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Some Simple Groups which are determined by the Set of their Character Degrees II

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To Guido Zappa on his 90th birthday

In part I of this paper we began to study the following conjecture.

CONJECTURE. Let H be a simple nonabelian group. If G is any group such that G and H have the same *set* of character degrees of irreducible characters over $\mathbb{C}$, then $G \cong H \times A$, where A is abelian.

In part I we proved this conjecture if H is a Suzuki group $\mathrm{Sz}(q)$ or (using an unpublished result by F. Lübeck) if $H = \mathrm{SL}(2, 2^f)$ for some f. In this paper we study the case $H = \mathrm{PSL}(2, p^f)$, where p is odd. As $\mathrm{PSL}(2,5) \cong \mathrm{PSL}(2,4) \cong A_5$, we can assume that $p^f > 5$.

The proof follows the same pattern as in part I. Also references to lemma 1 to lemma 6 refer to part I.

In section 4, we give a proof, which relies on the following lemma.

LEMMA 1. *Let p be a prime. We say that a group G has property (p), if the following holds:*

(1) *If $\chi \in \mathrm{Irr}\, G$, then $\chi(1)$ is a power of p or prime to p.*
(2) *There exists some $\chi \in \mathrm{Irr}\, G$ such that $\chi(1) = p^d > 1$.*

If G is a simple group with property (p), then either $G \cong \mathrm{PSL}(2, p^d)$, where $p^d > 3$, or $G \cong \mathrm{PSL}(2, 2^f)$, where $p = 2^f - 1$ on $p = 2^f + 1$, or $p = 3$ and $2^f = 8$.

(As a simple group has some even degree, so for $p = 2$ condition (2) can be omitted).

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Proof. a) $\mathrm{PSL}(2,q)$ has the degrees q, $q+1$, $q-1$ and if $2\not| q$ the odd one of $\frac{q\pm 1}{2}$. The cases, where $q+1$ on $q-1$ is a power of some prime, are the cases stated above. If the odd degree $\frac{q\pm 1}{2}$ is a prime-power p^a, then p^a and $2p^a$ are degrees, so condition (p) is not satisfied.

b) We have to show that all other simple groups do not have property (p) for any prime p. All quasi-simple groups, which have some prime-power degree, have been determined in Malle-Zaleskiĭ [10]. We have to go through the list of such groups, following the numbering in [10]. All statements about degrees of sporadic and other small simple groups come from the Atlas.

(I have to thank G. Malle for very detailed information).

(1) Let G be a simple group of Lie type of characteristic p and let χ be the Steinberg character of G with $\chi(1) = |G|_p = q^d$.

The list in Curtis-Iwahori-Kilmoyer [2] on p. 111 gives in most cases a degree, which is divisible by p, but not a power of p. The exceptions are the following:

$\mathrm{PSL}(2,q)$, as mentioned in a).

$\mathrm{B}_l\,(2) = \mathrm{Sp}(2l,2)$ and $p=2$. But by [2], p. 114 $\mathrm{B}_l\,(2)$ has the degree

$$\frac{2^{(l-1)^2}(2^{l-1}+1)(2^{l}+1)}{2\cdot 3},$$

which is even for $l \geq 3$, but not a power of 2. Observe that $\mathrm{B}_2(2) = \mathrm{Sp}(4,2) \cong S_6'$.

By [2], p. 114 $F_4(2)$ has the degree $2\cdot 3^2\cdot 5^2\cdot 13\cdot 17$.

$G_2(3)$ has the degrees 2^6 and 3^6, but also 56 and 21.

$B_2^1(q) = \mathrm{Sz}(q)$ has the degree $(q-1)\,2^n$, where $q = 2^{2n+1}$ (Suzuki [11]).

$G_2^1(q)$ has the degree $q(q^2-q+1)$, where $q = 3^{2n+1}$ (Ward [12], p. 87).

(Observe that as $A_8 \cong \mathrm{PSL}(4,2)$ the degree 2^6 of A_8 has been handled.)

(2) The case $G = \mathrm{PSL}(2,q)$ has been considered in section a).

(3) Let $G = \mathrm{PSL}(n,q)$, where $q>2$, n is an odd prime such that $(n,q-1)=1$ and $\chi(1) = \frac{q^n-1}{q-1}$ is a power of a prime (which happens sometimes). By Carter [1], 13.8 for $n \geq 4$ the group $\mathrm{PSL}(n,q)$ has the

degree

$$q^2 \frac{(q^n-1)}{q-1} \frac{(q^{n-3}-1)}{q^2-1}.$$

This is a multiple of the prime-power $\dfrac{q^n-1}{q-1}$, but also a multiple of q.

