

On some Classes of Divisible Modules.

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Dedicated to Guido Zappa on his 90th birthday

ABSTRACT - Commutative integral domains satisfying the property that h -divisible modules are finitely injective are characterized, as well as those Prüfer domains such that FP-injective modules are finitely injective. The latter property is also investigated for almost perfect domains.

0. Introduction.

It is a classical result due to Kaplansky [8] in 1952, that a commutative integral domain R is a Dedekind domain if and only if the two classes $\mathcal{D}(R)$ of divisible modules and $\mathcal{I}(R)$ of injective modules coincide. Thus, given a domain which is not a Dedekind domain, there is a strict inclusion $\mathcal{I}(R) \subset \mathcal{D}(R)$. In this case there are three further notable classes of R -modules falling between the classes $\mathcal{I}(R)$ and $\mathcal{D}(R)$, all attracting the interest of a number of researchers.

The first class, denoted by $\mathcal{FI}(R)$, consists of the *finitely injective* modules M (also called *locally injective* in [1] and [5]), defined by the property that every finite subset of M is contained in an injective submodule.

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The second class, denoted by $h\mathcal{D}(R)$, consists of the *h-divisible* modules, which are the epimorphic images of injective modules.

The third class, denoted by $\mathcal{FPI}(R)$, is the class of the FP-*injective* modules; it consists of the modules A such that $\text{Ext}_R^1(F, A) = 0$ for all finitely presented modules F . It is well known that the inclusions $\mathcal{I}(R) \subseteq \mathcal{FI}(R) \subseteq h\mathcal{D}(R) \cap \mathcal{FPI}(R) \subseteq \mathcal{D}(R)$ hold (see Corollary 1.3 below).

The containment relations between these classes of modules is shown in the diagram below. (Thanks to Dr D. K. Smith, Exeter, for the LaTeX code of this diagram.)

$$\begin{array}{ccc}
 & \mathcal{D}(R) & \\
 \hookleftarrow & & \hookrightarrow \\
 h\mathcal{D}(R) & & \mathcal{FPI}(R) \\
 \hookrightarrow & & \hookleftarrow \\
 & \mathcal{FI}(R) & \\
 & \cup & \\
 & \mathcal{I}(R) &
 \end{array}$$

In general, all these inclusions are strict, so one obtains genuine characterizations of integral domains by equating certain pairs in the diagram. We have already mentioned that $\mathcal{I}(R) = \mathcal{D}(R)$ characterizes Dedekind domains. The equality $\mathcal{I}(R) = \mathcal{FI}(R)$ characterizes Noetherian domains (this latter result holds for arbitrary rings R , see [15]); for these domains the equality $\mathcal{FI}(R) = \mathcal{FPI}(R)$ also holds (see [6, IX. 3.2]). The equality $h\mathcal{D}(R) = \mathcal{D}(R)$ characterizes Matlis domains, that is, the domains whose field of quotients Q has projective dimension ≤ 1 (see [12]). Finally, the equality $\mathcal{FPI}(R) = \mathcal{D}(R)$ characterizes Prüfer domains (see [13] and [6, IX. 3]).

The aim of this paper is to investigate the remaining two inclusions, that is to say, the relationships between the three classes $\mathcal{FI}(R)$, $h\mathcal{D}(R)$ and $\mathcal{FPI}(R)$ in the middle of the above diagram. Some results in this direction were proved by Facchini [5] for valuation domains. More precisely, he showed that, if R is an almost maximal valuation domain, then the equality $\mathcal{FI}(R) = h\mathcal{D}(R)$ holds, and that for such a domain R the equality $\mathcal{FI}(R) = \mathcal{FPI}(R)$ holds precisely if R is a Matlis domain.

The first goal of this paper is to prove that, for an arbitrary domain R , the two classes $\mathcal{FI}(R)$ and $h\mathcal{D}(R)$ coincide if and only if R is an almost maximal

Prüfer domain or, equivalently, an FSI domain (i.e. the classical quotient ring of every factor ring is self-injective, see [17]), this is our Theorem 2.4. This theorem extends Facchini's result to arbitrary domains and improves Olberding's result in [14], stating that a domain R such that all factor modules of its field of quotients Q are injective is an almost maximal Prüfer domain; in fact, in Theorem 2.4 we reach the same conclusion just assuming that all the quotients of Q are finitely injective. Our proof is inspired by the methods in [17] and uses completely different techniques with respect to Olberding's proof.

