{2l} W\Big) \oplus Z{\uparrow^H},$$

(where $l := 9k$) with $\mathrm{Hom}_{\mathbb{F}H}(W, Z{\uparrow^H}) \simeq \mathrm{Hom}_{\mathbb{F}M}(W{\downarrow_M}, Z) = 0$.

It remains to examine Case (5) in the situation in which H/L does not have a subgroup of index 2, and $W{\downarrow_M}$ is homogeneous. Observe that in this case $W{\downarrow_M}$ is simple because, by Lemma 1.3, if e denotes the multiplicity of a simple constituent of $W{\downarrow_M}$ in it, we have $e^2 \leq |H/M| = 3$. The composition length (as an $\mathbb{F}[KM]$-module) of $W{\downarrow_M}{\uparrow^{KM}} \simeq V{\downarrow_{KM}}$ can be 1, 2, or 3. We analyze the situation in each of the three cases.

Assume that the composition length of $V{\downarrow_{KM}}$ is 3, and set $V{\downarrow_{KM}} \simeq \simeq Z_1 \oplus Z_2 \oplus Z_3$ where the Z_i are simple $\mathbb{F}[KM]$-modules. Since we get

$$\mathrm{Hom}_{\mathbb{F}M}(W{\downarrow_M}, Z_i{\downarrow_M}) \simeq \mathrm{Hom}_{\mathbb{F}[KM]}(W{\downarrow_M}{\uparrow^{KM}}, Z_i) \simeq \mathrm{Hom}_{\mathbb{F}[KM]}(V{\downarrow_{KM}}, Z_i) \simeq \\ \simeq \mathrm{Hom}_{\mathbb{F}G}(V, Z_i{\uparrow^G}) \simeq \mathrm{End}_{\mathbb{F}G}(V),$$

the multiplicity of $W{\downarrow_M}$ in $\mathrm{soc}(Z_i{\downarrow_M})$ is $\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}G}(V)/\dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}M}(W{\downarrow_M})$. In particular, this number does not depend on i, so that the multiplicity of $W{\downarrow_M}$ in the socle of $V{\downarrow_M}$ is a multiple of 3. On the other hand, Mackey's Lemma gives

$$V{\downarrow_M} \simeq W{\downarrow_M}{\uparrow^{KM}}{\downarrow_M} \simeq W{\downarrow_M} \oplus \Big(\bigoplus_{j=1}^{d} W{\downarrow_L}{\uparrow^M} \Big),$$

where d is the number of double cosets of M in KM different from M. This number is given by $(|KM : M| - 1)/|M : L| = 3(|G : H| - 1)/|H : L|$, whence the multiplicity of $W{\downarrow_M}$ in the socle of $V{\downarrow_M}$ is congruent to 1 modulo 3. We thus reached a contradiction, so this case can not arise.

Let us now examine the case in which the composition length of $V{\downarrow_{KM}}$ is 2, so that we have $\dim V = 2k \dim X$. On the other hand, $\dim V$ is given by $|G : H| \dim W = s|G : H| \dim X$, where s denotes the composition length of $W{\downarrow_L}$. The conclusion is that s is even, say $2r$; therefore we get

$$W{\downarrow_L}{\uparrow^H} \simeq \bigoplus_{j=1}^{2r} (X{\uparrow^H}),$$

and of course we are done.