For $\mathrm{PSL}(3,q)$ we have the degrees q^2+q+1 and $q(q^2+q+1)$.

(4) Let $G=\mathrm{PSU}(n,q)$, where n is an odd prime such that $(n,q+1)=1$ and $\chi(1)=\dfrac{q^n+1}{q+1}$ is the power of a prime. By the Lusztig- Srinivasan correspondence between degrees of $\mathrm{PSL}(n,q)$ and $\mathrm{PSU}(n,q)$, replacing q by $-q$, as n is odd, we obtain from (3) for $n \geq 4$ the degree

$$q^2 \frac{(q^n+1)}{q+1} \frac{(q^{n-3}-1)}{q^2-1}$$

for $\mathrm{PSU}(n,q)$.

$\mathrm{PSU}(3,q)$ has the degrees q^2-q+1 and $q(q^2-q+1)$ (Klemm [9]).

(5) Let $G=\mathrm{Sp}(2n,q), n>1, q=r^k$ with an odd prime r, kn a power of 2 and $\chi(1)=\dfrac{q^n+1}{2}$ a power of a prime. By [1], p. 111 $\mathrm{Sp}(2n,q)$ has the degree

$$m=q\frac{(q^{n-1}+1)}{q+1} \frac{(q^n+1)}{2}.$$

As n is a power of 2, so $n-1$ is odd and hence

$$\frac{q^{n-1}+1}{q+1}=1-q+q^2-\ldots+q^{n-2}$$

is integral. Hence m is divisible by q and $\dfrac{q^n+1}{2}$.

(6) Let $G=\mathrm{Sp}(2n,3)$, $n>1$ a prime, $\chi(1)=\dfrac{3^n-1}{2}$ a power of a prime. By [2], p. 111 $\mathrm{Sp}(2n,3)$ has the degree

$$3\frac{(3^n-1)}{2} \frac{(3^{n-1}+1)}{2}.$$

(7) Let $G=A_n$, where $n=p^d+1 \geq 10$ and $\chi(1)=p^d$. Then A_n has also the degree.

$$(n-1)\frac{n-2}{2}=p^d\frac{(p^d-1)}{2}.$$

Observe that $A_8 \cong \mathrm{PSL}(4,2)$ has the degrees 7 and 2^6, but also 14. The cases

$A_5 \cong \mathrm{PSL}(2,5)$ and $A_6 \cong \mathrm{PSL}(2,9)$ have been considered in a).

(11) Let $G = \mathrm{Sp}(6,2)$ and $\chi(1) = 7$. Then G has the degree 21. (The degrees 2^6 and 3^3 of $\mathrm{Sp}(6,2)$ are treated in (1) and (17).)

(12) $A_6 \cong \mathrm{PSL}(2,9)$ has already been mentioned.

(14) M_{11} and M_{12} have the degree 11, but also 44 resp. 66.

(15) M_{11} and M_{12} have the degree 2^4, but also 44 resp. 66.
$\mathrm{PSL}(3,3)$ has the degree 2^4, but also 12.

(16) M_{24}, Co_2 and Co_3 have the degree 23, but also $11 \cdot 23$ resp. $99 \cdot 23$ resp. $8019 \cdot 23$.

(17) A_9, $\mathrm{Sp}(6,2)$ and ${}^2F_4(2)'$ have the degree 3^3, but also $2 \cdot 3 \cdot 7$ resp. 21 resp. $2 \cdot 3 \cdot 13$.

(18) $\mathrm{PSU}(3,3)$ has the degree 2^5, but also 28.

(19) $G_2(3)$ has the degree 2^6, but also $2 \cdot 3 \cdot 13$.
Hence all cases have been considered.

4. The group $\mathrm{PSL}(2,p^f)$, where $p \neq 2$.

For $p^f > 5$ we have

$$\mathrm{cd}\,\mathrm{PSL}(2,p^f) = \{1, p^f - 1, p^f, p^f + 1, r\},$$

where

$$r = \frac{p^f - 1}{2} \qquad \text{if } p^f \equiv -1 \pmod 4$$

$$r = \frac{p^f + 1}{2} \qquad \text{if } p^f \equiv 1 \pmod 4.$$

Observe that r is odd.

