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Complements of Connected Subgroups in Algebraic Groups.

P. PLAUMANN (*) - K. STRAMBACH (*) - G. ZACHER (**) (1)

To Professor Guido Zappa on his 90th birthday

ABSTRACT - The connected algebraic groups over an algebraically closed field whose lattice of closed connected subgroups is complemented or relatively complemented are determined.

Introduction.

If G is a connected algebraic group over an algebraically closed field, structural properties of the lattice ΛG of all closed connected subgroups of G turn out to be relevant in determining the algebraic structure of G. In the present note we classify the algebraic connected groups G with ΛG a complemented or relatively complemented lattice. Rosenlicht's theory of algebraic groups in the sense of A. Weil [12], allows us to reduce essentially our investigations, via the notion of a Λ-distributive pair, to the class of affine algebraic groups. We shall see that the class of connected algebraic groups G with ΛG complemented coincides with the class having in ΛG the Frattini subgroup ΦG trivial, while the class of connected algebraic groups with ΛG relatively complemented coincides with that of IM-groups, that is $\Phi[G/H] = 1$ for all intervals $[G/H]$ in ΛG. In abstract groups, the structure of solvable

(*) Indirizzo degli AA.: Mathematisches Institut Universität Erlangen-Nürnberg Bismarckstrasse 1 1/2, D-91054 Erlangen, Germany.
(**) Indirizzo dell'A.: Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, Via G. Belzoni, 7, I-35131 Padova, Italy.
(1) The authors are grateful to the University of Graz for the hospitality while working on parts of this manuscript. The third author acknowledges gratefully financial support from DAAD.

groups with complemented ([5], [15]) as well as with relatively complemented subgroup lattices ([9], [16]) is well understood and so also of *IM*-groups ([4], [10]); moreover all finite simple groups turn out to be complemented [3]. For more details on this topics we refer the interested reader to [14, Chapter 3].

In our approach an analog to theorem 3.1.10 in [14] plays a crucial part. Moreover a trivial Frattini subgroup imposes, as in abstract groups ([6], [17]), in algebraic groups severe structural restrictions on the normal nilpotent subgroups of ΛG. This fact is properly exploited in our investigaton of complemented, relatively complemented and *IM*-groups, which share all the property of having a trivial Frattini subgroup. Before presenting our main result we recall that a vector group V is a direct product of one-dimensional unipotent algebraic groups and that a P-group is an extension of a vector group V by a one-dimensional algebraic torus T which normalizes every subgroup of V but does not centralize any of these subgroups different from 1.

Our Main Theorem reads:

Given a connected algebraic group G over an algebraically closed field, let ΛG be the lattice of all closed connected subgroups of G. Then:

(i) *ΛG is complemented if and only if $\Phi G = 1$ if and only if G is an almost direct product of an abelian variety with an affine algebraic group L, where L is a semidirect product of a vector group V, which is a direct product of minimal normal subgroups of G with a reductive group.*

(ii) *ΛG is relatively complemented if and only if G is an IM-group if and only if G is an almost direct product of an abelian variety and an affine group L, where L is a direct product of a torus with a vector group or with a P-group.*

To avoid repetitions, in all what follows algebraic groups will be understood over algebraically closed fields, and a subgroup X of G always means an element of ΛG. The paper is subdivided into 4 sections. Section 1 contains prerequisities concerning the lattice ΛG, section 2 deals with properties of the Frattini subgroup and section 3 with those of ΛG for G a reductive connected group G. Finally in section 4 our main classification theorems of complemented, relatively complemented and *IM*-groups are presented. For basic notions on lattices, subgroup lattices and algebraic groups reference is made to [1], [14] and [7].

1. Prerequisites.

The set ΛG of closed connected subgroups of an algebraic group G forms a lattice if one takes for A, B in ΛG as meet $A \wedge B$ the connected component of the identity in $A \cap B$ and as join $A \vee B$ the algebraic subgroup generated by A and B.