Finally, let $V{\downarrow_{KM}}$ be simple, and let S be an $\mathbb{F}M$-module such that $\ker S$ contains L, and $|M : \ker S| = 2$. If $W{\downarrow_M} \otimes S \not\simeq W{\downarrow_M}$, then a contradiction arises as in the second paragraph of Case (5); therefore we must have $W{\downarrow_M} \otimes S \simeq W{\downarrow_M}$. If $W{\downarrow_M}$ is absolutely simple, then we apply Lemma 1.6 getting that $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}M}(W{\downarrow_M}, W{\downarrow_L}{\uparrow^M})$ is an even number; we reach now the desired conclusion, as $\mathrm{Hom}_{\mathbb{F}M}(W{\downarrow_M}, W{\downarrow_L}{\uparrow^M})$ is isomorphic to $\mathrm{Hom}_{\mathbb{F}H}(W, W{\downarrow_L}{\uparrow^H})$. We are left with the case in which $W{\downarrow_M}$ is not absolutely simple: in such a situation, the Theorem stated in the Introduction of [2] guarantees that $W{\downarrow_M}{\uparrow^H}$ is isomorphic to a direct sum of three copies of W, and also that $\mathrm{End}_{\mathbb{F}M}(W{\downarrow_M})$ has degree 3 as a field extension of $\mathbb{F}$. Observe that we can also assume $W{\downarrow_M}$ not induced from L, otherwise the composition length of $W{\downarrow_L}$ (as an $\mathbb{F}L$-module) is the even number $|M : L|$, and again we are done. Now, we get

$$\mathrm{Hom}_{\mathbb{F}M}(W{\downarrow_M}, V{\downarrow_M}) \simeq \mathrm{Hom}_{\mathbb{F}H}(W, V{\downarrow_M}{\uparrow^H}) \simeq \mathrm{Hom}_{\mathbb{F}H}(W, (V{\downarrow_{KM}})\downarrow_M{\uparrow^H}) \simeq$$

$$\simeq \mathrm{Hom}_{\mathbb{F}H}(W, V{\downarrow_{KM}}{\uparrow^G}{\downarrow_H}) \simeq \mathrm{Hom}_{\mathbb{F}H}(W, V{\downarrow_H} \oplus V{\downarrow_H} \oplus V{\downarrow_H}),$$

where the last isomorphism relation holds because of the following:

$$V{\downarrow_{KM}}{\uparrow^G} \simeq (W{\downarrow_M}{\uparrow^{KM}}){\uparrow^G} \simeq W{\downarrow_M}{\uparrow^G} \simeq (W{\downarrow_M}{\uparrow^H}){\uparrow^G} \simeq (W \oplus W \oplus W){\uparrow^G} \simeq V \oplus V \oplus V.$$

The conclusion so far is

$$\dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}M}(W{\downarrow_M}, V{\downarrow_M})/\dim_{\mathbb{F}}\mathrm{End}_{\mathbb{F}M}(W{\downarrow_M}) = 3\dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}H}(W, V{\downarrow_H})/3 =$$

$$= \dim_{\mathbb{F}}\mathrm{Hom}_{\mathbb{F}H}(W, V{\downarrow_H});$$

in other words, the multiplicity of W in the socle of $V{\downarrow_H}$ equals the multiplicity of $W{\downarrow_M}$ in the socle of $V{\downarrow_M}$. This completes the proof, as we can now use the inductive hypothesis and conclude that the latter multiplicity is odd. □

REMARK 2.1. Let G be a finite group, and $\mathbb{F}$ a field of prime characteristic. Denoting by n the order of G, we set $\mathbb{F}^{(n)}$ to be the (finite) subfield of the algebraic closure of $\mathbb{F}$ generated by the n-th roots of 1, and we define $\mathbb{F}_0 := \mathbb{F}^{(n)} \cap \mathbb{F}$. In this setting, Lemma 6 of [2] establishes what follows: for every subgroup X of G, and for every simple $\mathbb{F}X$-module U, there exists a simple $\mathbb{F}_0X$-module U_0, uniquely determined up to isomorphisms, such that $U \simeq U_0 \otimes_{\mathbb{F}_0} \mathbb{F}$ (we refer here to Definition 1.1b) of [3, VII]).