If $p^f \neq 9$, then $\mathrm{SL}(2,p^f)$ is the Schur covering group of $\mathrm{PSL}(2,p^f)$. The degrees of the properly projective irreducible representations of $\mathrm{PSL}(2,p^f)$ are $p^f - 1$, $p^f + 1$ and s, where

$$s = \frac{p^f + 1}{2} \qquad \text{if } p^f \equiv -1 \pmod 4$$

$$s = \frac{p^f - 1}{2} \qquad \text{if } p^f \equiv 1 \pmod 4.$$

(see Dornhoff [3], p. 228). But observe that $\mathrm{PSL}(2,9) \cong A_6$ has a Schur multiplier of order 6 and has some more degrees of projective representations (Atlas, p. 5).

THEOREM 3. *Suppose that* $p > 2$, $p^f > 5$ *and*

$$\operatorname{cd} G = \operatorname{cd} \mathrm{PSL}(2,p^f) = \{1, p^f - 1, p^f, p^f + 1, r\},$$

where

$$r = \begin{cases} \dfrac{p^f - 1}{2} & \text{if } p^f \equiv -1 \pmod 4 \\[2ex] \dfrac{p^f + 1}{2} & \text{if } p^f \equiv 1 \pmod 4). \end{cases}$$

Then

$$G \cong \mathrm{PSL}(2,p^f) \times A,$$

where A is abelian.

PROOF. *Step 1.* We have $G' = G''$:

Otherwise let G/N be a solvable, minimal nonabelian factor group of G.

Suppose at first that G/N is a q-group for some prime q. If $q = p$ and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = p^f - 1$, then $\chi_N \in \mathrm{Irr}(N)$ and by lemma 2 of part I therefore $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$. As now $p^f \in \operatorname{cd} G/N$, we obtain the forbidden degree

$$\chi(1)\,\tau(1) = (p^f - 1)\,p^f.$$

If $q \neq p$, we apply the same argument to $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = p^f$.

Hence by lemma 4 of part I we can assume that G/N is a Frobenius group with elementary abelian Frobenius kernel F/N. Then $|G/F| \in \operatorname{cd} G$ and $|F/N| = q^a$ for some prime q, where $|G/F|$ divides $q^a - 1$.

If

$$|G/F| \in \{p^f - 1, p^f, p^f + 1\},$$

then as $p^f > 3$ no proper multiple of $|G/F|$ is in $\operatorname{cd} G$. Hence by lemma 4 of part I, if $\chi \in \mathrm{Irr}\, G$ and $q \nmid \chi(1)$, then $\chi(1)$ divides $|G/F|$.

Suppose at first $|G/F| = p^f$, hence $q \neq p$. If $q \nmid p^f - 1$, we obtain the contradiction that $p^f - 1$ divides p^f. If $q \nmid p^f + 1$, we get the contradiction that $p^f + 1$ divides p^f. Hence q divides $p^f - 1$ and $p^f + 1$, hence $q = 2$. But as $2 \not| r$, hence lemma 4 of part I provides the contradiction that $r = \dfrac{p^f \pm 1}{2}$ divides p^f.

Suppose next that $|G/F| = p^f \pm 1$. If $q \neq p$, we take $\chi \in \operatorname{Irr} G$ such that $\chi(1) = p^f$ to obtain the contradiction that p^f divides $|G/F|$. If $q = p$, we obtain either the contradiction that $p^f + 1$ divides $p^f - 1$ or $p^f - 1$ divides $p^f + 1$. (Observe that $p^f > 3$).

There remains only the case that

$$|G/F| = r = \frac{p^f \pm 1}{2}.$$

By lemma 4 of part I, if $\psi \in \operatorname{Irr} F$, then *either*

$$|G/F|\psi(1) \in \operatorname{cd} G,$$

so $\psi(1) \leq 2$, *or* q divides $\psi(1)$.

Let $\chi \in \operatorname{Irr} G$ such that

$$\chi(1) = \begin{cases} p^f - 1 & \text{if } |G/F| = \dfrac{p^f + 1}{2} \\ p^f + 1 & \text{if } |G/F| = \dfrac{p^f - 1}{2} \end{cases}$$

As $2 \not| |G/F|$, so

$$\big(|G/F|, \chi(1)\big) = 1.$$

Therefore $\chi_F \in \operatorname{Irr} F$ and $\chi(1) > 2$. Hence q divides $\chi(1)$, so $q \neq p$. If $\tau \in \operatorname{Irr} G$ and $\tau(1) = p^f$, then also $\tau_F \in \operatorname{Irr} F$, but $q \not| \tau(1)$.