The second goal of this paper is to investigate when FP-injective modules are finitely injective. An easy consequence of Theorem 2.4 is that, for a Prüfer domain R , the equality $\mathcal{FI}(R) = \mathcal{FPI}(R)$ holds if and only if R is an almost maximal Matlis domain (Corollary 3.1). Furthermore, we will show that the above equality forces an almost perfect domain R to be Noetherian, provided that R is countable.

1. Preliminaries and review of known results.

All the rings considered are commutative with 1, even if some results in this paper hold over arbitrary associative rings. For all unexplained terminology and notation we refer to [6]. We recall the characterization of FP-injective modules over arbitrary rings given by Megibben [13] (see also [6, IX. 3.1]) that we shall use later on; recall that a module is absolutely pure if it is pure in every module in which is contained as a submodule.

THEOREM 1.1 (Megibben [13]). *For a module M over a ring R the following are equivalent:*

- (1) M is FP-injective;
- (2) M is absolutely pure;
- (3) M is pure in its injective hull.

In order to show that a finitely injective module is FP-injective, consider the following strong notion of purity which first appeared in a paper by Chase [4]. This notion was investigated by Rangaswamy and its collaborators [7, 15, 16], by Azumaya [1] and also by Zimmermann [19]. Following Zimmermann, we say that a submodule A of a module B is *s-pure* (short for *strongly pure*) if for every finite set of elements $x_1, \dots, x_n \in A$, there exists a homomorphism $\phi: B \rightarrow A$ such that $\phi x_i = x_i$ for all $i = 1, \dots, n$. (Note that s-pure submodules are called *locally split* in [1] and [5]). It is obvious that s-purity implies purity.

A module M is said to be *absolutely s-pure* if it is s-pure in every module in which it is contained as a submodule (absolutely s-pure modules are called *strongly absolutely pure* in [15]). Recall that an epimorphism $\pi : N \rightarrow M$ is said to be *locally split* if, for every finite set of elements $x_1, \dots, x_n \in M$, there exists a homomorphism $\sigma : M \rightarrow N$ such that $\pi\sigma x_i = x_i$ for all $i = 1, \dots, n$. The next theorem was partly proved in [15]. As usual, $E(M)$ will denote the injective hull of a module M .

THEOREM 1.2 (Ramamurthi and Rangaswamy [15]). *For a module M over a ring R the following are equivalent:*

- (1) M is finitely injective;
- (2) if A is a finitely generated submodule of a module B , every homomorphism from A to M extends to a homomorphism from B to M ;
- (3) M is absolutely s-pure;
- (4) M is s-pure in its injective hull;
- (5) M is a locally split epic image of a direct sum of injective modules;
- (6) every element of M is contained in an injective submodule of M .

PROOF. The equivalence of (1), (2), (3) and (6) is proved in [15]. Trivially (3) implies (4). Conversely, let M be a submodule of the module N . Then $E(N) = E(M) \oplus B$ and, given a fixed finitely generated submodule X of M , there exists a morphism $\phi : E(M) \rightarrow M$ such that $\phi(x) = x$ for every $x \in X$. Then ϕ extends to a morphism $\psi : E(N) \rightarrow M$ which, restricted to N , gives the required map. So (4) implies (3). Next, (1) obviously implies (5); for the converse, the same proof as in [19, Theorem 2.1, (6) \Rightarrow (5)] applies, replacing the term “pure-injective” by the term “injective”. To finish, we give a simple direct proof of (5) \Rightarrow (6). Let $\pi : \bigoplus_{i \in I} E_i \rightarrow M$ be a locally split epimorphism, with E_i injective for all i , and let $0 \neq x \in M$. By our hypothesis, there is a homomorphism $\sigma : M \rightarrow \bigoplus_{i \in I} E_i$ such that $\pi(\sigma(x)) = x$. So there is a finite subset J of I such that $\sigma(x) \in \bigoplus_{i \in J} E_i = E$. Let E' be the injective hull of $\sigma(Rx)$ in E . Then π restricted to $\sigma(Rx)$, and hence to E' , is monic. It now follows that $x \in \pi(E')$, which is an injective submodule of M . \square

COROLLARY 1.3. *If R is an integral domain, then the inclusion $\mathcal{FI}(R) \subseteq h\mathcal{D}(R) \cap \mathcal{FPI}(R)$ holds.*

PROOF. The inclusion $\mathcal{FI}(R) \subseteq \mathcal{FPI}(R)$ follows from the equivalence of (1) and (3) in Theorem 1.2, the equivalence of (1) and (2) in Theorem 1.1, and

the obvious fact that an absolutely s -pure module is absolutely pure. The inclusion $\mathcal{FI}(R) \subseteq h\mathcal{D}(R)$ follows from the equivalence of (1) and (5) in Theorem 1.2.