Any connected algebraic group G is the product $L_G D_G$ of two subgroups where L_G is the unique maximal connected affine algebraic subgroup of G and D_G the unique minimal connected algebraic subgroup of G such that G/D_G is an affine group. Using results in [12] one can state

PROPOSITION 1.1. *Let $G = L_G D_G$ be a given algebraic connected group. Then D_G is contained in the center of G and if $L_G \wedge D_G = 1$, then D_G is an abelian variety while $G/L_G \cap D_G$ is a direct product of the affine group $L_G/L_G \cap D_G$ and the abelian variety $D_G/L_G \cap D_G$.*

THEOREM 1.2. *Let G be an algebraic connected group. Then the following assertions are true:*

(i) *(L_G, D_G) is a distributive pair in ΛG, that is, for any H in ΛG one has*

$$H \wedge (L_G \vee D_G) = (H \wedge L_G) \vee (H \wedge D_G)$$

(ii) $\Lambda(G/L_G \wedge D_G) = \Lambda(L_G/L_G \wedge D_G) \times \Lambda(D_G/L_G \wedge D_G)$

(iii) *The mapping $\lambda : \Lambda G \to \Lambda L_G$, $H \mapsto L_H$ is a surjective meet-homomorphism satisfying*

(1) $H^\lambda = H \wedge L_G$
(2) $(H_1 \vee H_2)^\lambda \geq H_1^\lambda \vee H_2^\lambda$

(iv) *The mapping $\delta : \Lambda G \mapsto D_G$, $H \mapsto D_H$ is a surjective join-homomorphism satisfying*

(1) $H^\delta \leq H \wedge D_G$
(2) $(H_1 \wedge H_2)^\delta = H_1^\delta \wedge H_2^\delta$

PROOF. For $H \in \Lambda G$ one has $H \wedge L_G \leq H^\lambda$; on the other hand the affine group $H^\lambda \leq H$ lies in L_G, so (iii)(1) follows. Since the group HD_G/D_G is affine, $H/H \wedge D_G$ is affine too, thus (iv)(1) follows. From (iii)(1) and (iv)(1) one concludes that $H = (H \wedge L_G)(H \wedge D_G)$. Thus (i) and (ii) hold. From (iii)(1) follows that λ is a meet-homomorphism. The validity of (iii)(2) is obvious. Since $H_1 H_2^\delta / H_1^\delta H_2^\delta$ and $H_1^\delta H_2 / H_1^\delta H_2^\delta$ are respectively epimorphic images of H_1/H_1^δ, and H_2/H_2^δ in $H_1 \vee H_2 / H_1^\delta H_2^\delta$ and since a subdirect product of linear groups is linear $H_1 \vee H_2/H_1^\delta H_2^\delta$ is linear. Thus $H_1^\delta H_2^\delta \geq$ $\geq (H_1 \vee H_2)^\delta$. On the other hand $H_i \leq (H_1 \vee H_2)$ yields $H_i^\delta \leq (H_1 \vee H_2)^\delta$ hence $H_1^\delta H_2^\delta = (H_1 \vee H_2)^\delta$, i.e. δ is a join homomorphism.

There is a rational isomorphism from

$$H_1 \wedge H_2/H_1^\delta \wedge H_2 = H_1 \wedge H_2/H_1^\delta \wedge (H_1 \wedge H_2)$$

onto the linear group $(H_1 \wedge H_2)H_1^\delta/H_1^\delta$. Thus the group $(H_1 \wedge H_2)/H_1^\delta \wedge H_2$ is linear. Since a subdirect product of linear groups is linear, also $H_1 \wedge H_2/H_1^\delta \wedge H_2^\delta$ is linear i.e. (iv)(2) holds. □

Two connected algebraic groups G_1, G_2 are called *isogeneous* if there exists an algebraic group G and epimorphisms $q_i: G \to G_i$ such that ker q_i is finite [12]. According to Theorem 6 in [12] and Hilfssatz 1 in [11] one gets the following

LEMMA 1.3. *For a connected algebraic group G and a finite normal subgroup E of G, the canonical epimorphism $\pi: G \to G/E$ induces a lattice isomorphism from ΛG onto $\Lambda(G/E)$.*

If a connected algebraic group G is a product NH where N is a normal subgroup of G such that $N \wedge H = 1$, we call G an *almost semidirect product* of N with H. Lemma 1.3 allows often to discuss semidirect products instead of almost semidirect products.