The above result enables us to prove Theorem A in its full generality, without requiring that $\mathbb{F}$ is finite. In fact, assume the hypotheses of Theorem A as stated in the Introduction, and consider the modules W_0 and V_0 (an $\mathbb{F}_0H$-module and an $\mathbb{F}_0G$-module respectively) associated to W and V by means of [2, Lemma 6]. It is easy to check that W_0 and V_0 satisfy the hypotheses of Theorem A (for this purpose, it is convenient to take in account that the process of induction of modules "commutes" with the process of tensoring modules with a field extension); thus, as we proved Theorem A when the relevant field is finite, we can conclude that $(\dim_{\mathbb{F}_0}\mathrm{End}_{\mathbb{F}_0G}(V_0))/(\dim_{\mathbb{F}_0}\mathrm{End}_{\mathbb{F}_0H}(W_0))$ is an odd number (here we also applied Lemma 1.4). Now, using 1.12 and 1.1a) of [3, VII], we get

$$\dim_{\mathbb{F}}\mathrm{End}_{\mathbb{F}G}(V) = \dim_{\mathbb{F}}(\mathrm{End}_{\mathbb{F}_0G}(V_0) \otimes_{\mathbb{F}_0} \mathbb{F}) = \dim_{\mathbb{F}_0}\mathrm{End}_{\mathbb{F}_0G}(V_0)$$

and, similarly, $\dim_{\mathbb{F}}\mathrm{End}_{\mathbb{F}H}(W) = \dim_{\mathbb{F}_0}\mathrm{End}_{\mathbb{F}_0H}(W_0)$. Another appeal to 1.4 completes the argument.

We stress that, as mentioned in the Introduction, the assumption of W not being induced from L is crucial for Theorem A (see [5, 11.1]), *although that assumption is not needed when $|H/L|$ is odd.* In fact, assuming W induced from the $\mathbb{F}L$-module X, it is easy to see (using [3, VII, 4.12b)] and Clifford's Theorem) that the multiplicity of W as a composition factor in the socle of $V{\downarrow_H}$ is given by $|I_G(X) : I_H(X)|$, a divisor of the odd number $|G/L|$.

3. Form induction and tensor induction.

We start this section recalling some definitions and notation. For further details, we refer to [5, Introduction, Section 1 and Section 3].

DEFINITION 3.1. Let G be a group, $\mathbb{F}$ a field, V an $\mathbb{F}G$-module, and f a symplectic $\mathbb{F}$-form defined on (the underlying vector space of) V; if $f(u^g, v^g) = f(u, v)$ holds for all u, v in V and g in G, then f is called *G-invariant.*

DEFINITION 3.2. Let G be a group, H a subgroup of G, $\mathbb{F}$ a field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$. Assume that a G-invariant nonsingular symplectic $\mathbb{F}$-form f is defined on V, and that the following conditions hold:

(a) the restriction of f to $W \times W$, which is an H-invariant symplectic $\mathbb{F}$-form on W, is nonsingular;

(b) the translate W^g lies in $W^\perp$ for all g in G such that $W^g \neq W$;

(c) V is induced by W from H.

Then we say that V is *form-induced* by W (with respect to f) from H.

A map $P : H \to GL(d, \mathbb{F})$ is called a projective representation of H (of degree d, over the field $\mathbb{F}$) if the map $\bar{P}$, defined as the composite of P with the natural homomorphism of $GL(d, \mathbb{F})$ onto $PGL(d, \mathbb{F})$, is a group homomorphism. If P_1 and P_2 are projective representations of H having the same degree d, then they are called equivalent if $\bar{P}_2$ is the composite of $\bar{P}_1$ with an inner automorphism of $PGL(d, \mathbb{F})$; in this case, we write $\bar{P}_1 \simeq \bar{P}_2$.

Given two projective representations P and Q of H, having degrees c and d respectively, the symbol $P \otimes Q$ denotes the inner tensor product of P and Q (which is a projective representation of H whose degree is cd), whereas the symbol $P{\uparrow^{\otimes G}}$ denotes the projective representation of G which is tensor induced by P from H.

In order to achieve the desired results on form induction of modules, and consequently on tensor induction of representations, we need (together with Theorem A) a generalization of Theorem 9.10 in [5]. This generalization is only stated in that paper, so we present next a proof.