The next Proposition 1.5 was recently proved by Laradji [11], who answered Problem 33 in [6]; it improves the well known result (see [14, Lemma 2.4]) stating that a domain R is Prüfer provided that Q/I is injective for every ideal I of R (Q always denotes the field of quotients of R). We include the core of its proof for sake of completeness, making use of a lemma with a standard shifting argument. We will write $\text{Ext}(-, -)$ for $\text{Ext}_R(-, -)$, this will cause no ambiguity since we'll keep the ring R fixed.

LEMMA 1.4. *Given an ideal J of the domain R , Q/J is FP-injective if and only if $\text{Ext}^2(F, J) = 0$ for all finitely presented modules F .*

PROOF. From the exact sequence $0 \rightarrow J \rightarrow Q \rightarrow Q/J \rightarrow 0$ one gets

$$\text{Ext}^1(F, Q) = 0 \rightarrow \text{Ext}^1(F, Q/J) \rightarrow \text{Ext}^2(F, J) \rightarrow \text{Ext}^2(F, Q) = 0$$

so $\text{Ext}^1(F, Q/J) = 0$ if and only if $\text{Ext}^2(F, J) = 0$. \square

PROPOSITION 1.5 (Laradji [11]). *Let R be a domain with field of quotients Q . All the quotients of Q of the form $Q/(aR \cap bR)$ ($a, b \in R$) are FP-injective if and only if R is a Prüfer domain.*

PROOF. The sufficiency is well known (see, e.g., [6, IX. 3.4]). For the necessity it is enough to prove that a two-generated ideal $aR + bR$ of R is projective (see [6, I. 2.8]). Apply the functor $\text{Ext}^1(-, aR \cap bR)$ to the exact sequence

$$0 \rightarrow aR + bR \rightarrow R \rightarrow R/(aR + bR) \rightarrow 0$$

yielding

$$0 = \text{Ext}^1(R, aR \cap bR) \rightarrow \text{Ext}^1(aR + bR, aR \cap bR) \rightarrow \text{Ext}^2(R/(aR + bR), aR \cap bR).$$

The last term vanishes by Lemma 1.4, since $Q/(aR \cap bR)$ is FP-injective, hence the second Ext vanishes and consequently the exact sequence

$$0 \rightarrow aR \cap bR \rightarrow aR \oplus bR \rightarrow aR + bR \rightarrow 0$$

splits. Therefore $aR + bR$ is projective. \square

In the proof of the main Theorem 2.4 of the next section, a crucial role is played by commutative (von Neumann) regular rings R ; recall that these

are defined by the equational property that, for every $a \in R$, there exists an $x \in R$ such that $a = a^2x$. The following facts concerning commutative regular rings, freely used in the sequel, are well known: each factor ring and each localization at a multiplicative set of a regular ring is still regular; a local regular ring is a field. We will use also the two following results concerning regular rings. The first one is attributed by Lambeck to Johnson and Utumi.

LEMMA 1.6 ([10, Prop. 1, p. 102]). *Let E be an injective module over a ring R and A its endomorphism ring. Then the Jacobson radical $J = J(A)$ of A consists of the endomorphisms annihilating some essential submodule of E , A/J is regular and idempotents lift modulo J .*

The next result is attributed by Klatt and Levy to Osofsky.

LEMMA 1.7 ([9, Lemma 3.3]). *Let T be a commutative regular ring such that all its factor rings are self-injective. Then T is a finite direct product of fields.*

2. h -divisible versus finitely injective modules.

This section is devoted to providing the characterization of those domains for which h -divisible modules are finitely injective.

Our next lemma deals with the regular factor rings of Prüfer domains.