REMARK 1.4. We recall from [11] that if A is an abelian variety, then ΛA is a relatively complemented modular atomic lattice (for these notions see [1]).

2. The Frattini subgroup.

The Frattini subgroup ΦG of a connected algebraic group G is defined as the meet of all maximal elements in ΛG. Thus ΦG does not have a complement in G unless $\Phi G = 1$.

PROPOSITION 2.1. *Let G be a connected algebraic group and N a normal subgroup of G. Then $\Phi N \leq \Phi G$.*

PROOF. Assume $\Phi N \not\leq \Phi G$. Then there exists a maximal element M of G such that $M(\Phi N) = G$. But then $N = N \wedge M(\Phi N) = (\Phi N)(M \wedge N) =$ $= M \wedge N$, a contradiction since $N \neq M \wedge N$. □

THEOREM 2.2. *Let G be a connected algebraic group. Then*

(i) $\Phi D_G = D_G \wedge L_G$, $\Phi G = (\Phi L_G)(\Phi D_G) = \Phi L_G$

(ii) ΦL_G *is a unipotent group and* ΦD_G *is a product of one-dimensional central affine subgroups of G.*

PROOF. (i): Since D_G is a non splitting extension of the group $R = D_G \wedge L_G$ by an abelian variety, it follows from [13], Lemma 2 and Proposition 11 that the central group R is generated by one-dimensional affine subgroups. Let E be one of these subgroups and let M be a maximal element in ΛD_G. If $E \not\leq M$, then $D_G = EM$ and D_G/M would be a non trivial affine epimorphic image of D_G. So $\Phi D_G \geq R$ and since $\Phi(D_G/R) = 1$ (see Satz 6 in [11]) we get $D_G \wedge L_G = \Phi D_G \leq \Phi G$. Using theorem 1.2(ii) one gets $\Phi(G/\Phi D_G) = \Phi(L_G/\Phi D_G)$ and $\Phi G = \Phi L_G = (\Phi L_G)(\Phi D_G)$.
(ii): Let U be the unipotent radical of L_G and set ${L_G}^* = L_G/U$. Since ${L_G}^*$ is a reductive group, every normal subgroup in $\Lambda {L_G}^*$ is an almost direct factor of ${L_G}^*$ [7]. On the other hand $\Phi {L_G}^*$ is contained in every maximal normal element of $\Lambda {L_G}^*$. Thus one gets $\Phi {L_G}^* = 1$. It follows that $\Phi L_G \leq U$ is unipotent. The remaining part of (ii) follows from (i). □

COROLLARY 2.3. *Let G be an algebraic connected group. Then* $\Phi G = 1$ *if and only if* $\Lambda G = \Lambda L_G \times \Lambda D_G$, *where* D_G *is an abelian variety and* $\Phi L_G = 1$.

PROOF. If $\Phi G = 1$, then by theorem 2.2 one has $1 = D_G \wedge L_G = (\Phi L_G)(\Phi D_G)$. So by proposition 1.1 we have $\Lambda G = \Lambda L_G \times \Lambda D_G$ and D_G is an abelian variety. Conversely, if $\Lambda G = \Lambda L_G \times \Lambda D_G$, then $\Phi(G) = \Phi L_G \times \Phi D_G = 1$. □

Given a connected algebraic group G, the determination of the algebraic structure of $G/\Phi G$ is so reduced via corollary 2.3 to the case that G is an affine group. In this case an element $H \in \Lambda G$ is a nilpotent group if $H = H_u \times T$, T a torus.

LEMMA 2.4. *Let G be an affine algebraic group and let N be a normal subgroup of G such that* $N \leq \Phi G$. *Then* K/N *is a normal connected nilpotent subgroup of* G/N *if and only if K is a normal nilpotent element of* ΛG.