This contradiction shows $G' = G''$.

Step 2. If G'/M is a chief factor of G then

$$G'/M \cong \operatorname{PSL}(2, p^f):$$

As $G' = G''$, we have

$$G'/M = S_1 \times \ldots \times S_k,$$

where $S_i \cong S$ is simple and nonabelian. The degrees of S divide degrees of G, hence are prime to p or powers of p.

Suppose at first that $p \not| |S|$. Let $\chi \in \operatorname{Irr} G$ and $\chi(1) = p^f$. As $G' = G''$, so $\chi_{G'}$ cannot split in characters of G' of degrees 1. Hence

$$\chi_{G'} = \sum_i \varphi_i, \quad \varphi_i \in \operatorname{Irr} G'$$

and $\varphi_i(1) = p^r > 1$. As by our assumption $\varphi_i(1)$ is prime to $|G'/M|$, so $(\varphi_i)_M \in \operatorname{Irr} M$ and $\varphi_i\tau \in \operatorname{Irr} G'$ for all $\tau \in \operatorname{Irr}(G'/M)$. As G'/M is a non-abelian p'-group, there exists $\tau \in \operatorname{Irr}(G'/M)$ such that $\tau(1) > 1$ and $p \nmid \tau(1)$. But then $\varphi_i\tau$ has a "mixed" degree, a contradiction.

Hence $p||S|$. By the theorem of Ito and Michler there exists $\sigma \in \operatorname{Irr} S$ such that $p|\sigma(1)$. Hence $\sigma(1)$ is a power of p larger than 1. Therefore we can apply lemma 1 of this paper.

Case 1. Suppose $S \cong \mathrm{PSL}(2, p^m)$ for some m. Obviously $k = 1$, hence

$$G'/M \cong \mathrm{PSL}(2, p^m).$$

If $\psi \in \operatorname{Irr}(G'/M)$ and $\psi(1) = p^m$, then $\psi(1)$ divides $\chi(1)$ for some $\chi \in \operatorname{Irr} G$. Hence $\chi(1) = p^f$, so $m \leq f$.

Suppose $m < f$. We consider $\overline{G} = G/C_G(G'/M)$. Then $\overline{G}' \cong \mathrm{PSL}(2, p^m)$ and $|\overline{G} : \overline{G}'|$ divides $|\mathrm{Out}\ \mathrm{PSL}(2, p^m)| = 2m$.

Take at first $\psi_1 \in \operatorname{Irr} \overline{G}'$ such that $\psi_1(1) = p^m$. If we choose $\chi_1 \in \operatorname{Irr} \overline{G}$ such that $\left(\psi_1, (\chi_1)_{\overline{G}'}\right)_{\overline{G}'} > 0$, then $\chi_1(1) = p^f$ and

$$(\chi_1)_{\overline{G}'} = \sum_{i=1}^{k} \psi_1^{g_i},$$

where k divides $|\overline{G}/\overline{G}'|$. Hence $\chi(1) = p^f = kp^m$ divides $2mp^m$. As p is odd, so p^f divides mp^m.

Take $\psi_2 \in \operatorname{Irr} \overline{G}'$ such that $\psi_2(1) = p^m - 1$. As $\psi_2(1)$ divides some degree of G, so $p^m - 1$ divides $p^{2f} - 1$. Therefore m divides $2f$. As $m < f$, so $m \leq \frac{2}{3} f$. Hence

$$p^f \leq mp^m \leq \frac{2}{3} f p^{2f/3}.$$

This produces the contradiction

$$3^{f/3} \leq p^{f/3} \leq \frac{2}{3} f.$$

Hence $m = f$ and $G'/M \cong \mathrm{PSL}(2, p^f)$.

Case 2. Now we have to exclude the possibility that $S \cong \mathrm{PSL}(2, 2^s)$, where $p = 2^s - 1$ or $p = 2^s + 1$ or $p = 3$ and $S \cong \mathrm{PSL}(2, 8)$. Obviously $k = 1$, hence

$$G'/M \cong \mathrm{PSL}(2, 2^s).$$

We put $\overline{G} = G/C_G(G'/M)$. Then $|\overline{G} : \overline{G}'|$ divides $|\mathrm{Out}\,\mathrm{PSL}(2, 2^s)| = s$. Let $\psi \in \mathrm{Irr}\,\overline{G}'$ and $\psi(1) = p$ or $\psi(1) = 3^2$. If $\chi \in \mathrm{Irr}\,\overline{G}$ and χ is above ψ, then $\chi(1) = e\psi(1)$, where e divides s. As p divides $\chi(1)$, so $\chi(1) = p^f$ divides $s\psi(1)$.