LEMMA 2.1. *Let R be a Prüfer domain with an ideal I such that the factor ring R/I is regular. Then $I : I = R$.*

PROOF. We have that $I : I = \bigcap_{M \in \text{Max}(R)} I_M : I_M$ (see [6, I.2.6]). If I is not contained in M , then $I_M = R_M$, hence $I_M : I_M = R_M$. If $I \subseteq M$, then $R_M/I_M \cong (R/I)_M$ is a field, therefore I_M is a maximal ideal of the valuation domain R_M . Hence we see that $I_M : I_M = R_M$ in all cases. Therefore $I : I = \bigcap_{M \in \text{Max}(R)} I_M : I_M = \bigcap_{M \in \text{Max}(R)} R_M = R$. \square

In Lemma 2.1, we have considered ideals I whose endomorphism ring is canonically isomorphic to R . This property will be crucial in the proof of Theorem 2.4. Further examples of ideals I of a Prüfer domain R satisfying the condition $I : I = R$ are the locally archimedean ideals. We will also use

the simple fact that, if I satisfies this condition then so does the product of I and an invertible ideal. The following lemma shows a consequence of this condition and the fact that the R -module Q/I is finitely injective. Note that for a module M and subset A of R , $M[A]$ denotes the annihilator of A in M .

LEMMA 2.2. *Let I be an ideal of a domain R such that $I : I = R$ and Q/I is finitely injective. Then R/I is self-injective.*

PROOF. Since Q/I is finitely injective, R/I is contained in an injective summand E of Q/I . So we have the inclusions:

$$R/I \subseteq E[I] \subseteq (Q/I)[I] = (I : I)/I = R/I.$$

Therefore R/I equals $E[I]$, which is an injective R/I -module. \square

Recall now that a proper non-zero ideal I of a ring R is said to be a *waist* ideal if, given any ideal J of R , either $J \supseteq I$ or $J \subseteq I$; this is equivalent to saying that, given an element $t \in R \setminus I$, $tR \supseteq I$. The next lemma is well known, but we include its proof for the sake of completeness.

LEMMA 2.3. *A ring R containing a waist ideal I is indecomposable.*

PROOF. Assume that $R = A \oplus B$, with A and B ideals and $A \neq 0$. If $A \subseteq I$, then necessarily $B \subseteq I$, so $I = R$ but this is absurd, since I is proper by definition. If $A \not\subseteq I$, then $B \subseteq I$, otherwise $0 \neq I \subseteq A \cap B$. Thus $B \subseteq A$ and consequently $B = 0$, therefore R is indecomposable. \square

We are now in position to prove the main result of this section.

THEOREM 2.4. *For a domain R with field of quotients Q the following are equivalent:*

- (1) *R is an almost maximal Prüfer domain;*
- (2) *h -divisible modules are finitely injective;*
- (3) *every epic image of Q is finitely injective.*

PROOF. Assume that (1) holds. As the torsion part of an h -divisible module splits and a torsion-free h -divisible module is injective, it is enough to prove that torsion h -divisible modules are finitely injective. If R is an almost maximal Prüfer domain, torsion modules have primary decomposition, so it is enough to consider the local case of R an almost maximal valuation domain. But then (2) follows from [5, Theorem 3].

Clearly (2) \Rightarrow (3).

Let us assume (3), that is to say suppose that Q/K is a finitely injective module for every submodule $K \subseteq Q$. Since finitely injective modules are FP-injective, R is a Prüfer domain by Proposition 1.5. We claim that every localization R_S at a multiplicative subset S of R also satisfies condition (3). In fact, let K be an R_S -submodule of Q and $x \in Q/K$. Then $Q/K = B \oplus C$, with B an injective R -module containing x . For each $s \in S$, the multiplication by s induces an automorphism of B , as $sB = B$ and $B[s] = 0$, since $(Q/K)[s] = 0$; hence B is an R_S -submodule of Q/K , so it is injective also as R_S -module (see [6, IX.1.(F)]). Thus Q/K is a finitely injective R_S -module, as desired. As a consequence of this, we infer that every localization R_P of R at a maximal ideal P is an almost maximal valuation domain. Indeed, R_P is a valuation domain satisfying condition (3); the finitely injective R_P -module Q/R_P is indecomposable, hence injective; this implies that R_P is almost maximal (see [6, IX. 4.4]). In view of the characterization of almost maximal Prüfer domains in [3] as locally almost maximal h -local domains (see also [6, IV. 3.9]), it will now be enough to prove that R is h -local, that is to say, it is of finite character and every non-zero prime ideal is contained in a unique maximal ideal.

Let $0 \neq x \in R$. As $Rx : Rx = R$, R/Rx is self-injective by Lemma 2.2. Let J/Rx denote the Jacobson radical of the ring R/Rx . Then, applying Lemma 1.6 to the injective module R/Rx over itself, we deduce that the ring R/J is regular. Since R/J has the same number of maximal ideals as R/Rx , in order to prove that x belongs to only finitely many maximal ideals of R , we must prove that R/J has only finitely many maximal ideals, namely, it is a finite product of fields. By Lemma 1.7, it is enough to show that all the factor rings of R/J are self-injective. So, let $J \subseteq I \subseteq R$ and consider the factor ring R/I , which is still regular. Lemma 2.2 implies that R/I is self-injective, provided that $I : I = R$. But this is ensured by Lemma 2.1. Henceforth we conclude that R has finite character.