PROOF. Assume $K/N \lhd G/N$ and K/N nilpotent. Then $K = K_u T$, where T is a maximal torus of K, and by theorem 2.2 (ii) one has $N \leq K_u$. Since $NT \lhd G$, by the Frattini argument one has $\mathcal{N}(T)N = G$. Hence

$\mathcal{N}(T) = G$ since $K = K_u \times T$. For the converse, $K = K_u \times T \lhd G$ and by theorem 2.2 (ii) $N \leq K_u$ hence $K/N \lhd G/N$ and K is nilpotent. □

REMARK 2.5. Let FG be the Fitting subgroup of G, i.e. the join of all normal nilpotent elements of ΛG. Using theorem 2.2 (ii) the lemma 2.4 tells us that $FG/\Phi G$ is the Fitting subgroup of $G/\Phi G$.

LEMMA 2.6. *Given the connected affine algebraic group G, let $N/\Phi G$ be a normal nilpotent subgroup of $G/\Phi G$. Then $G/\Phi G$ splits over $N/\Phi G$ and $N/\Phi G$ is a direct product of minimal normal subgroup of $G/\Phi G$.*

PROOF. Since $\Phi(G/\Phi G) = 1$, by proposition 2.1 one has $\Phi(N/\Phi G) = 1$. As $N/\Phi G$ is the intersection of subgroups of codimension 1, it is a direct product of a vector group and a torus. Take a subgroup $A/\Phi G$ in $\Lambda(N/\Phi G)$ which is maximal with respect to properties to be generated by minimal normal subgroups in $\Lambda(N/\Phi G)$ and having a complement $C/\Phi G$ in $\Lambda(G/\Phi G)$. By induction assume that A is properly contained in N. Then $1 \neq C \wedge N/\Phi G \lhd CN/\Phi G = G/\Phi G$. Let $B/\Phi G$ be a minimal connected normal subgroup of $G/\Phi G$ contained in $C \wedge N/\Phi G$. Since $\Phi(G/\Phi G) = 1$, there also exists a maximal connected subgroup $M/\Phi G$ such that $M \wedge B/\Phi G = 1$. Now by construction $(M \wedge C) \wedge AB = \Phi G$, while $(M \wedge C)AB = (MB \wedge C)A = CA = G$, a contradiction to the maximality of A. □

3. Affine groups.

LEMMA 3.1. *Let G be a connected affine algebraic group and take $D \in \Lambda G$ such that D lies in only finitely many but in at least two maximal elements of ΛG. Let $\mathcal{C}$ be a conjugacy class of elements of ΛG which are not all contained in only one maximal element of ΛG containing D. Then there is a $C \in \mathcal{C}$ such that $G = C \vee D$.*

PROOF. Let $\mathfrak{M} = \{M_1, \dots, M_\ell\}$ be the set of maximal elements of ΛG which contain D and let C_0 be an arbitrary element of $\mathcal{C}$. By hypothesis we have $\ell > 1$ and the sets $X_i = \{g \in G \mid C_0^g \leq M_i\}$ are non empty closed subsets of G (see [2] I.1, p. 52). Since G is an irreducible variety, there is an element $g \in G \setminus \bigcup_i X_i$. For such an element one has $C_0^g \vee D = G$. □

PROPOSITION 3.2. *A maximal torus T of a connected affine reductive algebraic group G is contained in only finitely many maximal elements of ΛG.*

PROOF. By ([2], 13.1, p. 163) the set $\mathcal{B}^T$ of all Borel subgroups containing T is finite, and each element of $\mathcal{B}^T$ contains only finitely many T-invariant connected closed unipotent subgroups (see [7], 28.1, p. 170). From this it follows that T lies in only finitely many parabolic subgroups of G (see [7], 28.3, p. 172). The maximal elements of ΛG are parabolic or reductive (see [7], 30.4, p. 187). We denote with $\mathcal{R}^T$ the set of all maximal elements of ΛG which are reductive and contain T. It follows from ([7], 28.1, p. 170) that T normalizes only finitely many connected unipotent subgroups of G. Hence the Borel subgroups of the groups in the family $\mathcal{R}^T$ form a finite set. As every element of $\mathcal{R}^T$ is generated by two of its Borel subgroups (see [7], 26.2, Corollary C, p. 160) the family $\mathcal{R}^T$ is finite, too. □