THEOREM 3.3. *Let G be a solvable group, H a subgroup of G having odd index, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module which carries a G-invariant nonsingular symplectic $\mathbb{F}$-form f, and W a submodule of $V{\downarrow_H}$ such that $V \simeq W{\uparrow^G}$. Assume that W is induced from the normal core L of H in G. Then there exists a submodule Z of $V{\downarrow_H}$ such that f does not vanish on Z, $V \simeq Z{\uparrow^G}$, and Z has odd multiplicity as a composition factor in* $\mathrm{soc}(V{\downarrow_H})$.

PROOF. We proceed by induction on $|G : H|$. If H is a maximal subgroup of G, then we get the conclusion applying [5, 9.10]; thus we shall assume that there exists a proper subgroup E of G such that H is properly contained in E. Now, V is induced by W from H, so we get $V \simeq (W{\uparrow^E}){\uparrow^G}$; denoting by R the module $W{\uparrow^E}$, we have that V is induced by R from E, and R is in turn induced from a normal subgroup of G contained in E (which is L). We conclude that R is induced from the normal core of E in G and, since $|G : E|$ is odd, we can apply the inductive hypothesis (we can certainly assume that R is a submodule of $V{\downarrow_E}$) and find a submodule S of $V{\downarrow_E}$ such that f does not vanish on S, $V \simeq S{\uparrow^G}$, and S has odd multiplicity as a composition factor in $\mathrm{soc}(V{\downarrow_E})$.

Next, we know that there exists a submodule X of $V{\downarrow_L}$ such that $V \simeq X{\uparrow^G}$; by Mackey's Lemma we get

$$V{\downarrow_E} \simeq \bigoplus_{t\in\mathcal{T}} \left((X^t){\uparrow^E}\right)$$

where $\mathcal{T}$ is a set of representatives for the double cosets in G of L and E. Since each of the $(X^t){\uparrow^E}$ induces V from E and is therefore simple, we have that S is isomorphic, as an $\mathbb{F}E$-module, to one of those. We conclude that S is induced from L, hence also from the normal core of H in E. Therefore we can use again the inductive hypothesis, obtaining that there exists a submodule Z of $S{\downarrow_H}$ such that f does not vanish on Z, $S \simeq Z{\uparrow^E}$, and Z has odd multiplicity as a composition factor in $\mathrm{soc}(S{\downarrow_H})$. Now, putting together the two steps, we see that Z satisfies the required conditions. $\square$

We are now in a position to extend Theorem 10.1 and Theorem 10.2 of [5]. The two theorems below are only stated, as a proof of them can be obtained arguing as in 10.1 and 10.2 of [5], just replacing Theorem 9.7 and Theorem 9.10 of [5] with Theorem A and Theorem 3.3 of this paper.

THEOREM 3.4. *Let G be a solvable group, H a subgroup of G having odd index, $\mathbb{F}$ a finite field, V a simple $\mathbb{F}G$-module, and W a submodule of $V{\downarrow_H}$. Denoting by L the normal core of H in G, assume that G/L is a Frobenius group with Frobenius complement H/L. Assume also that V carries a G-invariant nonsingular symplectic $\mathbb{F}$-form f which does not vanish on W. If V is induced by W from H, then V is also form-induced from H (with respect to f).*

THEOREM 3.5. *Let G be a solvable group, H a subgroup of G having odd index, and D a faithful, primitive, tensor-indecomposable representation of G. Denoting by L the normal core of H in G, assume that G/L is a Frobenius group with Frobenius complement H/L. Assume also that we have $\bar{D}{\downarrow_H} \simeq \overline{P_1 \otimes P_2}$, where P_1 and P_2 are projective representations of H. If $\deg P_2$ is not 1, and $(\deg P_2)^{|G:H|}$ is a divisor of $\deg D$, then we have $(\deg P_2)^{|G:H|} = \deg D$, and there exists a projective representation P of H such that $\bar{D} \simeq \overline{P{\uparrow}^{\otimes G}}$ holds.*

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