If $\psi(1) = p$, we obtain $s \geq p^{f-1}$ and then

$$p = 2^s \pm 1 \geq 2^{p^{f-1}} \pm 1,$$

which implies $f = 1$.

Suppose at first that $p = 2^s + 1$. If $\tau \in \mathrm{Irr}\,\overline{G}'$ and $\tau(1) = 2^s - 1$, then $\tau(1)$ divides $p - 1 = 2^s$ or $p + 1 = 2^s + 2$. This implies

$$2^s + 2 = 2(2^s - 1),$$

a contradiction as $p^f = p = 2^s + 1 > 5$.

Next suppose that $p = 2^s - 1$. If $\tau \in \mathrm{Irr}\,\overline{G}'$ and $\tau(1) = 2^s + 1$, we again obtain that $2^s + 1$ divides $p + 1 = 2^s$ or $p - 1 = 2^s - 2$, in both cases a contradiction. Hence there remains only the possibility that $p = 3$ and $S \cong \mathrm{PSL}(2, 8)$. We take $\psi \in \mathrm{Irr}\,\overline{G}'$ such that $\psi(1) = 3^2$ and $\chi \in \mathrm{Irr}\,\overline{G}$ such that χ is above ψ. Then $\psi(1) = 3^2$ divides $\chi(1) = 3^f$ and $\chi(1)$ divides $3\psi(1) = 3^3$. As $7 \in \mathrm{cd}\,\mathrm{PSL}(2, 8)$, so 7 divides some degree of $\overline{G}$. As

$$\mathrm{cd}\,\overline{G} \subseteq \mathrm{cd}\,G = \{1, 3^f - 1, 3^f + 1, r\},$$

where $r = \dfrac{1}{2}(3^f \pm 1)$, so $f = 3$ and

$$\mathrm{cd}\,G = \{1, 13, 26, 27, 28\}.$$

But there exists $\rho \in \mathrm{Irr}\,\overline{G}'$ such that $\rho(1) = 8$, and 8 does not divide any degree of G.

Hence this case is impossible.

Step 3. If $\vartheta \in \mathrm{Irr}\,M$, then $I_{G'}(\vartheta) = G'$ and therefore $M' = [M, G']$:

Suppose $I = I_{G'}(\vartheta) < G'$ for some $\vartheta \in \mathrm{Irr}\,M$ and

$$\vartheta^I = \sum_i \varphi_i, \quad \varphi_i \in \mathrm{Irr}\,I.$$

Then $\varphi_i^{G'} \in \mathrm{Irr}\,G'$, hence

$$|G' : I| \cdot \varphi_i(1) \in \mathrm{cd}\,G'.$$

The proper subgroups of $G'/M \cong \mathrm{PSL}(2, p^f)$ are of indices at least $p^f + 1$, with the exceptions (observe $p^f > 5$) that

$$\begin{aligned} p^f &= |G' : I| = 7 && \text{and} && I/M \cong S_4, \\ p^f &= |G' : I| = 11 && \text{and} && I/M \cong A_5, \\ p^f &= 9,\ |G' : I| = 6 && \text{and} && I/M \cong A_5. \end{aligned}$$

(Huppert [6], p. 214.) The last possibility is excluded as 6 does not divide any degree of PSL(2, 9). So in all cases $\varphi_i(1) = 1$, hence φ_i is an extension of ϑ to I. Therefore

$$(\varphi_i\tau)^{G'} \in \text{Irr } G'$$

for all $\tau \in \text{Irr } I/M$. So $|G' : I| \cdot \tau(1)$ divides some degree of G.

If $|G' : I| = p^f + 1$, then I/M is metabelian of order $p^f(p^f - 1)/2$, so there exists $\tau \in \text{Irr } I/M$ such that $\tau(1) = \frac{1}{2}(p^f - 1)$. But this provides a contradiction as $\frac{1}{2}(p^f + 1)(p^f - 1)$ does not divide any degree of G.