LEMMA 3.3. *Let $S \neq 1$ be a torus in a simple algebraic group G. Then there is a torus of G which is a complement of S in ΛG.*

PROOF. If G is a simple abelian variety then there is nothing to show. Let T be a maximal torus of an simple affine algebraic group G. From proposition 3.2 we know that T lies only in finitely many maximal elements of ΛG. Thus we may apply lemma 3.1 to T and to the conjugacy class $\{S^g | g \in G\}$. Hence T has a supplement S^g for some $g \in G$, and so S has $T^{g^{-1}}$ as a supplement in G. But then if C is a complement of $T^{g^{-1}} \wedge S$ in $T^{g^{-1}}$, then C is a complement of S in ΛG. □

LEMMA 3.4. *A connected unipotent subgroup H of a reductive algebraic group G has a complement in G.*

PROOF. Let B be a Borel subgroup of G containing H. According to ([7], 26.2, corollary C) there exists a Borel subgroup B^- such that $B \wedge B^-$ is a maximal torus of G. We set $P = H \vee B^-$. Since $[G/B^-]$ is a Boolean algebra (see [7], Theorem, p. 179) there exists a parabolic subgroup L of G such that $P \wedge L = B^-$ and $P \vee L = G$. We have $H \wedge L \leq P \wedge L = B^-$; hence $H \wedge L \leq H \wedge B^- = 1$ and $H \vee L = H \vee B^- \vee L = P \vee L = G$. □

LEMMA 3.5. *Let G be a connected algebraic group and let $A_1, A_2 \in \Lambda G$ such that $G = A_1 A_2$. For an $H \in \Lambda G$ assume that there exists a complement C_1 of $H \wedge A_1$ in ΛA_1 and a complement C_2 of $(H \vee A_1) \wedge A_2$ in ΛA_2. If $C_1 C_2 = C_2 C_1$, then $C_1 C_2$ is a complement of H in ΛG.*

PROOF. Put $D_1 = H \cap A_1$ and $D_2 = (H \vee A_1) \cap A_2$. The groups $E_i = C_i \cap D_i$, $(i = 1, 2)$, are finite subgroups of A_i. Suppose $x \in H \cap C_1 C_2$.

Since there are elements $c_i \in C_i$, $(i = 1, 2)$, with $x = c_1 c_2$, one has

$$c_2 = c_1^{-1} x \in (H \vee C_1) \cap C_2 \leq (H \vee A_1) \cap A_2 \cap C_2 = D_2 \cap C_2 = E_2.$$

Thus $xc_2^{-1} = c_1 \in H \cap C_1 = H \cap A_1 \cap C_1 = D_1 \cap C_1 = E_1$. It follows that $H \cap C_1 C_2$ is contained in the finite set $E_1 E_2$ and this implies $H \wedge C_1 C_2 = 1$. Having obtained this, the rest of the proofs is verbatim as in ([14], 3.1.4) □

THEOREM 3.6. *For a reductive connected algebraic group G the lattice ΛG is complemented.*

PROOF. We know that G decomposes in an almost direct product $G = G_1 G_2 \ldots G_t$, where G_i is a torus or a simple algebraic group. From lemma 3.5 it follows that ΛG is complemented if each ΛG_i is complemented. So take $H \in \Lambda G_i$. If H is unipotent, then according to lemma 3.4 it has a complement in ΛG_i. If H is not unipotent, then H contains a torus $T \neq 1$. It follows from lemma 3.3 that in G there is a torus W which is a supplement of T in ΛG. But then a complement of $H \wedge W$ in W is a complement of H in ΛG. The conclusion follows. □

4. The main theorems.

THEOREM A: *Let G be a connected algebraic group over an algebraically closed field. Then the following statements are equivalent:*

(i) *ΛG is a complemented lattice*

(ii) $\Phi(G) = 1$

(iii) *G is an almost direct product of an abelian variety A with an affine algebraic group L which is a semidirect product of the unipotent radical R_u and a reductive group K. Moreover, R_u is a direct product of minimal normal vector groups of G.*

(iv) *In ΛG there is a normal nilpotent subgroup N such that N is the direct product of connected minimal normal subgroups of G and N has a complement K in ΛG with ΛK complemented.*

PROOF. The statement (ii) follows easily from (i), as was already remarked in the beginnig of section 2.