Finally, we will show that every non-zero prime ideal P of R is contained in a unique maximal ideal. Let us assume, by way of contradiction, that there are two maximal ideals $M_1 \neq M_2$ containing P . Let $S = R \setminus (M_1 \cup M_2)$ and consider the localization R_S of R at S . The ring R_S has exactly two maximal ideals and satisfies condition (3), as shown above. So we may assume that M_1 and M_2 are the only maximal ideals of R . Let $J = M_1 \cap M_2$ be the Jacobson radical of R and pick a non-zero element $x \in P \subseteq M_1 \cap M_2$. Then, as in the preceding proof, R/Rx is self-injective with J/Rx as its Jacobson radical, so R/J is regular whence $J : J = R$ by Lemma 2.1. Since $Jx : Jx = J : J$, R/Jx is also self-injective,

by Lemma 2.2, and contains $Rx/Jx \cong R/J \cong R/M_1 \oplus R/M_2$ inside its socle. Thus R/Jx contains as a proper direct summand the injective envelope of R/M_1 , hence it is not indecomposable. In order to obtain a contradiction, it is enough to show, by Lemma 2.3, that P/Jx is a waist ideal in R/Jx . Actually, we will prove the stronger condition that P is a waist ideal of R . To this end pick a $t \in R \setminus P$, then we have that $R_{M_i}t \supseteq R_{M_i}P$, $i = 1, 2$ since these localisations are valuation rings and $P \subseteq M_1 \cap M_2$. So $Rt \supseteq P$ showing that P is indeed a waist ideal. This completes the proof of the theorem. \square

REMARK. Looking carefully at the proof of (3) \Rightarrow (1) in Theorem 2.4 above, one can see that the following condition, weaker than (3), need only be assumed: the quotients of Q of the form $Q/(aR \cap bR)$ are FP-injective, and, given any localization R_S at a multiplicative subset S of R and an ideal J of R_S satisfying the condition $J : J = R_S$, Q/J is finitely injective as an R -module or, equivalently, as an R_S -module.

We already recalled in the Introduction the theorem by Brandal [3] and Olberding [14] stating that a domain R is an almost maximal Prüfer domain if and only if all the quotients of Q are injective (see also [6, IX.5.3]). Olberding characterized almost maximal Prüfer domains in [14] in many different ways, weakening the preceding characterization, but always dealing with injective quotients of Q . We also note that the domain case of [17, Theorem B] characterizes almost maximal Prüfer domains as the fractionally self-injective domains.

3. FP-injective versus finitely injective modules.

This section is devoted to the investigation of those domains whose FP-injective modules are finitely injective. We start with an easy consequence of Theorem 2.4, which characterizes these domains among Prüfer domains.

COROLLARY 3.1. *Let R be a Prüfer domain. Then all FP-injective modules are finitely injective if and only if R is a Matlis almost maximal domain.*

PROOF. Recall that over Prüfer domains the equality $\mathcal{FPI}(R) = \mathcal{D}(R)$ holds. Assume that R is a Matlis almost maximal Prüfer domain. Then all divisible modules are h -divisible, and h -divisible modules are finitely in-

jective by Theorem 2.4; hence FP-injective modules are finitely injective. Conversely, assume that all the FP-injective modules are finitely injective. Then all divisible modules are such, hence R is almost maximal by Theorem 2.4, and is Matlis as divisible modules are h -divisible. \square

We derive immediately the following noteworthy consequence.

COROLLARY 3.2. *For a domain R all divisible R -modules are finitely injective if and only if R is a Matlis almost maximal Prüfer domain.* \square

Corollary 3.1 extends to Prüfer domains Facchini's result in [5] for valuation domains. As Facchini already noted, answering a question posed by Azumaya [1], Corollary 3.1 shows that the equality $\mathcal{FI}(R) = \mathcal{FPI}(R)$ does not force a domain R to be Noetherian, even if it is a Matlis domain. However, this happens for certain Matlis domains, namely, almost perfect domains satisfying some cardinal conditions, as we are going to show.