In the remaining exceptional cases we can choose

$$\begin{array}{llll} \tau(1) = 3 & \text{if} \quad p^f = 7 & \text{and} & I/M \cong S_4, \\ \tau(1) = 4 & \text{if} \quad p^f = 11 & \text{and} & I/M \cong A_5. \end{array}$$

This produces the forbidden degrees $3 \cdot 7$ resp. $4 \cdot 11$ of G'.

Hence $I_{G'}(\vartheta) = G'$ for all $\vartheta \in \text{Irr } M$. Therefore by lemma 6 of part I we obtain $M' = [M, G']$.

Step 4. We have $|M/M'| \leq 2$:

By lemma 6 of part I, $|M/M'|$ is bounded by the order of the Schur multiplier of $\text{PSL}(2, p^f)$. Therefore $|M/M'| \leq 2$ if $p^f \neq 9$ (Huppert [6], p. 646). But the Schur multiplier of $\text{PSL}(2, 9) \cong A_6$ has order 6.

Suppose $p^f = 9$ and $|M/M'| > 2$. From the Atlas, p. 5 we obtain the degrees of the irreducible characters of G'/M, which do not have M/M' in their kernel, namely

$$3,\ 6,\ 9,\ 15 \quad \text{if} \quad |M/M'| = 3\,,$$
$$6,\ 12 \quad \text{if} \quad |M/M'| = 6.$$

In this case we have

$$\text{cd } G = \text{cd PSL}(2, 9) = \{1, 5, 8, 9, 10\}.$$

As 6 does not divide any degree of G, so $|M/M'| > 2$ is not possible in the case $p^f = 9$.

Therefore $|M/M'| \leq 2$ in all cases.

Step 5. We claim that $\text{cd } M \subseteq \{1, 2\}$ and hence $M'' = E$:

Suppose $\vartheta \in \text{Irr } M$ and $\vartheta(1) > 1$. If ϑ allows an extension ϑ_0 to G', then $\vartheta_0\,\tau \in \text{Irr } G'$ for all $\tau \in \text{Irr } (G'/M)$. Taking $\tau(1) = p^f + 1$, we obtain

$$(p^f + 1)\vartheta(1) \in \text{cd } G'.$$

But as $\vartheta(1) > 1$, so $(p^f + 1)\vartheta(1)$ does not divide any degree of G. Hence ϑ does not allow any extension to G'.

If $\varphi \in \operatorname{Irr} G'$ and $(\varphi_M, \vartheta)_M > 0$, then, as $I_{G'}(\vartheta) = G'$ by lemma 3*c*) of part I, $\varphi = \vartheta_0\, \tau_0$, where ϑ_0 and τ_0 are characters of irreducible projective representations of G', $\vartheta_0(1) = \vartheta(1)$ and τ_0 is the character of an irreducible, projective, non ordinary representation of $G'/M \cong \mathrm{PSL}(2, p^f)$. The degrees of these representations are $p^f - 1$, $p^f + 1$ and s, where

$$s = \frac{p^f + 1}{2} \qquad \text{if } p^f \equiv -1 \pmod 4,$$

$$s = \frac{p^f - 1}{2} \qquad \text{if } p^f \equiv 1 \pmod 4.$$

(See Dornhoff [3], p. 228.) As $\vartheta(1) > 1$, so

$$(p^f \pm 1)\, \vartheta(1) \notin \operatorname{cd} G'.$$

There remains only the possibility that $s\vartheta(1) \in \operatorname{cd} G'$, which implies $\vartheta(1) \leq 2$. Hence $\operatorname{cd} M \subseteq \{1, 2\}$, and therefore $M'' = E$ (Isaacs [8], p. 202).

Step 6. We have $\operatorname{cd} G/M = \operatorname{cd} G$:
As

$$\operatorname{cd} G'/M = \{1, p^f - 1, p^f, p^f + 1, r\}$$

and $\operatorname{cd} G/M \subseteq \operatorname{cd} G$, we see immediately that

$$p^f - 1, p^f, p^f + 1 \in \operatorname{cd} G/M.$$

Take $\chi \in \operatorname{Irr} G$ such that

$$\chi(1) = r = \frac{p^f \pm 1}{2}.$$

As r is odd and $\operatorname{cd} M \subseteq \{1, 2\}$ by step 5, so

$$\chi_M = \sum_i \lambda_i, \quad \lambda_i(1) = 1.$$

Hence $M' \leq \ker \chi$. If $M = M'$, then $\chi \in \operatorname{Irr} G/M$ and

$$\chi(1) = r \in \operatorname{cd} G/M.$$

Suppose $|M/M'| = 2$. As $G' = G''$, so

$$\chi_{G'} = \sum_i \rho_j, \quad \rho_j \in \mathrm{Irr}\, G',\ \rho_j(1) > 1.$$