If $\Phi G = 1$, then (iii) is an easy consequence of corollary 2.3 and lemma 2.6.

Putting $N = AR_u$ in (iii) one obtains (iv).

Let N be a connected closed normal nilpotent subgroup of G satisfying (iv). Then N is an almost direct product of of minimal connected subgroups N_i, normal in G. Since he center Z_i of N_i is normal in G, we conclude that N is commutative.

Let $B \in \Lambda G$ be a normal subgroup of G and let C be maximal in ΛG with respect to being normal in G and having finite intersection with B. If N is not contained in BC, then there is i such that N_i lies not in BC. Since $B \triangleleft G$, the group BC is normal in G and $BC \cap N_i$ is finite because of the minimality of N_i. Hence BCN_i is an almost direct product of B, C, N_i. It follows that $CN_i \cap B$ is finite and that C is a proper subgroup of CN_i, contradicting the maximality of C. Thus C has a complement in ΛG.

Take $H \in \Lambda G$ arbitrary. As ΛK is complemented, there is a complement D of $H \wedge K$ in ΛK. Since $N \triangleleft G$ is commutative, we see that $B_1 = (H \vee K) \wedge N$ is normal n N as well as in $H \vee K$. Hence B_1 is normal in $NK = G$. As we have seen in the last paragraph there is a complement C of B_1 in ΛN which is normal in G. Since $C \wedge D \leq N \wedge K$ is finite, we conclude from Lemma 3.5 that CD is a complement of H in ΛG. □

The two following statements are immediate consequences of Theorem A.

COROLLARY 4.2. *A connected affine algebraic group over an algebraically closed field of characteristic 0 has a complemented lattice ΛG if and only if its unipotent radical is a vector group.*

COROLLARY 4.3. *Let G be a connected algebraic group. Then ΛG is complemented if and only if each characteristic subgroup of ΛG has a complement in G.*

COROLLARY 4.4. *An almost direct product G of sequence $G_1, \dots, G_n$ of connected algebraic groups is a complemented group if and only if ΛG_i is a complemented lattice for every i.*

PROOF. Use lemma 1.3 and observe that $\Phi G = \Phi G_1 \times \cdots \times \Phi G_n$. Then use theorem A. □

A connected algebraic group G is called an *IM*-group if one has $\Phi[G/H] = 1$ for every interval $[G/H]$ in ΛG. This means that the meet of all maximal elements of the interval $[G/H]$ is H. By theorem A the class of *IM*-groups is a subclass of the class of complemented connected algebraic groups.

Lemma 4.5. *If G is a connected algebraic IM-group, then G is solvable.*

Proof. If the proposition is not true, then there are reductive groups, which are counter-examples. Among them we choose a group G in which the Borel subgroups have minimal dimension. Let B be a Borel subgroup of G and let U be the unipotent radical of B. Let B_1 a Borel subgroup of G with unipotent radical U^- such that $U \wedge U^- = 1$ and $U \vee U^- = G$ ([7], 26.2 Corollary C, p. 160 and 27.5 Theorem, p. 167). Let Ω be the set of all maximal elements in the interval $[G/U]$. If $M \in \Omega$ is parabolic, it contains B since B is the normalizer of U in G. It follows that not all elements of Ω are parabolic, otherwise $B = U$. Let $M \in \Omega$ be reductive ([7], 30.4, p. 187). Then U is the unipotent radical of some Borel subgroup B_2 of M. Let B_3 be a Borel subgroup of M with unipotent radical U_3 such that $U \wedge U_3 = 1$ and $U \vee U_3 = M$ hold. Because the dimension of M is minimal, the groups U^- and U_3 have the same dimension. But this implies the contradiction $G = U \vee U^- = U \vee U_3 = M$. □

A connected algebraic group G is called a *t*-group if the relation of normality is transitive in ΛG.