We introduce the following notation: if P is a maximal ideal of a commutative ring R , let us denote by $\kappa(P)$ the cardinality of $R_P/PR_P \cong R/P$, and by $a(P)$ the dimension of P/P^2 as R/P -vector space.

LEMMA 3.3. *Let R be a commutative ring of cardinality $\kappa \geq \aleph_0$. If $\kappa(P)^{a(P)} > \kappa$ for some maximal ideal P , then the strict inclusion $\mathcal{FI}(R) \subset \mathcal{FPI}(R)$ holds.*

PROOF. First we note that $\kappa \geq \kappa(P)$, hence $\kappa(P)^{a(P)} > \kappa$ can happen only if $a(P) \geq \aleph_0$. By [6, I.8.8], there exists a pure submodule N of $E = E(R/P)$ of cardinality $\leq \kappa$, which is FP-injective by Theorem 1.1. If we prove that E has cardinality strictly bigger than κ , then N is a proper submodule of E and, from this fact, it will follow that N cannot be finitely injective, since E is indecomposable. Hence $N \in \mathcal{FPI}(R) \setminus \mathcal{FI}(R)$, as desired.

In order to prove that E has cardinality strictly bigger than κ , it is enough to show that $E[P^2]$ has cardinality $> \kappa$. Now we have:

$$E[P^2]/E[P] \cong \operatorname{Hom}_R(P/P^2, E) \cong \prod_{a(P)} R/P$$

and the cardinality of $\prod_{a(P)} R/P$ is $\kappa(P)^{a(P)} > \kappa$. Hence $E[P^2]$ has cardinality strictly greater than κ as required. \square

We are now in the position to prove that for a countable non-Noetherian almost perfect domain R the strict inclusion $\mathcal{FI}(R) \subset \mathcal{FPI}(R)$ holds. Recall that a domain R is said to be *almost perfect* (see [2]) if all its proper quotients are perfect; as proved in [2], this amounts to say that R is h -local and locally almost perfect.

THEOREM 3.4. *For a countable almost perfect domain R the equality $\mathcal{FI}(R) = \mathcal{FPI}(R)$ holds if and only if R is Noetherian.*

PROOF. Only the proof of the necessity is needed. Thus, let us assume that R is a countable almost perfect non-Noetherian domain. Since R is h -local, there is at least one maximal ideal P such that R_P is non-Noetherian. Note that R_P has cardinality \aleph_0 as well. Recalling that torsion R_P -modules are semiartinian, from [2, Theorem 5.2] there follows that PR_P/P^2R_P has infinite dimension over R_P/PR_P , hence $a(P) \geq \aleph_0$. There follows that $\kappa(P)^{a(P)} \geq 2^{\aleph_0} > \aleph_0$, so Lemma 3.3 applies. \square

We can now furnish an easy example of a domain such that four of the five classes of divisible modules described in the Introduction are different.

EXAMPLE. Let R be a local non-Noetherian countable almost perfect domain. Then the strict inclusions $\mathcal{I}(R) \subset \mathcal{FI}(R) \subset \mathcal{FPI}(R) \subset \mathcal{D}(R)$ hold: the first one since R is non-Noetherian, the second one by Theorem 3.4, the latter since R is not Prüfer (see [2]). However, as almost perfect domains are Matlis domains, $h\mathcal{D}(R) = \mathcal{D}(R)$. Two concrete examples for such a domain R are obtainable in the following way. Let K be a countable field containing a subfield F such that $[K : F] = \aleph_0$. The domain $R = F + XK[X]_{(X)}$ satisfies the required conditions (see [2]). The latter example for such a domain R is taken from [18]. Let V be a countable non-discrete archimedean valuation domain. If V contains a field K , set $R = K + zV$, where z is an element of V of positive value; otherwise V contains a prime number p of positive value, in which case set $R = \mathbb{Z}_p + pV$. For the proof that in both cases R is almost perfect see [18].

Note that Lemma 3.3 does not apply, for instance, to the local almost perfect domain $R = F + XK[X]_{(X)}$, where $F = \mathbb{C}$ and $K = \mathbb{C}(Y)$. In this case $\kappa = \kappa(P) = 2^{\aleph_0}$ and $a(P) = [K : F] = \aleph_0$, so that $\kappa(P)^{a(P)} = 2^{\aleph_0} = \kappa$.

The characterization of general domains R such that the equality $\mathcal{FI}(R) = \mathcal{FPI}(R)$ holds remains an open problem. It is easy to see that this happens if and only if all torsion FP-injective R -modules are finitely injective and the torsion submodules of the FP-injective modules split.

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