If ρ_j is faithful on M/M', then $\rho_j(1)$ is one of the values $p^f \pm 1$ or s. But as $\rho_j(1)$ divides $\chi(1) = r$, this is impossible. Hence $M \leq \ker \rho_j$, so $\chi \in \mathrm{Irr}\, G/M$ also in this case.

Step 7. We now claim $G/M = G'/M \times C_{G/M}(G'/M)$:

The characters of G'/M are all invariant under G, for otherwise by fusion of characters of G'/M we obtain a forbidden degree which is a proper multiple of p^f or $p^f \pm 1$, or by fusion of the two characters of degree $r = \frac{1}{2}(p^f \pm 1)$ we lose the degree r, which by step 6 is a degree of G/M. As the characters of G'/M separate the conjugacy classes, so G preserves the conjugacy classes of G'/M. Hence by a theorem of Feit and Seitz [4], G induces only inner automorphisms on G'/M. (As $G'/M \cong \mathrm{PSL}(2, p^f)$, this can be seen directly.) Therefore

$$G/M = G'/M \times C_{G/M}(G'/M).$$

Step 8. We have $M = E$, and hence

$$G \cong \mathrm{PSL}(2, p^f) \times A,$$

where A obviously is abelian:

By step 4 we know $|M/M'| \leq 2$. Suppose $|M/M'| = 2$. If $p^f \neq 9$, then G'/M' is the uniquely determined Schur covering group of $G'/M \cong \mathrm{PSL}(2, p^f)$, so

$$G'/M' \cong \mathrm{SL}(2, p^f).$$

This also is true for $p^f = 9$ and $|M/M'| = 2$.

We put $C/M = C_{G/M}(G'/M)$. If $x \in G'/M'$ and $\mathrm{ord}\, x = p$, $y \in C/M'$, then $x^y \in xM/M'$. As $\mathrm{ord}\, x^y = p > 2$, this implies $x^y = x$. Therefore C/M' centralizes G'/M'. Hence by step 7

$$G/M' = G'/M' \cdot C/M'$$

is a central product with amalgamated subgroup $M/M' = \langle zM' \rangle$. Take $\psi \in \mathrm{Irr}\, C/M'$ such that $\psi(z) = -\psi(1)$. If $\chi \in \mathrm{Irr}\, G'/M'$ and $\chi(z) = -\chi(1)$, then

$$\chi\psi \in \mathrm{Irr}\,(G'/M' \times C/M')$$

and

$$\chi(z)\,\psi(z^{-1}) = \chi(1)\,\psi(1).$$

Hence $(z, z^{-1}) \in \ker \chi\,\psi$. Therefore $\chi\psi$ is a character of

$$(G'(M' \times C/M')/\langle(zM', z^{-1}M')\rangle \cong G'/M' \cdot C/M' = G/M'.$$

Hence $\chi(1)\,\psi(1) \in \mathrm{cd}\,G$. The characters χ of G'/M' with M/M' not in their kernel have the degrees $p^f - 1$, $p^f + 1$ and s. Hence $(p^f - 1)\,\psi(1)$, $(p^f + 1)\,\psi(1)$ and $s\psi(1)$ are in $\mathrm{cd}\,G$. This implies $\psi(1) = 1$ and then the contradiction $s \in \mathrm{cd}\,G$. Hence $|M/M'| = 1$, so by step 5 $M = M' = = M'' = E$. □

Added during proof. Recently I was informed of the following results: Let G be nonsolvable.

a) If the degree graph $\Delta(G)$ has 3 components, then $G \cong \mathrm{PSL}\,(2, 2^f) \times \times A$, where A is abelian.

b) If $\Delta(G)$ has 2 components, then $\mathrm{PSL}\,(2, p^f)$ is the only nonsolvable composition factor of G.

(M. L. Lewis, D. L. White, J. Algebra **266** (2003), 51-76 and **283** (2005), 80-92.)

As $\Delta(G)$ for solvable G has at most 2 components, so *a*) proves theorem 2 of [7].

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