Lemma 4.6. *Let G a connected algebraic IM-group and A, B elements of ΛG such that $A \triangleleft B \triangleleft G$. Then $A \triangleleft G$, i.e. G is a t-group.*

Proof. Let $\Omega = \{M \in \Lambda G | A \leq M, M \text{ minimal in } \Lambda G\}$ and $\Delta = \{M \in \Omega | B \leq M\}$. Then $A = \bigcap_{M \in \Omega \setminus \Delta} M$ and for each $M \in \Omega \setminus \Delta$ one has $MB = G$. Thus for $g \in G$ there are $m \in M$, $b \in B$ such that $g = mb$. Hence $(M \wedge B)^g = M^b \wedge B \geq A^b = A$, i.e. $A \triangleleft G$. □

Lemma 4.7. *If G is an algebraic P-group, Then ΛG is a projective space. Moreover, if the characteristic of the ground field* k *is positive, then G has dimension at most 2.*

Proof. Let $G = VT$ be a P-group. If *char* (k) $= 0$, then any subspace of the Lie algebra $\mathcal{L}G$ of G is a subalgebra. But by ([7], 13.1) the lattice of subalgebras of $\mathcal{L}G$ and ΛG are isomorphic.

Assume that char k > 0 and $\dim G > 2$. Then for every 1-dimensional $U \in \Lambda G$ there is a 1-dimensional $E \in \Lambda G$ such that $U \cap E$ is finite. The automorphism groups T_U respectively T_E induced by T on U respectively E are one-dimensional tori. The group T_U leaves the finite group $U \cap E$

invariant and hence centralizes it, a contradiction. Hence for $V \neq 1$ we have $\dim G = 2$ and ΛG is a projective line. □

THEOREM B: *Let G be a connected algebraic group. Then the following statements are equivalent:*

(i) *ΛG is a relatively complemented lattice*
(ii) G is an IM-group
(iii) *G is a solvable complemented t-group*
(iv) *G is an almost direct product of an abelian variety by an affine group L, where $L = V \times C$ or $L = P \times C, V$ a vector group, C a torus and $P = VT$ a P-group.*

PROOF. If ΛG is relatively complemented, then for $H \in \Lambda G$ one has $\Phi[G/H] = 1$. So G is an IM-group.

If G is an IM-group, then by Theorem A one has $\Phi G = 1$. It follows from Lemma 4.5 and Lemma 4.6, that G is a complemented solvable t-group.

If G satisfies (iii), then by Theorem A it is an almost direct product of an abelian variety A and a solvable affine t-group L with ΦL. So L_u is a commutative group. Since L is a t-group and L_u is commutative, it follows that $H \leq L_u \triangleleft L$ implies $H \triangleleft L$. So L has the structure indicated in (iv).

If G is as in (iv), then by Corollary 2.3 one has $\Lambda G = \Lambda L \times \Lambda A$. Since ΛA is relatively complemented (see Remark 1.4), all we have to show is that L is a relatively complemented group. This is clear if L is commutative, because then $\Lambda L = \Lambda V \times \Lambda AT$ ([11], Satz 5). Assume that G is a counterexample in which the dimension of L is minimal. Then$L$ is not commutative, i. e. $L = VT \times A$ with $T \neq 1$. Theorem A shows that L is complemented. The structure given in (iv) is inherited by all closed connected subgroups of L as well as by all epimorphic images of L. Since G is a counterexample, there exists an atom $H \in \Lambda G$ such that HG and the interval $[G/H]$ is not relatively complemented. As HG one has $VT = VH$. So using Lemma 4.7 one sees that $[G/H] \simeq \Lambda(V \times C)$ is a complemented lattice, a contradiction. □

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